CHAPTER V

THE NILPOTENT ALGEBRAS

In this chapter we classify the multiplications in nilpotent algebras of dimensions 1,2 and3. Then we prove a theorem which tells when there exists an isomorphism between a nilpotent algebra and the quotient algebra of the polynomial algebra $K_0[x]$ by the radical (x^{n-1})

Theorem 5.1 : Let A be a nilpotent algebra of dimension 1 over a field K. Then $A^2=\{0\}$, or equivalently $xy=0$ for all x,y in A.

Proof : Since A is a nilpotent algebra over K, there exists a $k>0$ such that $A^k=\{0\}$. Next, we shall prove that $A^2\neq A$. To prove this, suppose that $A^2=A$. Then we have

$$
\begin{aligned}
A^k = A^{k-2}.A^2 &= A^{k-2}.A \\
= A^{k-3}.A^2 &= A^{k-3}.A \\
= A^{k-4}.A^2 &= A^{k-4}.A \\
&\text{-- -- -- -- -- -- --} \\
= A^2.A^2 &= A^2.A \\
= A.A &= A^2 \\
= A &
\end{aligned}
$$

That is $A=A^k=\{0\}$ which contradicts the hypothesis that dimension of A is 1. Therefore $A^2\subset A$ which implies that dim $A^2=0$. This completes the proof of the theorem.

Q.E.D.

<u>Theorem 5.2</u> : Let A be a nilpotent algebra of dimension 2 over a field K. If the multiplication in A is nontrivial, then it is unique (up to isomorphism).

<u>Proof</u> : We can similarly prove as in Theorem 5.1 that $A \supset A^2 \supset A^3 \supset \dots \supset A^k = \{0\}$. Therefore the dimension of A^2 is 1 or 0. If dimension $A^2 = 0$, then this is the trivial case, so we may assume that dimension of A^2 is 1 which implies that $A^3 = \{0\}$. Since dimension $A^2 = 1$, we may let $e_2 \neq 0$ be a basis of A^2. For dimension of A is 2, we can have $e_1 \neq 0$ independent to e_2 such that e_1, e_2 is a basis of A. For $x = a_1e_1 + a_2e_2$, $y = b_1e_1 + b_2e_2$, $\{a_i, b_j\}$ $i,j = 1,2,3$, $\subset K$, we have

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_2b_1e_2e_1 + a_2b_2e_2^2.$$

Since $e_1e_2, e_2e_1 \in A^3 = \{0\}$ and $e_2^2 \in A^4 = \{0\}$,

(1) $\quad xy = a_1b_1e_1^2.$

If $e_1^2 = 0$, then $xy = 0$ for all x,y in A. Therefore $e_1^2 \neq 0$ and we may let $e_1^2 = e_2$. Hence (1) becomes

$$xy = a_1b_1e_2, \text{ for all } x,y \text{ in } A$$

Therefore the nontrivial multiplication in A is unique (up to isomorphism).

<u>Q.E.D.</u>

Next, we consider the case where a nilpotent algebra A over a field K has dimension 3. We can similarly prove (as we did in Theorem 5.1) that $A \supset A^2 \supset A^3 \supset \ldots \supset A^k = \{0\}$. Thus we see that dimension $A^2=2$ or 1 or 0. Dimension $A^2=0$ is the trivial case, so we just consider the case where dimension $A^2=1$, or dim $A^2=2$. If dimension A^2 is 2, then dimension A^3 is 1 or 0 and $A^4=\{0\}$. If dimension $A^2=1$, then $A^3=\{0\}$.

Now, let us start by investigating the case where the dimension of A^2 is 2 and $A^3=\{0\}$. Therefore, we may let e_1 and e_2 be a basis of A^2, and then let e_3 be linearly independent with respect to e_1 and e_2 such that e_1, e_2, e_3 forms basis of A. For x,y inA we can write

$$x = a_1e_1 + a_2e_2+a_3e_3,$$
$$y = b_1e_1 + b_2e_2+b_3e_3, \qquad \{ai, bj\} \subset K,\ i,j=1,2,3,$$

Hence,

$$xy = a_1b_1e_1^2+a_1b_2e_1e_2+a_1b_3e_1e_3+a_2b_1e_2e_1 +a_2b_2e_2^2+a_2b_3e_2e_3+a_3b_1e_3e_1+a_3b_2e_3e_2+a_3b_3e_3^2.$$

Since $e_1^2, e_2^2, e_1e_2, e_2e_1 \in A^4=\{0\}$ and $e_1e_3, e_3e_1, e_2e_3, e_3e_2 \in A^3=\{0\}$, we have

$$xy = a_3b_3e_3^2.$$

and consequently, dimension of A^2 is 1. This contradicts the hypothesis that dimension $A^2=2$, so this case is impossible.

Next, we shall consider the other multiplication cases of a nilpotent algebra of dimension 3. Let us begin with a definition.

<u>Definition 5.3</u> : Let A be an algebra with multiplication o

and B be an algebra with multiplication $*$. Then the multiplications in A and B are <u>isomorphic</u> iff there exists a linear, 1-1, function f of A onto B such that $f(x \circ y) = f(x) * f(y)$.

Let A be an algebra of dimension 3 over $\mathbb{R}$. Suppose that $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ are two distinct bases of A respectively, then claim that the linear mapping $f: A \to A$ such that

(I)
$$f(e_1) = (km)^{1/3} e'_1,$$
$$f(e_2) = \frac{m}{(km)^{1/3}} e'_1 + \left(\frac{-n}{k}\right) e'_2,$$
$$f(e_3) = e'_3,$$

for $k \neq 0$, $m \neq 0$, $n \neq 0$ in R, is 1-1 and onto. To see that f is 1-1 and onto we need only show that the determinant of the coefficients on the right side is not zero. See proof in [3].

$$\det f = \det \begin{bmatrix} (km)^{1/3} & 0 & 0 \\ \frac{m}{(km)^{1/3}} & \left(\frac{-n}{k}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= (km)^{1/3}\left(\frac{-n}{k}\right)$$

Which is not zero for $k \neq 0, m \neq 0, n \neq 0$. Therefore f is linear, 1-1 and onto function on A.

Next we shall show that the following linear maps of A to itself are 1-1 and onto by showing that their determinants are not 0.

(II)
$$f(e_1) = k_1 e'_2,$$
$$f(e_2) = k_2 e'_1,$$
$$f(e_3) = k_3 e'_3, \quad k_i \in \mathbb{R} \text{ and } k_i \neq 0,\ j=1,2,3,$$

f is 1-1, onto, since

$$\det\left[f\right] = \det\begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

$$= -k_1k_2k_3 = 0$$

(III) $f(e_1) = k_1e'_1+k_2e'_2,$

$f(e_2) = k_3e'_2,$

$f(e_3) = e'_3,$ $\{k'_i = 0\} \subset \mathbb{R},\ i = 1,2,3,$

f is 1-1, onto since

$$\det\left[f\right] = \det\begin{bmatrix} k_1 & k_2 & 0 \\ 0 & k_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = k_1k_3 = 0$$

(IV) $f(e_1) = k_1e'_1,$

$f(e_2) = k_2e'_1 + k_2e'_2,$

$f(e_3) = e'_3,$ $\{k'_i = 0\} \subset \mathbb{R},\ i = 1,2,3,$

f is 1-1 and onto, since

$$\det\left[f\right] = \det\begin{bmatrix} k_1 & 0 & 0 \\ k_2 & k_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = k_1k_3 = 0$$

(V) $f(e_1) = e'_1+e'_2,$

$f(e_2) = -e'_1+e'_2,$

$f(e_3) = e'_3,$

f is 1-1, onto, since

$$\det\left[f\right] = \det\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 = 0$$

<u>Theorem 5.4</u> : Let A be a nilpotent algebra of dimension 3 over the field $\mathbb{R}$. If dimension of A^2 is 2 and dimension of $A^3 = 1, A^4 = \{0\}$, then the multiplication in A is uniquely determined up to isomorphism.

<u>Proof</u> From the hypothesis that dimension A = 3, dimension $A^2=2$, dimension $A^3=1$ and $A^4=\{0\}$, we may let $\{e_1,e_2,e_3\}$ be a basis in A such that $\{e_2,e_3\}$ is a basis of A^2 and e_3 is a basis of A^3. For each x,y in A we may write

$$x = a_1e_1+a_2e_2+a_3e_3,$$
$$y = b_1e_1+b_2e_2+b_3e_3, \quad \{ai, bj\} \subset \mathbb{R}, i,j=1,2,3,$$

and thus we obtain

$$xy = a_1b_1e_1^2+a_1b_2e_1e_2+a_1b_3e_1e_3+a_2b_1e_2e_1+a_2b_2e_2^2$$
$$+a_2b_3e_2e_3+a_3b_1e_3e_1+a_3b_2e_3e_2+a_3b_3e_3^2.$$

Since $e_2^2, e_1e_3, e_3e_1 \in A^4 = \{0\}, e_2e_3, e_3e_2 \in A^5 = \{0\}$ and $e_3^2 \in A^6 = \{0\}$, we have

$$xy = a_1b_1e_1^2+a_1b_2e_1e_2+a_2b_1e_2e_1.$$

Since $e_1^2 \in A^2$, we can write $e_1^2 = k_1e_2+k_2e_3$ for some $k_1, k_2 \in \mathbb{R}$ and since $e_1e_2, e_2e_1 \in A^3$ we get $e_1e_2=k_3e_3$ and $e_2e_1=k_4e_3$ for some k_3, k_4 in $\mathbb{R}$. That is, the multiplication xy can be expressed in the form:

$$xy = a_1b_1 (k_1e_2+k_2e_3)+a_1b_2k_3e_3+a_2b_1k_4e_3, \text{ i.e.}$$

(*) $$xy = k_1a_1b_1e_2+(k_2a_1b_1+k_3a_1b_2+k_4a_2b_1)\ e_3.$$

We begin the final step of the proof with an observation about k_1,k_2,k_3,k_4. Since dimension of $A^2=2$, the case $k_1=0$ and the case $k_2=k_3=k_4=0$ cannot occur. The proof now proceeds with 7 cases.

Case 1. In this first case, we consider the multiplication (*) when $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 = k_4 = 0$. In particular,

$$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 , \quad k_1, a_1, b_1, k_2 \in \mathbb{R}. \text{ Therefore,}$$

$$xy = a_1 b_1 (k_1 e_2 + k_2 e_3).$$

This formula holds for all x,y in A and since $k_1 e_2 + k_2 e_3$ is a vector in A, we have dimension $A^2 = 1$ which contradicts the hypothesis. Therefore this first case is impossible.

Case 2. For this second case, we shall investigate the multiplication (*) when $k_1 \neq 0$, $k_4 \neq 0$ and $k_2 = k_3 = 0$. That is

(2.1) $$xy = k_1 a_1 b_1 e_2 + k_4 a_2 b_1 e_3 , \quad k_1, k_4, a_1, a_2, b_1 \in \mathbb{R}$$

Our objective is to check whether A is associative under the multiplication in this case. To do this, let

$$z = c_1 e_1 + c_2 e_2 + c_3 e_3, \quad \{c_i\} \subset \mathbb{R}, \; i = 1,2,3 \text{ and then}$$

consider (xy)z and x(yz). We have,

$$(xy)z = (a_1 e_1 + a_2 e_2 + a_3 e_3)(b_1 e_1 + b_2 e_2 + b_3 e_3)(c_1 e_1 + c_2 e_2 + c_3 e_3)$$

(2.1) asserts that

$$(xy)z = [k_1 a_1 b_1 e_2 + k_4 a_2 b_1 e_3](c_1 e_1 + c_2 e_2 + c_3 e_3)$$
$$= k_4 k_1 (a_1 b_1) c_1 e_3$$

whereas, on the other hand

$$x(yz) = (a_1 e_1 + a_2 e_2 + a_3 e_3)[(b_1 e_1 + b_2 e_2 + b_3 e_3)(c_1 e_1 + c_2 e_2 + c_3 e_3)]$$

$$= (a_1 e_1 + a_2 e_2 + a_3 e_3)(k_1 b_1 c_1 e_2 + k_4 b_2 c_1 e_3)$$

$$= 0$$

Hence A is not associative under the multiplication (2.1) in this case, or equivalently, the multiplication in this case is impossible.

<u>Case 3</u>. Assuming $k_1 \neq 0, k_3 \neq 0$, $k_2 = k_4 = 0$ it follows that

$$xy = k_1 a_1 b_1 e_2 + k_3 a_1 b_2 e_3$$

This case is similar to the second case in that the same method of proof shows that A is not associative under this multiplication. Therefore the multiplication in this case is impossible.

<u>Cose 4</u>. We begin this case by expressing $k_1 \neq 0$, $k_3 \neq 0, k_4 \neq 0$ and $k_2 = 0$. The multiplication (*) becomes,

$$xy = k_1 a_1 b_1 e_2 + (k_3 a_1 b_2 + k_4 a_2 b_1) e_3.$$

Now let e_1', e_2', e_3' be another basis of A such that $e_1' = e_1$, $e_2' = k_1 e_2$, $e_3' = k_1 k_3 e_3$, then for

$$x = a_1' e_1' + a_2' e_2' + a_3' e_3',$$

$$y = b_1' e_1' + b_2' e_2' + b_3' e_3', \ \{a_i', b_j'\} \subset \mathbb{R}, \ i,j = 1,2,3, \text{ we get}$$

$$xy = a_1' b_1' (e_1')^2 + a_1' b_2' e_1' e_2' + a_2' b_1' e_2' e_1'.$$

But we have, $(e_1')^2 = e_1^2 = k_1 e_2 + k_2 e_3 = e_2'$,

$$e_1' e_2' = k_1 e_1 e_2 = k_1 k_3 e_3 = e_3',$$

$$e_2' e_1' = k_1 e_2 e_1 = k_1 k_4 e_3 = \frac{k_4}{k_3} e_3'.$$

Therefore

$$xy = a_1' b_1' e_2' + (a_1' b_2' + \frac{k_4}{k_3} a_2' b_1') e_3'$$

To check the associative law we let $z = c_1' e_1' + c_2' e_2' + c_3' e_3'$, $\{c_i'\} \subset \mathbb{R}$, $i = 1,2,3$, It follows that

$$(xy)z = [(a_1' e_1' + a_2' e_2' + a_3' e_3')(b_1' e_1' + b_2' e_2' + b_3' e_3')]$$

$$(c_1' e_1' + c_2' e_2' + c_3' e_3')$$

$$= \left[a_1'b_1'e_2' + \left(a_1'b_2' + \frac{k_4}{k_3}a_2'b_1'\right) e_3'\right](c_1'e_1' + c_2'e_2' + c_3'e_3')$$

$$= \frac{k_4}{k_3}(a_1'b_1')c_1'e_3',$$

on the other hand,

$$x(yz) = (a_1'e_1' + a_2'e_2' + a_3'e_3')\left[(b_1'e_1' + b_2'e_2' + b_3'e_3')\right.$$
$$\left.(c_1'e_1' + c_2'e_2' + c_3'e_3')\right]$$
$$= (a_1'e_1' + a_2'e_2' + a_3'e_3')\left[b_1'c_1'e_2' + \left(b_1'c_2' + \frac{k_4}{k_3}b_2'c_1'\right)e_3'\right]$$
$$= a_1'b_1'c_1'e_3'.$$

To have $(xy)z = x(yz)$, we must have

$$\frac{k_4}{k_3}\, a_1'b_1'c_1' = a_1'b_1'c_1'.$$

That is $\frac{k_4}{k_3} = 1$. Therefore in this case the multiplication of x,y in A can be expressed as

$$xy = a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3'.$$

Case 5. Set $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$, and $k_3 = 0$. Then the multiplication (*) becomes,

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_2b_1)e_3.$$

The same method of the proof in the second case shows that A is not associative under this multiplication. Therefore, the multiplication in this case is impossible.

Case 6. In this case we have that $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ and $k_4 = 0$. Then from (*) the multiplication xy is

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2)e_3.$$

Similarly to case 2, we can prove that A is not associative under this multiplication. Therefore, the multiplication in this case is impossible.

Case 7. For this final case, let $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ and $k_4 \neq 0$. Then the multiplication (*) is

$$(7.1) \qquad xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_3 a_1 b_2 + k_4 a_2 b_1) e_3.$$

To check associativity, we let $z = c_1 e_1 + c_2 e_2 + c_3 e_3$, $\{c_i\}\ i = 1,2,3 \in R$. Then (7.1) implies that

$$\begin{aligned}(xy)z &= \left[(a_1 e_1 + a_2 e_2 + a_3 e_3)(b_1 e_1 + b_2 e_2 + b_3 e_3)\right] (c_1 e_1 + c_2 e_2 + c_3 e_3) \\ &= \left[k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_3 a_1 b_2 + k_4 a_2 b_1) e_3\right] (c_1 e_1 + c_2 e_2 + c_3 e_3) \\ &= k_4 (k_1 a_1 b_1) c_1 e_3\end{aligned}$$

whereas,

$$\begin{aligned}x(yz) &= (a_1 e_1 + a_2 e_2 + a_3 e_3)\left[(b_1 e_1 + b_2 e_2 + b_3 e_3) (c_1 e_1 + c_2 e_2 + c_3 e_3)\right] \\ &= (a_1 e_1 + a_2 e_2 + a_3 e_3)\left[k_1 b_1 c_1 e_2 + (k_2 b_1 c_1 + k_3 b_1 c_2 + k_4 b_2 c_1) e_3\right] \\ &= k_3 a_1 (k_1 b_1 c_1) e_3\end{aligned}$$

Since A is an associative algebra, we must have

$$(xy)z = x(yz).$$

That is

$$k_1 k_4 a_1 b_1 c_1 = k_1 k_3 a_1 b_1 c_1, \{k_i, a_i, b_i\} \subset R,\ i = 1,2,3,4.$$

Therefore, $k_3 = k_4$ (or else A is not associative). Hence, the multiplication in this case becomes

$$(7.2) \qquad xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_3 a_1 b_2 + k_3 a_2 b_1) e_3.$$

Furthermore, we claim that the multiplication in this case is isomorphic to the multiplication in case 4. In case 4 we have

$$(4.1) \qquad x \circ y = a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3'$$

where $\qquad x = a_1'e_1' + a_2'e_2' + a_3'e_3'$,

$\qquad y = b_1'e_1' + b_2'e_2' + b_3'e_3'$, $\{a_i', b_j'\} \subset \mathbb{R}$, $i,j=1,2,3$.

Let $f: A \to A$ be a function defined by

$$f(e_1) = (k_1k_3)^{1/3}e_1',$$
$$f(e_2) = \frac{k_3}{(k_1k_3)^{1/3}}e_2' - \frac{k_2}{k_1}e_3',$$
$$f(e_3) = e_3', \qquad k_1, k_2, k_3 \in \mathbb{R},$$

for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ in $\mathbb{R}$. Then (7.2) implies, that

$$f(xy) = f\left[k_1a_1b_1e_2 + (k_2a_1b_1 + k_3a_1b_2 + k_3a_2b_1)e_3\right]$$
$$= k_1a_1b_1\left[\frac{k_3}{(k_1k_3)^{1/3}}e_2' - \frac{k_2}{k_1}e_3'\right] + (k_2a_1b_1 + k_3a_1b_2 + k_3a_2b_1)e_3'$$
$$= (k_1k_3)^{2/3}a_1b_1e_2' + k_3(a_1b_2 + a_2b_1)e_3'.$$

On the other hand, the multiolication (4.1) implies that

$$f(x)of(y) = f(a_1e_1 + a_2e_2 + a_3e_3)f(b_1e_1 + b_2e_2 + b_3e_3)$$
$$= \left[(k_1k_3)^{1/3}a_1e_1' + \left(\frac{k_3}{(k_1k_3)^{1/3}}e_2' - \frac{k_2}{k_1}e_3'\right)a_2 + a_3e_3'\right]$$
$$\left[(k_1k_3)^{1/3}b_1e_1' + \left(\frac{k_3}{(k_1k_3)^{1/3}}e_2' - \frac{k_2}{k_1}e_3'\right)b_2 + b_3e_3'\right]$$
$$= \left[(k_1k_3)^{1/3}a_1e_1' + \frac{k_3}{(k_1k_3)^{1/3}}a_2e_2' + \left(-\frac{k_2}{k_1}a_2 + a_3\right)e_3'\right]$$
$$\cdot\left[(k_1k_3)^{1/3}b_1e_1' + \frac{k_3}{(k_1k_3)^{1/3}}b_2e_2' + \left(-\frac{k_2}{k_1}b_2 + b_3\right)e_3'\right]$$
$$= (k_1k_3)^{2/3}a_1b_1e_2' + k_3(a_1b_2 + a_2b_1)e_3'.$$

That is $f(xoy) = f(x)\, f(y)$. This result, together with the previous argument about the map f in case I implies that these two multiplications are isomorphic.

Therefore, we have already proved that the multiplication in a nilpotent algebra A of dimension 3 over a field $\mathbb{R}$ with dimension of $A^2=2$, dimension $A^3=1$, and $A^4=0$, is uniquely determined up to isomorphism.

Q.E.D.

Remark : Suppose A is a nilpotent algebra of dimension 3 with dimension $A^2=1$ and $A^3=\{0\}$. Let $\{e_1,e_2,e_3\}$ and $\{e_1',e_2',e_3'\}$ be bases in A such that e_3 and e_3' are in A^2. Moreover, let $f: A\to A$ be an isomorphism. then $f: A^2\to A^2$. Therefore, $f(e_3)\in A^2$. Consequently, we may write

$$f(e_1) = m_1e_1' + m_2e_2' + m_3e_3',$$
$$f(e_2) = p_1e_1' + p_2e_2' + p_3e_3',$$
$$f(e_3') = qe_3', \quad \{m_i, p_j, q\}\subset\mathbb{R}, j,i=1,2,3.$$

Now we begin our discussion of multiplications in a 3-dimensional nilpotent algebra A over $\mathbb{R}$ with dimension $A^2=1$, by choosing a basis e_1,e_2,e_3 in A such that $e_3\in A^2$. First, note that there is never any need to check associativity in this case since $A^3=\{0\}$. For each x,y in A we have

$$x= a_1e_1+a_2e_2+a_3e_3,$$
$$y= b_1e_1+b_2e_2+b_3e_3, \qquad \{a_i, b_j\}\subset\mathbb{R},\ i=1,2,3$$

It follows that the multiplication is

$$xy= a_1b_1e_1^2+a_1b_2e_1e_2+a_1b_3e_1e_3+a_2b_1e_2e_1+a_2b_2e_2^2$$
$$+a_2b_3e_2e_3+a_3b_1e_3e_1+a_3b_2e_3e_2+a_3b_3e_3^2.$$

Since $e_1e_3, e_3e_1, e_2e_3, e_3e_2 \in A^3 = \{0\}$ and $e_3^2 \in A^4 = \{0\}$, then

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_2b_1e_2e_1 + a_2b_2e_2^2.$$

Since $e_1^2, e_1e_2, e_2e_1, e_2^2 \in A^2$, we may write

$$\begin{aligned} e_1^2 &= k_1e_3, \\ e_1e_2 &= k_2e_3, \\ e_2e_1 &= k_3e_3, \\ e_2^2 &= k_4e_3, \quad \text{for some } k_i \in \mathbb{R},\ i=1,2,3. \end{aligned}$$

Therefore,

$$(**) \qquad xy = (k_1a_1b_1 + k_2a_1b_2 + k_3a_2b_1 + k_4a_2b_2)e_3$$

Our task is to classify the multiplications xy by studying k_1, k_2, k_3, k_4.

We observe that the case where $k_1 = k_2 = k_3 = k_4 = 0$ cannot happen since the dimension of $A^2 = 1$. Therefore, we consider the following cases.

<u>Case 1</u>. If $k_1 \neq 0$ and $k_2 = k_3 = k_4 = 0$, then the multiplication (* *) becomes

$$xy = k_1a_1b_1e_3.$$

As in Theorem 5.4, we may choose a new basis $e_1' = e_1, e_2' = e_2, e_3' = k_1e_3$.

Therefore,

$$xy = a_1'b_1'(e_1')^2 + a_1'b_2'e_1'e_2' + a_2'b_1'e_2'e_1' + a_2'b_2'(e_2')^2,$$

for

$$\begin{aligned} x &= a_1'e_1' + a_2'e_2' + a_3'e_3', \\ y &= b_1'e_1' + b_2'e_2' + b_3'e_3', \quad \{a_i', b_j'\} \subset \mathbb{R},\ i,j=1,2,3. \end{aligned}$$

Since $(e_1')^2 = e_1^2 = k_1e_3 = e_3'$

$$e_1'e_2' = e_1e_2 = k_2e_3 = 0$$

$$e_2'e_1' = e_2e_1 = k_3e_3 = 0$$

$$(e_2')^2 = e_2^2 = k_4e_3 = 0$$

then

(1.1) $xy = a_1'b_1'e_3'$.

<u>Case 2.</u> Let $k_4 \neq 0$ and $k_1 = k_2 = k_3 = 0$. Then the multiplication (**) can be written as

(2.1) $$xoy = k_4a_2b_2e_3.$$

We assert that the multiplication in this case is isomorphic to the multiplication in case 1. Let f be the linear map of A to itself defined by

$$f(e_1') = e_2,$$

$$f(e_2') = e_1,$$

$$f(e_3') = k_4e_3, \quad k_4 \in \mathbb{R}.$$

Then by the argument in page 42 case II we know that f is 1-1 and onto. Multiplication (2.1) implies that

$$\begin{aligned} f(x)of(y) &= f(a_1'e_1'+a_2'e_2'+a_3'e_3')of(b_1'e_1'+b_2'e_2'+b_3'e_3') \\ &= (a_2'e_1+a_1'e_2+k_4a_3'e_3)o\ (b_2'e_1+b_1'e_2 \\ &\quad + k_4b_3'e_3) \\ &= k_4a_1'b_1'e_3, \end{aligned}$$

whereas, the multiplication (1.1) implies that

$$\begin{aligned} f(xy) &= f(a_1'b_1'e_3') \\ &= k_4a_1'b_1'e_3. \end{aligned}$$

$f(xy) = f(x)of(y)$ and these two multiplications are isomorphic.

<u>Case 3.</u> In this case we assume that $k_3 \neq 0$, $k_1 = k_2 = k_4 = 0$. This, together with (**), implies that

$$xy = k_3 a_2 b_1 e_3.$$

Like the other cases we choose a new basis $e_1' = e_1$, $e_2' = e_2$, $e_3' = k_3 e_3$ and get the result,

(3.1) $$xy = a_2' b_1' e_3',$$

where $$x = a_1' e_1' + a_2' e_2' + a_3' e_3' ,$$

$$y = b_1' e_1' + b_2' e_2' + b_3' e_3', \quad \{a_i' , b_j'\} \subset \mathbb{R}, i,j=1,2,3.$$

Notice that A is not a commutative algebra over R under this multiplication, but A is commutative under the multiplication (1.1) in case 1. Therefore, the multiplication in this case is not isomorphic to the one in case 1 (and ∴ in case 2).

<u>Case 4.</u> Starting with the assumption that $k_2 \neq 0$, $k_1 = k_3 = k_4 = 0$, we can write (**) as

(4.1) $$x \circ y = k_2 a_1 b_2 e_3.$$

This multiplication is isomorphic to the multiplication (3.1) in case 3. To show this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_2,$$

$$f(e_2') = e_1,$$

$$f(e_3') = k_2 e_3, \quad k_2 \in \mathbb{R}.$$

We already proved that case II on page 42 is a 1-1, onto map so f is a 1-1, onto map. Then the multiplication (3.1) in case 3 implies that

$$f(xy) = f(a_2' b_1' e_3'),$$

$$= k_2 a_2' b_1' e_3,$$

whereas, on the other hand, (4.1) implies that

$$f(x) \text{ of } (y) = f(a_1'e_1'+a_2'e_2'+a_3'e_3') \circ$$
$$f(b_1'e_1'+b_2'e_2'+b_3'e_3')$$
$$= (a_2'e_1+a_1'e_2+k_2a_3'e_3)\circ(b_2'e_1+b_1'e_2+k_2b_3'e_3)$$
$$= k_2a_2'b_1'e_3.$$

Therefore, it is immediate that these two multiplications are isomorphic.

Case 5. Assume that $k_1 \neq 0$, $k_2 \neq 0$ and $k_3=k_4=0$ in this case. Then (**) becomes

(5.1) $$xoy = k_1a_1b_1e_3 + k_2a_1b_2e_3.$$

We claim that this multiplication is isomorphic to the multiplication (3.1) in case 3. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1) = k_1e_1' + e_2',$$
$$f(e_2) = k_2e_2',$$
$$f(e_3) = e_3', \qquad k_1,k_2 \in \mathbb{R}.$$

Then, the multiplication (5.1) implies that

$$f(xoy) = f\left[(k_1a_1b_1 + k_2a_1b_2)\, e_3\right]$$
$$= (k_1a_1b_1 + k_2a_1b_2)\, e_3',$$

and we use the multiplication (3.1) in case 3 page 53 to get

$$f(x)f(y) = f(a_1e_1+a_2e_2+a_3e_3)f(b_1e_1+b_2e_2+b_3e_3)$$
$$= \left[(k_1a_1+k_2a_2)e_1'+a_1e_2'+a_3e_3'\right]$$
$$\left[(k_1b_1+k_2b_2)e_1'+b_1e_2'+b_3e_3'\right]$$
$$= a_1(k_1b_1+k_2b_2)e_3'$$
$$= (k_1a_1b_1+k_2a_1b_2)e_3'.$$

This with the property of f in case III page 43 implies that these two multiplications are isomorphic.

<u>Case 6.</u> Let $k_3 \neq 0$, $k_4 \neq 0$, $k_1 = k_2 = 0$. Then from (**) we have

$$(6.1) \qquad xoy = (k_3 a_2 b_1 + k_4 a_2 b_2) e_3 .$$

This multiplication is isomorphic to the multiplication in case 3. To prove this, let f: $A \rightarrow A$ be a linear map difined by

$$f(e_1) = k_3 e_1',$$
$$f(e_2) = k_4 e_1' + e_2',$$
$$f(e_3) = e_3', \qquad k_3, k_4 \in \mathbb{R}.$$

Then f is a 1-1, onto map by the case IV page 43. This with the multiplication (6.1) implies that

$$f(xoy) = f\left[(k_3 a_2 b_1 + k_4 a_2 b_2) e_3\right]$$
$$= (k_3 a_2 b_1 + k_4 a_2 b_2) e_3',$$

whereas, from the multiplication (3.1) of case 3 page 53, we have

$$f(x)f(y) = f(a_1 e_1 + a_2 e_2 + a_3 e_3) f(b_1 e_1 + b_2 e_2 + b_3 e_3)$$
$$= \left[a_2 e_1' + (k_3 a_1 + k_4 a_2) e_2' + a_3 e_3'\right]\left[b_2 e_1' + (k_3 b_1 + k_4 b_2) e_2' + b_3 e_3'\right]$$
$$= (k_3 a_1 + k_4 a_2) b_2 e_3'$$
$$= (k_3 a_1 b_2 + k_4 a_2 b_2) e_3'.$$

That is f(xoy) = f(x)f(y), and consequently these two multiplications are isomorphic.

<u>Case 7.</u> We begin this case with the assumption that $k_1 \neq 0$, $k_3 \neq 0$ and $k_2 = k_4 = 0$, then from (**) we have,

$$xoy = (k_1a_1b_1+k_3a_2b_1)\ e_3.$$

We claim that this multiplication is isomorphic to the multiplication in case 3. Let $f: A\to A$ be the linear map defined by

$$f(e_1) = e_1'+k_1e_2' \ ,$$
$$f(e_2) = k_3e_2',$$
$$f(e_3) = e_3'.$$

Then we have that f is a linear, 1-1, onto map by the case III page 43. Moreover

$$f(xoy) = f\left[(k_1a_1b_1+k_3a_2b_1)e_3\right]$$
$$= (k_1a_1b_1+k_3a_2b_1)e_3',$$

whereas, the multiplication (3.1) of case 3 page 53 gives

$$f(x)f(y) = f(a_1e_1+a_2e_2+a_3e_3)f(b_1e_1+b_2e_2+b_3e_3)$$
$$= \left[a_1e_1'+(k_1a_1+k_3a_2)e_2'+a_3e_3'\right]$$
$$\left[b_1e_1'+(k_1b_1+k_3b_2)e_2'+b_3e_3'\right]$$
$$= (k_1a_1+k_3a_2)b_1e_3'$$
$$= (k_1a_1b_1+k_3a_2b_1)e_3'.$$

Therefore, these two multiplications are isomorphic.

<u>Case 8.</u> In this case we take $k_2\neq 0$, $k_4\neq 0$, $k_1=k_3=0$ in (**). This assumption , together with (**), implies that

(8.1) $$xoy = (k_2a_1b_2+k_4a_2b_2)e_3.$$

As in the above cases, we can prove that this multiplication is isomorphic to the muliplication (3.1) in case 3. We let $f: A\to A$ be the linear map defined by

$$f(e_1) = k_2e_2',$$

$$f(e_2) = e'_1+k_4e'_2,$$

$$f(e_3) = e'_3.$$

By (IV) page 43, f is a 1-1, onto map. We have from (8.1) that

$$f(xoy) = f\left[(k_2a_1b_2+k_4a_2b_2)e_3\right]$$
$$= (k_2a_1b_2+k_4a_2b_2)e'_3,$$

and, by using (3.1) of case 3 page 53, we get

$$f(x)f(y) = f(a_1e_1+a_2e_2+a_3e_3)f(b_1e_1+b_2e_2+b_3e_3)$$
$$= \left[a_2e'_1+(k_2a_1+k_4a_2)e'_2+a_3e'_3\right]$$
$$\left[b_2e'_1+(k_2b_1+k_4b_2)e'_2+b_3e'_3\right]$$
$$= (k_2a_1+k_4a_2)b_2e'_3$$
$$= (k_2a_1b_2+k_4a_2b_2)e'_3.$$

That is f(xoy) = f(x)f(y), these two multiplication are isomorphic.

In the proof of the next cases, it will be useful to have the following definitions and lemma.

<u>Definition 5.5</u> : The <u>center</u> C of an algebra A is the set

$$C = \{x\in A \mid xy = yx = 0 \ \forall y\in A\}.$$

By the left-center C_L of A and the right - center C_R of A we mean

that

$$C_L = \{x\in A \mid xy = 0, \forall y\in A\}$$

and

$$C_R = \{x\in A \mid yx = 0, \forall y\in A\}.$$

<u>Lemma 5.6</u> : Let A and B be finite dimensional algebras over a field R with multiplication o and *respectively. Suppose that these two multiplications are isomorphic,

with respect to the function $f: A \rightarrow A$, then f takes the center (left center, right center) C (C_L, C_R) of A isomorphically onto the center (leftcenter, right center) C' (C'_L, C'_R) of B.

Proof By the definition of center, we have

$$C = \{ x \in A \mid xy = yx = 0, \forall y \in A \}.$$

and

$$C' = \{ x' \in B \mid x'y' = y'x' = 0, \forall y' \in B \}.$$

Let $x \in C$, consider f(x). Since f is an isomorphism of A onto B, then for all $y' \in B$ we can find a unique $y \in A$ such that $f(y) = y'$.

Therefore

$$f(x) * y' = f(x) * f(y).$$

By using the definition of isomorphism of multiplications, we have

$$\begin{aligned} f(x) * y' &= f(x) * f(y) \\ &= f(xoy) \\ &= f(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} y' * f(x) &= f(y) * f(x) \\ &= f(yox) \\ &= f(0) \\ &= 0 \end{aligned}$$

That is $f(x) * y' = y' * f(x) = 0$ for all y' in B, and hence $f(x) \in C'$.

Now, let $x' \in C'$, therefore $x' * y' = y' * x' = 0$ for all y' in B. Since f is onto, we can find a unique $x \in A$ such that $f(x) = x'$. To prove that $x \in C$, suppose otherwise, i.e $x \notin C$, then there exists a $y \in A$ such that $xoy \neq 0$ or $yox \neq 0$. Since f is an isomorphism, the kernel of $f = \{0\}$, implying that $f(xoy) \neq 0$ if $xoy \neq 0$. Hence, $f(x) * f(y) \neq 0$ for some $y \in A$. This implies that $x' = f(x) \notin C'$ which is a contradiction (we can similarly prove that $f(x) \notin C'$ for $yox \neq 0$). Therefore, if $f(x) \in C$ we must have $x \in C$. Using the fact that f is an isomorphism of A onto B and the proof above we can conclude that f takes C isomorphically on to C'.

We can use the same method as above to prove the same result for the left and right centers of A and B.

<u>Q.E.D.</u>

Now we continue to the next cases.

<u>Case 9</u>. Keeping our earlier notation, we have from (**) with $k_2 \neq 0$, $k_3 \neq 0$, $k_1 = k_4 = 0$ that

$$xoy = (k_2 a_1 b_2 + k_3 a_2 b_1) e_3,$$

for $x = a_1 e_1 + a_2 e_2 + a_3 e_3$, $y = b_1 e_1 + b_2 e_2 + b_3 e_3$, $\{a_i, b_j\} \subset \mathbb{R}, i, j = 1, 2, 3$

Like the other previous cases, we may choose a new basis $e''_1 = e_1$, $e''_2 = e_2$, $e''_3 = k_2 e_3$ such that

$$xoy = \left(a''_1 b''_2 + \frac{k_3}{k_2} a''_2 b''_1\right) e''_3,$$

for $x = a''_1 e''_1 + a''_2 e''_2 + a''_3 e''_3$, $y = b''_1 e''_1 + b''_2 e''_2 + b''_3 e''_3$,

$\{a''_i, b''_j\} \subset \mathbb{R}$, $i, j = 1, 2, 3$

Let $k'' = \frac{k_3}{k_2}$, then we have

(9.1) $\quad x \circ y = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3$, for some $k'' \neq 0$ in $\mathbb{R}$.

We claim that the multiplication (9.1) is not isomorphic to the multiplications in case 1 and case 3. First, we shall prove that it is not isomorphic to case 1. Since, we have, in case 1, that

(1.1) $\quad xy = a'_1 b'_1 e'_3$

for $x = a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3$, $y = b'_1 e'_1 + b'_2 e'_2 + b'_3 e'_3$, $\{a'_i, b'_j\} \subset \mathbb{R}$, $i,j = 1,2,3$. Therefore, the center C of A under the multiplication (1.1) is generated by e_2 and e_3, that is $C = [e_2, e_3]$. Hence, the deimension of C is 2. But the center C' of A under the multiplication (9.1) is generated by e_3 and the dimension of C' is 1. These imply that the center C cannot be isomorphic to the center C' and hence, these two multiplications are not isomorphic.

Secondly we shall prove that the multiplication in case 3 is not isomorphic to the multiplication in case 9. We begin by recalling that the multiplication in case 3 is

(3.1) $\quad xy = a'_2 b'_1 e'_3$,

for $x = a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3$, $y = b'_1 e'_1 + b'_2 e'_2 + b'_3 e'_3$, $\{a'_i, b'_j\} \subset \mathbb{R}$, $i,j = 1,2,3$. We can see that the left center C_L of A under the multiplication (3.1) is $C_L = [e_1, e_3]$ whereas the left center C'_L of A under the multiplication (9.1) is $C'_L=[e_3]$. Therefore, the dimensions of C_L and C'_L are not equal, and consequently they cannot be isomorphic. Thus the multiplication (3.1) is not isomorphic to the multiplication (9.1).

Suppose that e'_1, e'_2, e'_3 is another basis of A such that we have the multiplication

(9.2) $\qquad x*y = (a'_1b'_2 + k'a'_2b'_1)e'_3, \quad k'_1 \neq 0$ in $\mathbb{R}$,

for $x=a'_1e'_1+a'_2e'_2+a'_3e'_3$, $y= b'_1e'_1+b'_2e'_2+b'_3e'_3$, $\{a'_i, b'_j\} \subset \mathbb{R}$, $i,j=1,2,3$.

We claim that the multiplication (9.1) and (9.2) are isomorphic iff $k' = k''$ or $k' = \frac{1}{k''}$. First we assume that the multiplication (9.1) and (9.2) are isomorphic. Therefore, we can find a linear, 1-1, onto function f: $A \to A$ such that

(9.3) $\qquad f(x*y) = f(x)of(y)$.

This function f is in the form

$$f(e'_1)= m_1e''_1 + m_2e''_2 +m_3e''_3,$$

$$f(e'_2)= p_1e''_1 + p_2e''_2 +p_3e''_3,$$

$$f(e'_3)= qe''_3, \quad \text{for } \{m_i, p_j, q\} \subset \mathbb{R},\ i,j=1,2,3, q\neq 0 \text{ in } \mathbb{R}.$$

Since th formula (9.3) holds for all x,yin A. Let $x=e'_1, y=e'_1$, then (9.2),(9.1) and (9.3) imply that

(1) $\qquad m_1m_2(1+k'') = 0$

If $x = e'_1$, $y = e'_2$, then

(2) $\qquad m_1p_2 + k''m_2p_1= q.$

If $x = e'_2$, $y = e'_1$, then

(3) $\qquad m_2p_1 + k''m_1p_2= q.k'.$

If $x = e'_2$, $y = e'_2$, then

(4) $$p_1p_2(1+k'') = 0.$$

Suppose that $k'' = -1$, then from (2) we have

(5) $$m_1p_2 - m_2p_1 = q,$$

and from (3) implies that

(6) $$m_2p_1 - m_1p_2 = q.k'.$$

Adding (5) and (6) together we get

$$q\ (1+k') = 0$$

Since $q \neq 0$ (or else ker $f \neq 0$ which is a contradiction), $1+k'=0$. That is $k' = -1$.

Suppose that $k'' \neq -1$, then $1+k'' \neq 0$. Therefore (1) and (4) imply that $m_1=0$ or $m_2=0$ and $p_1=0$ or $p_2=0$. If $m_1=0$ and $m_2=0$, then $f(e_1')=m_3e_3''$ and f is not an isomorphism. Therefore $m_1=0$ or $m_2=0$ and not both. Suppose that $m_1=0$, then $m_2 \neq 0$.

From (2) we have

$$k''m_2p_1 = q,$$

whereas (3) implies that

$$m_2p_1 = q.k',$$

Therefore,

$$k''qk' = q.$$

Since $q \neq 0$, then $k' = \frac{1}{k''}$.

Similarly, if $m_2=0$, then $m_1 \neq 0$ and we have from (2) that

$$m_1p_2 = q.$$

whereas (3) implies that

$$k'm_1p_2 = q.k''.$$

Therefore,

$$k'q = qk''$$

That is $k' = k''$. Therefore, if (9.1) is isomorphic to (9.2), then $k' = k''$ or $k' = \frac{1}{k''}$.

Conversely, suppose that $k' = k''$. We let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1'',$$
$$f(e_2') = e_2'',$$
$$f(e_3') = e_3'',$$

Then (9.1) implies that

$$\begin{aligned} f(x) o f(y) &= f(a_1'e_1'+a_2'e_2'+a_3'e_3') o f(b_1'e_1'+b_2'e_2'+b_3'e_3') \\ &= (a_1'e_1''+a_2'e_2''+a_3'e_3'') o (b_1'e_1''+b_2'e_2''+b_3'e_3'') \\ &= (a_1'b_2'+k''a_2'b_1')e_3'', \end{aligned}$$

whereas (9.2) implies that

$$\begin{aligned} f(x*y) &= f\left[(a_1'b_2'+k'a_2'b_1')e_3'\right] \\ &= (a_1'b_2'+k'a_2'b_1')e_3''. \end{aligned}$$

Therefore $f(x*y) = f(x) o f(y)$ for $k'=k''$, and f is 1-1, onto from case II page 42. Hence, the multiplications (9.1) and (9.2) are isomorphic.

Suppose further that $k' = \frac{1}{k''}$. Let $f: A \to A$ be defined by

$$f(e_1') = k'e_2'',$$
$$f(e_2') = e_1'',$$
$$f(e_3') = e_3'', \qquad k' \neq 0 \text{ in } \mathbb{R}.$$

Then (9.2) implies that,

$$f(x*y) = f\left[(a_1'b_2'+k'a_2'b_1')e_3'\right]$$

$$= (a_1'b_2'+k'a_2'b_1')e_3'',$$

and on the other hand, (9.1) implies that

$$f(x)of(y) = f(a_1'e_1'+a_2'e_2'+a_3'e_3')of\ (b_1'e_1'+b_2'e_2'+b_3'e_3')$$
$$= (a_2'e_1''+k'a_1'e_2''+a_3'e_3'')o(b_2'e_1''+k'b_1'e_2''+b_3'e_3'')$$
$$= \left[k'a_2'b_1'+k''(k'a_1')b_2'\right]e_3''$$
$$= (a_1'b_2'+k'a_2'b_1')e_3'',\ \text{since } k''=\frac{1}{k'}.$$

That is $f(x*y) = f(x)o\ f\ y$ for $k'=\frac{1}{k''}$. Case (II) page 42 implies that f is 1-1 and onto. Therefore these multiplications are isomorphic. Thus the multiplications (9.1) and (9.2) are isomorphic iff $k'=k''$ or $k'=\frac{1}{k''}$.

Case 10. Let $k_1\neq 0, k_4\neq 0, k_2=k_3=0$, Then (**) becomes

(10.1) $x*y = (k_1a_1b_1+k_4a_2b_2)e_3$.

If $\frac{k_1}{k_4}<0$, then we let $h = -\frac{k_1}{k_4}$. That is $h>0$, we can choose a new basis $e_1'=e_2' =\sqrt{h}\ e_2$ (take the positve square root.), $e_3' = k_1e_3$, and get

(10.2) $$x*y = (a_1'b_1' - a_2'b_2')e_3',$$

for $x = a_1'e_1'+a_2'e_2'+a_3'e_3'$, $y = b_1'e_1'+b_2'e_2'+b_3'e_3'$,
$\left\{a_i' , b_j'\right\}\subset \mathbb{R}$, $i,j = 1,2,3$.

Consider the multiplication (9.1) of case 9 page 60. If $k''=1$ we have

(9.3) $$xoy = (a_1''b_2'' + a_2''b_1'')\ e_3'',$$

for $x = a_1''e_1''+a_2''e_2''+a_3''e_3''$, $y = b_1''e_1''+b_2''e_2''+b_3''e_3''$,
$\left\{a_i'' ,b_j''\right\}\subset \mathbb{R}$, $i,j = 1,2,3$.

We claim that the multiplications (10.2) and (9.3) are isomorphic. To prove this, let $f: A\to A$ be the linear map defined by

$$f(e_1') = e_1''+e_2'',$$

$$f(e_2') = -e_1''+e_2'',$$

$$f(e_3') = 2e_3'',$$

then case V page 43 implies that f is 1-1 and onto mapping on A. The multiplication (10.2) implies that

$$f(x*y) = f\left[(a_1'b_1' - a_2'b_2')\ e_3'\right]$$
$$= 2(a_1'b_1' - a_2'b_2')\ e_3'',$$

whereas, the multiplication (9.3) implies that

$$f(x)of(y) = f(a_1'e_1'+a_2'e_2'+a_3'e_3')of(b_1'e_1'+b_2'e_2'+b_3'e_3')$$
$$= \left[(a_1'-a_2')e_1''+(a_1'+a_2')e_2''+2a_3'e_3''\right]o\left[(b_1'-b_2')e_1'' +(b_1'+b_2')e_2''+2b_3'e_3''\right]$$
$$= \left[(a_1'-a_2')(b_1'+b_2')+(a_1'+a_2')(b_1'-b_2')\right]e_3''$$
$$= 2(a_1'b_1'-b_2'b_2')e_3'',$$

that is $f(x*y) = f(x)of(y)$, or equivalently these two multiplications are isomorphic.

Next, if $\frac{k_1}{k_4}>0$ in case(10.1), then we may choose a new basis $e_1' = e_1, e_2' = \sqrt{\frac{k_1}{k_4}}e_2$, (take the positive square root) $e_3' = k_1e_3$ such that (10.1) becomes

(10.3) $$x*y = (a_1'b_1'+a_2'b_2')e_3',$$

for $x= a_1'e_1'+a_2'e_2'+a_3'e_3'$, $y= b_1'e_1'+b_2'e_2'+b_3'e_3'$, $\{a_i', b_j'\}\subset R$, i, j = 1,2,3.

This multiplication is not isomorphic to the multiplication in case 1. Since the center C of A under the multiplication in case 1 is $C=[e_2,e_3]$ and dimension of C is 2, whereas the center C' of A under the multiplication (10.3) is $C' = [e_3]$ and dimension of C' is 1. Moreover, the algebra A is not commutative under the multiplication (3.1) of case 3 page 53, but A is commutative under the multiplication (10.3). Therefore the multiplications (10.3) and (3.1) cannot be isomorphic. Next, we claim that the multiplication (10.3) is not isomorphic to the multiplication (9.1) of case 9. Recalling that the multiplication (9.1) is

(9.1) $\quad x o y = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3, \quad k'' \neq 0$ in $\mathbb{R}$,

for $x = a''_1 e''_1 + a''_2 e''_2 + a''_3 e''_3$, $y = b''_1 e''_1 + b''_2 e''_2 + b''_3 e''_3$,

$\{a''_i, b''_j\} \subset \mathbb{R}$, $i,j = 1,2,3$.

Suppose to the contrary that these two multiplications are isomorphic, then we can find a linear, 1-1, onto map $f: A \to A$ such that

(10.4) $\quad f(xoy) = f(x) * f(y)$,

and f is in the form

$$f(e''_1) = m_1 e'_1 + m_2 e'_2 + m_3 e'_3,$$

$$f(e''_2) = p_1 e'_1 + p_2 e'_2 + p_3 e'_3,$$

$$f(e''_3) = q e'_3, \quad \{m_i, p_j\} \subset \mathbb{R},\ i,j = 1,2,3,\ q \neq 0 \text{ in } \mathbb{R}.$$

Since, (10.4) holds for all x,y in A. Then, if $x = e''_1$, $y = e''_1$, (10.3), (9.1), (10.4) imply that

$$m_1^2 + m_2^2 = 0$$

But m_1, m_2 is in $\mathbb{R}$, therefore $m_1 = 0$ and $m_2 = 0$. Hence $f(e''_1) = m_3 e'_3$, where e''_1 is in A and e''_3 is in A^2, and f is not an isomorphism . This is a contradiction. That is the multiplications (9.1) and (10.3) cannot be isomorphic.

<u>Case 11</u>. In this case we assume that $k_2 \neq 0$, $k_3 \neq 0$, $k_4 \neq 0$, $k_1 = 0$, then the multiplication (**) becomes

(11.1) $$x*y = (k_2 a_1 b_2 + k_3 a_2 b_1 + k_4 a_2 b_2)\, e_3$$

If we choose a new basis $e'_1 = \frac{k_4}{k_2} e_1$, $e'_2 = e_2$, $e'_3 = k_4 e_3$, then it is immediate that

$$x*y = (a'_1 b'_2 + \frac{k_3}{k_2}\, a'_2 b'_1 + a'_2 b'_2) e'_3,$$

for $x = a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3$, $y = b'_1 e'_1 + b'_2 e'_2 + b'_3 e'_3$,

$\left\{ a'_i\ ,\ b'_j \right\} \subset \mathbb{R}$ $i, j = 1,2,3$. Let $k' = \frac{k_3}{k_2}$, then

(11.2) $$x*y) = (a'_1 b'_2 + k' a'_2 b'_1 + a'_2 b'_2) e'_3, \text{ for } k' \neq 0 \text{ in } \mathbb{R}.$$

Suppose $k' \neq -1$, then claim that this multiplication is isomorphic to the multiplication (9.1) of case 9 page 60 whenever $k' = k''$. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e'_1) = e''_1,$$
$$f(e'_2) = \frac{1}{(1+k')}\, e''_1 + e''_2,$$
$$f(e'_3) = e''_3, \quad k' \neq 0, -1 \text{ in } \mathbb{R}.$$

In (9.1) of case 9, we have

$$x o y = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3, \quad k'' \neq 0 \text{ in } \mathbb{R}.$$

for $x = a''_1 e''_1 + a''_2 e''_2 + a''_3 e''_3$, $y = b''_1 e''_1 + b''_2 e''_2 + b''_3 e''_3$,

$\left\{ a''_i, b''_j \right\} \subset \mathbb{R}$, $i, j, = 1,2,3$.

Therefore, with $k' = k''$ we have

$$\begin{aligned} f(x) o f(y) &= f(a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3) o f(b'_1 e'_1 + b'_2 e'_2 + b'_3 e'_3) \\ &= \left[(a'_1 + \frac{1}{(1+k')}\ a'_2) e''_1 + a'_2 e''_2 + a'_3 e''_3 \right] o \\ &\quad \left[(b' + \frac{1}{(1+k')} b'_2)\ e''_1 + b'_2 e''_2 + b'_3 e''_3 \right] \\ &= \left[(a'_1 + \frac{1}{(1+k')} a'_2) b'_2 + k'' a'_2 (b'_1 + \frac{1}{(1+k')} b'_2) \right] e''_3 \\ &= (a'_1 b'_2 + k'' a'_2 b'_1 + a'_2 b'_2) e''_3, \end{aligned}$$

whereas, the multiplication (11.2) implies that

$$f(x*y) = f(a_1'b_2'+k'a_2'b_1'+a_2'b_2')e_3'$$
$$= (a_1'b_2'+k'a_2'b_1'+a_2'b_2')e_3''.$$

That is $f(x*y) = f(x)of(y)$ whenever $k'=k''$ and $k'\neq -1$. Consequently, the multiplication (11.2) with $k'\neq -1$is isomorphic to the multiplication in case 9.

Suppose $k'=-1$, then (11.2) becomes

(11.3) $\quad x*y = (a_1'b_2'-a_2'b_1'+a_2'b_2')e_3'.$

We can easily see that the algebra A is not commutative under the multiplication (11.3) while A is commutative under the multiplication in case 1 and case 10. Therefore the multipliplication (11.3) cannot be isomorphic to the multiplication in case 1 and case 10. Moreover, the left center C_L of A under the multiplication (11.3) is e_3 and hence C_L has dimension 1. Therefore the multiplication (11.2) cannot be isomorphic to multiplication (3.1)of case 3 under which the left center C_L' is e_1,e_3 and has dimension 2. Furthermore, claim that the multiplication (11.3) is not isomorphic to the multiplication in case 9. Recalling that the multiplication in case 9 is

(9.1) $\quad xoy = (a_1''b_2''+k''a_2''b_1'')e_3''$, $k''\neq 0$ in R, for

$$x = a_1''e_1''+a_2''e_2''+a_3''e_3'', \ y = b_1''e_1''+b_2''e_2''+b_3''e_3'',$$
$$\{a_i'', b_j''\} \subset R, i,j=1,2,3.$$

Suppose that these two multiplications are isomorphic, then there exists a linear map $f: A\rightarrow A$ which is 1-1, onto and

(11.4) $\quad f(xoy) = f(x)*f(y).$

This function f is in the form,

$$f(e_1'') = m_1e_1'+m_2e_2'+m_3e_3',$$
$$f(e_2'') = p_1e_1'+p_2e_2'+p_3e_3', \quad \{mi,pi\}_{i=1,2,3} \subset \mathbb{R},$$
$$f(e_3'') = qe_3', \quad q\neq 0 \text{ in } \mathbb{R}.$$

The formula (11.4) holds for all x,y in A. Therefore, if $x=e_1''$, $y=e_1''$ (11.3),(9.1) and (11.4) imply that

(1) $$m_2^2= 0$$

If $x=e_2''$, $y=e_2''$, then

(2) $$p_2^2 = 0.$$

From (1) and (2) we can see that $m_2=0$ and $p_2=0$. Therefore

$$\det f = \begin{bmatrix} m_1 & 0 & m_3 \\ p_1 & 0 & p_3 \\ 0 & 0 & q \end{bmatrix} = \theta,$$

that is f is hot a 1-1, onto mapping which is a contradiction. Hence the multiplications (11.3) and (9.1) are not isomorphic.

Case 12. Let $k_1\neq 0$, $k_2\neq 0$, $k_3\neq 0$ and $k_4=0$, then (**) can be written as

(12.1) $$xoy = (k_1a_1b_1+k_2a_1b_2+k_3a_2b_1)e_3.$$

Like the other cases. We choose a new basis $e_1''=e_1$, $e_2''=\frac{k_1}{k_2}e_2$, $e_3''=k_1e_3$ and get

$$xoy = (a_1''b_1'' + a_1''b_2'' + \frac{k_3}{k_2}a_2''b_1'')e_3'',$$

for $x = a_1''e_1''+a_2''e_2''+a_3''e_3''$, $y= b_1''e_1''+b_2''e_2''+b_3''e_3''$, $\{a_i'',b_j''\}\subset\mathbb{R}$, i,j=1,2,3.

Let $k'' = \frac{k_3}{k_2}$, then

(12.2) $$xoy = (a_1''b_1''+a_1''b_2''+k''a_2''b_1'')e_3'', \quad k''\neq 0 \text{ in } \mathbb{R}.$$

Claim that this multiplication is isomorphic to the multiplication (11.1) in case 11 whenever $k'=\frac{1}{k''}$.

Recalling that the multiplication (11.1) is

$$(11.1) \qquad (x*y) = (a_1'b_2'+k'a_2'b_1'+a_2'b_2')e_3', \quad k'\neq 0 \text{ in } \mathbb{R},$$

for $x = a_1'e_1'+a_2'e_2'+a_3'e_3'$, $y = b_1'e_1'+b_2'e_2'+b_3'e_3'$,

$\{a_i', b_j'\} \subset \mathbb{R}$, $i,j = 1,2,3.$

Let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_2'',$$
$$f(e_2') = k'e_1''+(1-k')e_2'',$$
$$f(e_3') = e_3'', \qquad k\neq 0 \text{ in } \mathbb{R}.$$

The multiplication (12.2) with $k''=\frac{1}{k'}$ implies that

$$f(x)of(y) = f(a_1'e_1'+a_2'e_2'+a_3'e_3')of(b_1'e_1'+b_2'e_2'+b_3''e_3')$$
$$= \left[k'a_2'e_1''+(a_1'+(1-k')a_2')e_2''+a_3'e_3''\right] o \left[k'b_2'e_1''+(b_1'+(1-k')b_2')e_2''+b_3'e_3''\right]$$
$$= \left[(k'a_2')(k'b_2')+(k'a_2')(b_1'+(1-k')b_2')+k''(a_1'+(1-k')a_2')(k'b_2')\right]e_3''$$
$$= (a_1'b_2'+k'a_2'b_1'+a_2'b_2')e_3'',$$

whereas the multiplication (11.1) implies that

$$f(x*y) = f\left[(a_1'b_2'+k'a_2'b_1'+a_2'b_2')e_3'\right]$$
$$= (a_1'b_2'+k'a_2'b'+a_2'b_2')e_3''.$$

That is $f(x*y) = f(x)of(y)$ and f is 1-1, onto from case IV page 43.

Therefore the multiplication in case 12 is isomorphic to the multiplications in case 11.

<u>Case 13</u>. Suppose that $k_1\neq 0, k_3\neq 0, k_4\neq 0, k_2=0$, then the multiplication (**) can written as

$$x*y = (k_1a_1b_1+k_3a_2b_1+k_4a_2b_2)e_3.$$

By choosing a new basis $e_1'''=e_1, e_2'''=\frac{k_1}{k_3}e_2, e_3'''=k_1e_3$, we may thus write

$$x*y = (a'''_1 b'''_1 + a'''_2 b'''_1 + \frac{k_1 k_4}{k_3^2} a'''_2 b'''_2) e'''_3$$

for $x = a'''_1 e'''_1 + a'''_2 e'''_2 + a'''_3 e'''_3$, $y = b'''_1 e'''_1 + b'''_2 e'''_2 + b'''_3 e'''_3$, $\{a'''_i, b'''_j\} \subset \mathbb{R}$, $i,j=1,2,3$. Let $k''' = \frac{k_1 k_4}{k_3^2}$, then we have

(13.1) $\quad x*y = (a'''_1 b'''_1 + a'''_2 b'''_1 + k''' a'''_2 b'''_2) e'''_3$, for $k''' \neq 0$ in $\mathbb{R}$.

Recalling that the multiplication (9.1) in case 9 is (9.1) $xoy = (a''_1 b''_2 + k'' a''_2 b''_1) e''_3$, for $k'' \neq 0$ in $\mathbb{R}$ and $x = a''_1 e''_1 + a''_2 e''_2 + a''_3 e''_3$, $y = b''_1 e''_1 + b''_2 e''_2 + b''_3 e''_3$, $\{a''_i, b''_i\}$ $i=1,2,3 \subset \mathbb{R}$. We claim that the multiplications (13.1) and (9.1) are isomorphic iff $k''' = \frac{-k''}{(1-k'')^2}$, $k'' \neq \pm 1$. Toprove this, we first assume that these two multiplications are isomorphic, then there exists a linear map $f; A \rightarrow A$ such that $f(x*y) = f(x)of(y)$ and f is in the form

$$f(e'''_1) = m_1 e''_1 + m_2 e''_2 + m_3 e''_3,$$
$$f(e'''_2) = p_1 e''_1 + p_2 e''_2 + p_3 e''_3, \quad \{mi, pi\}\ i=1,2,3 \subset \mathbb{R},$$
$$f(e'''_3) = q e''_3, \qquad q \neq 0 \text{ in } \mathbb{R}.$$

Therefore (13.1),(9.1) and $f(x*y)$ $f(x)of(y)$ imply that, for $x = e'''_1, y = e'''_1$, we have

(1) $\quad m_1 m_2 + k'' m_2 m_1 = q.$

For $x = e'''_1$, $y = e'''_2$, we have

(2) $\quad m_1 p_2 + k'' m_2 p_1 = 0.$

For $x = e'''_2$, $y = e'''_1$, we have

(3) $\quad p_1 m_2 + k'' p_2 m_1 = q.$

For $x = e'''_2$, $y = e'''_2$, we have

(4) $\quad p_1 p_2 + k'' p_2 p_1 = k''' q.$

Since $q \neq 0$, equation (1) implies that $k'' \neq -1$.

From (2) and (3) we have

(5) $$m_1 p_2 (k''^2 - 1) = q k''.$$

Since q and k" are not zero, and $k'' \neq \pm 1$, (5) implies

that $$m_1 = \frac{q k''}{p_2 (k''^2 - 1)}.$$

Therefore, (1) implies that

$$m_2 = \frac{q}{m_1 (1+k'')} = \frac{q p_2 (k''^2 - 1)}{q k'' (1+k'')} = \frac{p_2 (k''-1)}{k''}$$

From (2) and (3) we have,

$$m_2 p_1 (1 - k''^2) = q.$$

That is $$p_1 = \frac{q}{m_2 (1 - k''^2)} = \frac{q k''}{(1 - k''^2) . p_2 (k''-1)}$$

Substituting p_1 in (4) we have

$$p_2 = \frac{k''' q}{p_1 (1+k'')} = \frac{k''' q (1-k''^2) p_2 (k''-1)}{(1+k'') . q k''}$$

That is

$$k'' = -k''' (1-k'')^2$$

$$k''' = \frac{-k''}{(1-k'')^2}$$

and $k'' \neq 0, \pm 1$.

Conversely, suppose that $k''' = \frac{-k''}{(1-k'')^2}$ and $k'' \neq 0, \pm 1$. Let $f: A \to A$ be the linear map defined by

$$f(e_1''') = e'' + \frac{e_2''}{(1+k'')},$$

$$f(e_2''') = \frac{e_1''}{(1-k'')} - \frac{k''}{(1-k''^2)} . \quad e_2,$$

$$f(e_3''') = e_3''.$$

Then (13.1) implies that

$$f(x*y) = f\left[(a_1''' b_1''' + a_2''' b_1''' + k''' a_2''' b_2''') e_3'''\right]$$

$$= (a_1''' b_1''' + a_2''' b_1''' + k''' a_2''' b_2''') e_3''$$

$$= \left[a_1''' b_1''' + a_2''' b_1''' - \frac{k''}{(1-k''^2)} \; a_2''' b_2'''\right] e_3'',$$

whereas (9.1) implies that

$$
\begin{aligned}
f(x)of(y) &= f(a'''_1e'''_1+a'''_2e'''_2+a'''_3e'''_3)\,o\,f(b'''_1e'''_1+b'''_2e'''_2+b'''_3e'''_3) \\
&= \left[\left(a'''_1+\frac{a'''_2}{(1-k'')}\right)e''_1+\left(\frac{a'''_1}{(1+k'')}-\frac{a'''_2k''}{(1-k''^2)}\right)e''_2+a'''_3e''_3\right]o \\
&\quad \left[\left(b'''_1+\frac{b'''_2}{(1-k'')}\right)e''_1+\left(\frac{b'''_1}{(1+k'')}-\frac{b'''_2k''}{(1-k''^2)}\right)e''_2+b'''_3e''_3\right] \\
&= \left[\left(a'''_1+\frac{a'''_2}{(1-k'')}\right)\left(\frac{b'''_1}{(1+k'')}-\frac{b'''_2k''}{(1-k''^2)}\right)\right. \\
&\quad \left.+k''\left(b'''_1+\frac{b'''_2}{(1-k'')}\right)\left(\frac{a'''_1}{(1+k'')}-\frac{a'''_2k''}{(1-k''^2)}\right)\right]e''_3 \\
&= \left(a'''_1b'''_1+a'''_2b'''_1-\frac{k''}{(1-k'')^2}\,a'''_2b'''_2\right)e''_3.
\end{aligned}
$$

That is $f(x*y)=f(x)of(y)$ and since f is 1-1 and onto (see page 43) we can have that the multiplication (13.1) is isomorphic to (9.1).

Under the assumption above that $k'''=\dfrac{-k''}{(1-k'')^2}$ we can see that for a given number k''' we can find k'' to make (13.1) isomorphic to (9.1) only if $k'''\neq 0$ and $k'''<\frac{1}{4}$. Therefore we have to consider (13.1) when $k'''\geq\frac{1}{4}$.

We claim that the multiplication (13.1) is isomorphic to the case 11. iff $k'''=\frac{1}{4}$. Recalling first that the multiplication in case 11 is

(11.3) $$xoy = (a'_1b'_2 - a'_2b'_1 + a'_2b'_2)e'_3,$$

for $x = a'_1e'_1+a'_2e'_2+a'_3e'_3$, $y = b'_1e'_1+b'_2e'_2+b'_3e'_3$,

$\{a'_i,\ b'_i\}$ $i = 1,2,3 \subset \mathbb{R}$.

Suppose that the multiplications (13.1) and (11.3) are isomorphic then we can find a linear mapping $f: A\to A$ such that $f(x*y)=f(x)f(y)$ for all x,y in A. The mapping f is in the form

$$f(e'''_1) = m_1e'_1 + m_2e'_2 + m_3e'_3,$$

$$f(e'''_2) = p_1e'_1 + p_2e'_2 + p_3e'_3, \ \{mi, pi\} \ i = 1,2,3 \subset \mathbb{R},$$

$$f(e'''_3) = q\ e'_3, \qquad q \neq 0 \text{ in } \mathbb{R}.$$

Therefore, if $x = e'''_1$, $y = e'''_1$, the multiplication (11.3) and (13.1) implies that

(1) $m_2^2 = q,$

If $x = e'''_1$, $y = e'''_2$, then

(2) $m_1p_2 - m_2p_1 + m_2p_2 = 0.$

If $x = e'''_2$, $y = e'''_1$, then

(3) $m_2p_1 - m_1p_2 + m_2p_2 = q$

If $x = e'''$, $y = e'''$, then

(4) $p_2^2 = qk'''$

From (2) and (3) we have that

(5) $2m_2p_2 = q.$

That is $m_2 = \frac{q}{2p_2}.$

Representing m_2 in (1) we have

$$\frac{2}{4p_2^2} = q$$

and reperesenting p_2 that is in the equation (4), we get

$$\frac{q^2}{4\ q\ k'''} = q$$

Therefore,

$$k''' = \frac{1}{4}.$$

Conversely, suppose that $k''' = \frac{1}{4}$, then let $f: A \to A$ be the linear map defined by

$$f(e'''_1) = e'_2,$$

$$f(e_1''') = \frac{1}{2}(e_1'+e_2')$$

$$f(e_3''') = e_3',$$

then (13.1) implies that

$$f(x*y) = f\left[(a_1'''b_1'''+a_2'''b_1'''+k'''a_2'''b_2''')e_3'''\right]$$

$$= (a_1'''b_1'''+a_2'''b_1'''+\frac{1}{4}a_2'''b_2''')e_3'.$$

whereas, on the other hand (11.3) implies that

$$f(x)of(y) = f(a_1'''e_1'''+a_2'''e_2'''+a_3'''e_3''')of(b_1'''e_1'''+b_2'''e_2'''+b_3'''e_3''')$$

$$= \left[\frac{1}{2}a_2'''e_1'+(a_1'''+\frac{a_2'''}{2})e_2'+a_3'''e_3'\right]o\left[\frac{b_2'''}{2}e_1' + (b_1'''+\frac{b_2'''}{2})e_2' + b_3'''e_3'\right]$$

$$= \left[(\frac{a_2'''}{2})(b_1'''+\frac{b_2'''}{2})-(a_1'''+\frac{a_2'''}{2})(\frac{b_2'''}{2})+(a_1'''+\frac{a_2'''}{2})(b_1'''+\frac{b_2'''}{2})\right]e_3'$$

$$= (a_1'''b_1'''+a_2'''b_1'''+\frac{1}{4}a_2'''b_2''')e_3'$$

we thus see that (13.1) with $k''' = \frac{1}{4}$ isomorphic to (11.3).

Therefore it is left to consider (13.1) when $k''' > \frac{1}{4}$. This case is not isomorphic to case 9 and case 11 by the above proofs. Under the multiplication (13.1) with $k''' > \frac{1}{4}$ we can see that the algebra A is not commutative. But the algebra A is commutative under the multiplication in case 1 and case 10. Therefore the multiplication (13.1) with $k''' > \frac{1}{4}$ is not isomorphic to the multiplication in case 1 and case 10. Next, we can observe that the left C_L of the algebra A under the multiplication in case 3 is generated by e_1 and e_3 and hence C_L has dimension 2, whereas the left center C_L' of A under the multiplication (13.1) is generated by e_3 and has dimension 1. Thus the multiplication (13.1) cannot be isomorphic to one of the case (3).

Furthermore, suppose that e'_1, e'_2, e'_3 is another basis of A such that

(13.2) $\quad xoy = (a'_1b'_1 + a'_2b'_1 + k'a'_2b'_2)e'_3$, $k' \neq 0$ in $\mathbb{R}$,

for $x = a'_1e'_1 + a'_2e'_2 + a'_3e'_3$, $y = b'_1e'_1 + b'_2e'_2 + b'_3e'_3$, $\{a'_i, b'_i\}$ $i = 1,2,3 \subset \mathbb{R}$.

We claim that the multiplication (13.1) and (13.2) are isomorphic iff $k' = k'''$. To prove this, we first suppose that these two multiplications are isomorphic. Therefore, there exists a linear mapping $f: A \to A$ defined by

$$f(e'_1) = m_1e'''_1 + m_2e'''_2 + m_3e'''_3,$$

$$f(e'_2) = p_1e'''_1 + p_2e'''_2 + p_3e'''_3, \quad \{mi, pi\} \quad i=1,2,3 \subset \mathbb{R},$$

$$f(e'_3) = q\,e'''_3, \qquad q \neq 0 \text{ in } \mathbb{R},$$

such that $f(xoy) = f(x)*f(y)$.

Hence, for $x = e'_1$, $y = e'_1$, we have

(1) $\quad m_1^2 + m_2m_1 + k'''m_2^2 = q.$

For $x = e'_1$, $y = e'_2$, we have

(2) $m_1p_1 + m_2p_1 + k'''m_2p_2 = 0.$

For $x = e'_2$, $y = e'_1$, we have

(3) $m_1p_1 + m_1p_2 + k'''m_2p_2 = q.$

For $x = e'_2$, $y = e'_2$, we have

(4) $p_1^2 + p_1p_2 + k'''p_2^2 = q\,k'.$

Take $p_1 \times (1) - m_1 \times (2)$, we get

(5) $\quad k'''m_2(m_2p_1 - m_1p_2) = q\,p_1.$

Take (3) - (2), we get

(6) $\qquad m_1p_2 - m_2p_1 = q.$

Therefore, from (5) and (6), we have

(7) $$k''' m_2 = -p_1.$$

Take $m_1\times(4)-p_1\times(3)$, we get

$$k''' p_2(m_1 p_2 - m_2 p_1) = q(k''' m_1 - p_1).$$

This with (6) imply that

(8) $$k''' p_2 = (k' m_1 - p_1).$$

Take $m_2\times(3)-p_2\times(1)$, we get

$$m_1(m_2 p_1 - m_1 p_2) = q(m_2 - p_2).$$

This, together with (6), gives

(9) $$m_1 = p_2 - m_2.$$

Take. $m_2\times(4)-p_2\times(2)$, we get

$$p_1(m_2 p_1 - m_1 p_2) = qk' m_2,$$

that is

(10) $$p_1 = -k' m_2.$$

If $m_2=0$, then $p_1=0$ from (7) and (10). Therefore (8) and (9) imply that

$$k''' = k'.$$

If $m_2\neq 0$, then (7) and (10) imply that

$$k''' = k'.$$

Conversely, if $k''' = k'$, let $f: A\to A$ be a linear map defined by

$$f(e_1') = e_1''',$$
$$f(e_2') = e_2''',$$
$$f(e_3') = e_3''',$$

Then (13.1) implies that

$$\begin{aligned} f(x)*f(y) &= f(a_1'e_1'+a_2'e_2'+a_3'e_3')*f(b_1'e_1'+b_2'e_2'+b_3'e_3') \\ &= \left[a_1'e_1'''+a_2'e_2'''+a_3'e_3'''\right]*\left[b_1'e_1'''+b_2'e_2'''+b_3'e_3'''\right] \\ &= (a_1'b_1'+a_2'b_1'+k'''a_2'b_2')e_3''' \end{aligned}$$

$$= (a_1'b_1'+a_2'b_1'+k'a_2'b_2')e_3''',$$

whereas, (13.2) implies that

$$f(xoy) = f\left[(a_1'b_1'+a_2'b_1'+k'a_2'b_2')e_3'\right]$$

$$= (a_1'b_1'+a_2'b_1'+k'a_2'b_2')e_3'''.$$

These, together with the property of f in case page 43, we have that (13.1) and (13.2) are isomorphic.

Case 14. Suppose that $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$, and $k_3 = 0$, then the multiplication (**) becomes

$$xoy = (k_1a_1b_1+k_2a_1b_2+k_4a_2b_2)e_3,$$

using the same procedure as before, we may choose a new basis $e_1'=e_1$, $e_2' = \frac{k_1}{k_2}e_2$, $e_3' = k_1e_3$ and obtain

$$xoy = (a_1'b_1'+a_1'b_2'+\frac{k_1k_4}{k_2^2}a_2'b_2')e_3',$$

for $x = (a_1'e_1'+a_2'e_2'+a_3'e_3')$, $y = b_1'e_1'+b_2'e_2'+b_3'e_3'$,

$\{a_i', b_i'\} \subset \mathbb{R}$, $i = 1,2,3$.

Let $\frac{k_1k_4}{k_2^2} = k'$, then

(14.1) $$xoy = (a_1'b_1'+a_1'b_2'+k'a_2'b_2')e_3'.$$

We claim that (14.1) is isomorphic to (13.1) in page 74, whenever $k' = k'''$ for all k' in $\mathbb{R}$. To show this, let $f: A \to A$ be the linear map defined by

$$f(e_1''') = e_1',$$

$$f(e_2''') = e_1'-e_2',$$

$$f(e_3''') = e_3',$$

Then (13.1) of case 13 page 74 implies that

$$f(x*y) = f\left[(a_1'''b_1'''+a_2'''b_1'''+k'''a_2'''b_2''')e_3'''\right]$$

$$= (a_1'''b_1'''+a_2'''b_1'''+k'''a_2'''b_2''')e_3',$$

whereas, (14.1) with $k' = k'''$ implies that

$$f(x)of(y) = f(a_1'''e_1'''+a_2'''e_2'''+a_3'''e_3''')of(b_1'''e_1'''+b_2'''e_2'''+b_3'''e_3''')$$
$$= \left[(a_1'''+a_2''')e_1'-a_2'''e_2'+a_3'''e_3'\right] o \left[(b_1'''+b_2''')e_1' -b_2'''e_2'+b_3'''e_3'\right]$$
$$= \left[(a_1'''+a_2''')(b_1'''+b_2''')+(a_1'''+a_2''')(-b_2''')+k''' (-a_2''')(-b_2''')\right]e_3'$$
$$= (a_1'''b_1'''+a_2'''b_1'''+k'''a_2'''b_2''')e_3'.$$

Therefore we are done since f is 1-1 and onto (seepage 43) imply that the cases 13 and 14 are isomorphic.

<u>Case 15</u>. Finally we turn to the case where all k_1, k_2, k_3, k_4 are not zero. With this assumption and (**) we obtain.

$$x*y = (k_1a_1b_1+k_2a_1b_2+k_3a_2b_1+k_4a_2b_2)e_3.$$

We choose a new basis e_1', e_2', e_3' such that $e_1' = e_1, e_2' = (k_2e_1-k_1e_2)$, $e_3' = e_3$. Then we can see that

$$(e')^2=e_1^2=k_1e_3 = k_1e_3'.$$
$$e_1'e_2' = e_1(k_2e_1-k_1e_2)=k_2e_1^2-k_1e_1e_2=k_2k_1e_3-k_1k_2e_3= 0,$$
$$e_2'e_1' = (k_2e_1-k_1e_2)e_1=k_2e_1^2-k_1e_2e_1=k_2k_1e_3-k_1k_3e_3$$
$$= k_1(k_2-k_3)e_3',$$
$$(e')^2= (k_2e_1-k_1e_2)^2= k_2^2e_1^2-k_2k_1e_1e_2-k_1k_2e_2e_1+k_1^2e_2^2$$
$$= k_2^2k_1e_3-k_2k_1k_2e_3-k_1k_2k_3e_3+k_1^2k_4e_3.$$
$$= k_1(k_1k_4-k_2k_3)e_3',$$

and hence

$$(15.1)\ x*y =\left[k_1a_1'b_1'+k_1(k_2-k_3)a_2'b_1'+k_1(k_1k_4-k_2k_3)a_2'b_2'\right]e_3',$$

for $x = a_1'e_1'+a_2'e_2'+a_3'e_3', y = b_1'e_1'+b_2'e_2'+b_3'e_3'$,

$\{a_i', b_i'\}$ $i = 1,2,3 \subset \mathbb{R}$. and all k_1, k_2, k_3, k_4 are not zero.

We have at least one zero so we are back to a previous case.

As a conclusion, the nilpotent algebra A over a field $\mathbb{R}$ with dimension A = 3, dimension $A^2 = 1$ and $A^3 = \{0\}$, possesses an infinite number of mon - isomorphic multiplications which can be divided into 6 classes. That is, for each $x = a_1e_1+a_2e_2+a_3e_3$, $y = b_1e_1+b_2e_2+b_3e_3$, $\{a_i, b_i\}_{i = 1,2,3} \subset \mathbb{R}$, we have

(1) $xy = a_1b_1e_3$,

(2) $xy = a_2b_1e_3$,

(3) $xy = (a_1b_2+ka_2b_1)e_3$, $|k| \geqslant 1$ in $\mathbb{R}$.

(4) $xy = (a_1b_1+a_2b_2)e_3$.

(5) $xy = (a_1b_2-a_2b_1+a_2b_2)e_3$.

(6) $xy = (a_1b_1+a_2b_1+ka_2b_2)e_3, k > \frac{1}{4}$ in $\mathbb{R}$.

Furthermore, we shall prove a theorem about the isomorphism between a nilpotent algebra and a quotient algebra of a polynomial algebra by an ideal. We shall begin our discussion with a definition.

<u>Definition 5.7</u> : A nilpotent algebra A over a field K is called a free nilpotent algebra iff for each x,y in A

$$xy = 0 \Rightarrow \exists\ 0<k<n \text{ such that } x \in A^k \text{ and } y \in A^{n-k}$$

The converse condition is trivially true.

<u>Theorem 5.8</u> : A free nilpotent algebra A over a field K ($A^n = \{0\}$ for some smallest positive integer n>1) with dimension of A = n - 1, is isomorphic to the quotient algebra of a polynomial algebra by an ideal i.e. $A \cong K_o[x]/(x^{n-1})$.

__Proof.__ First, we claim that $A \supset A^2 \supset A^3 \supset \ldots \supset A^n = \{0\}$. Suppose instead that $A^m = A^{m+1}$ for some $m<n$, then we can see that

$$A^{m+2} = A^{m+1} \cdot A = A^{m+1} = A^m$$
$$A^{m+3} = A^{m+2} \cdot A = A^{m+1} = A^m$$
$$\text{-------------------------}$$
$$A^n = A^m,$$

which implies that $A^m = \{0\}$. But this contradicts to that n is the smallest positive integer such that $A^n = \{0\}$. Therefore,

$$A \supset A^2 \supset A^3 \supset \ldots \supset A^n = \{0\}.$$

Since dimension $A=n-1$, then the above result yields that dimension of $A^2 = n-2$, dimension of $A^3 = n-3, \ldots,$ dimension of $A^{n-1} = 1$.

Let $x \neq 0$ be in $A \setminus A^2$, then $x^{n-1} \in A^{n-1}$. Suppose that $x^{n-1} = 0$, then $x.x^{n-2} = 0$. This contradicts the hypothesis that $xy = 0 \Rightarrow \exists\, 0 < k < n$ such that $x \in A^k$, $y \in A^{n-k}$. Hence, $x^{n-1} \neq 0$, let $e = x$, then e^{n-1} is a basis of A^{n-1}. Consider e^{n-2}, we claim that e^{n-2} is independent of e^{n-1}. Suppose instead that $e^{n-2} = ae^{n-1}$ for some a in K and $a \neq 0$. Then

$$e^{n-3}(e - ae^2) = 0.$$

Since $e^{n-3} \in A^{n-3}$ and $e - ae^2 \in A \setminus A^2$, then this contradicts the hypothesis. Therefore, e^{n-2} is independent of e^{n-1}. Hence, e^{n-2}, e^{n-1}, forms a basis of A^{n-2}.

By repeating the same method as above we have that $e, e^2, \ldots, e^{n-1}$ is a basis of A.

Next, we look at $K_o[x]/(x^{n-1})$. For y $K_o[x]/(x^{n-1})$ we can write

$$y = a_1x + a_2x^2 + \dots\ a_{n-1}x^{n-1},\ \{a_i\} \subset K,\ i = 1,2,3,\dots n$$

Now, let f: $K_o[x]/(x^{n-1}) \longrightarrow A$ be a mapping such that

$$f(x^k) = e^k \quad \text{for } k = 1,2,\dots n-1.$$

It is obvious that f is a linear, 1-1 and onto mapping. Next, we will show that f is a homomorphism. Let $y,z \in K_o[x]/(x^{n-1})$, then

$$y = a_1x+a_2x^2+\dots +a_{n-1}x^{n-1},\ z = b_1x+b_2x^2+\dots+b_{n-1}x^{n-1}$$

for $\{a_i, b_i\} \subset K$, $i = 1,2,\dots n-1$. Then

$$\begin{aligned}
f(yz) &= f\left[(a_1x+a_2x^2+\dots+ a_{n-1}x^{n-1})\right.\\
&\qquad \left.(b_1x+b_2x+\dots+b_{n-1}x^{n-1})\right]\\
&= f\left[(a_1b_1)x^2+(a_1b_2+a_2b_1)x^3+\right.\\
&\qquad +(a_1b_3+a_2b_2+a_3b_1)x^4+\dots\\
&\qquad \left.\dots + (a_1b_{n-2}+a_2b_{n-3}+\dots+ a_{n-2}b_1)x^{n-1}\right]\\
&= a_1b_1f(x^2) + (a_1b_2+a_2b_1)f(x^3) +\dots\\
&\qquad +(a_1b_{n-2}+a_2b_{n-3}+\dots+a_{n-2}b_1)f(x^{n-1})\\
&= a_1b_1e^2+(a_1b_2+a_2b_1)e^3+\dots+(a_1b_{n-2}+a_2b_{n-3}\\
&\qquad +a_{n-2}b_1)e^{n-1}\\
&= (a_1e+a_2e^2+\dots+a_{n-1}e^{n-1})(b_1e+b_2e^2+\dots+b_{n-1}e^{n-1})\\
&= \left[a_1f(x)+a_2f(x^2)+\dots+a_{n-1}f(x^{n-1})\right]\\
&\qquad \left[b_1f(x)+b_2f(x^2)+\dots+b_{n-1}f(x^{n-1})\right]\\
&= f(a_1x+a_2x^2+\dots+a_{n-1}x^{n-1})f(b_1x+b_2x^2+\dots+b_{n-1}x^{n-1})\\
&= f(y)f(z).
\end{aligned}$$

Therefore, A is isomorphic to $K_o[x]/(x^{n-1})$ and the theorem is proved.

Q.E.D.