CHAPTER IV

THE SEMISIMPLE ALGEBRAS

The material of this chapter is based on reference [1].

In this chapter we study the structure of semisimple algebras. We begin by recalling a definition which is used repeatedly in this chapter. A ring R is called right Artinian if it satisfies the descending chain condition on right ideals or, equivalently, every nonempty set of right ideals of R possesses a minimal member. We have already shown in Chapter I that every algebra A finite dimensional over a field K is a ring with minimum condition.

Lemma 4.1 : Let A be a ring and suppose that for some $a \in A$, $a^2 - a$ is nilpotent. Then either a is nilpotent or, for some polynomial q(x) with integer coefficients $e = aq(a)$ is a nonzero idempotent of A.

Proof : Since $a^2 - a$ is nilpotent, there exists $k > 0$ such that

$$(a^2 - a)^k = 0.$$

By expanding this, we get

$$a^{2k} + (-1)^{k-1} ka^{k+1} + (-1)^{k-2} \frac{k(k-1)}{2} a^{k+2} + \dots + (-1)^k a^k = 0$$

Therefore,

$$a^k = (-1)^{k+1} a^{2k} + (-1^2)^k \; ka^{k+1} + \dots + (-1)^{k+2} 2ka^{2k-1}$$

$$= a^{k+1} p(a)$$

where p(x) is a polynomial having integer coefficients.

Therefore,

$$a^k = a^{k+1}p(a)$$

$$= a^k . ap(a)$$

$$= (a^{k+1}p(a))ap(a)$$

$$= a^{k+2}p(a)^2$$

Continuing this process we get

$$a^k = a^{2k}p(a)^k$$

Let $e = a^k p(a)^k$

If a is not nilpotent, then $a^k \neq 0$ for all k.

Hence $e \neq 0$ and

$$e^2 = a^{2k}p(a)^{2k} = a^k p(a)^k = e$$

Therefore, $\exists$ a polynomial q(x) with integer coefficients such that $e = aq(a)$ is a nonzero idempotent element.

Q.E.D.

Using this lemma we can prove the following theorem.

Theorem 4.2 : In a right Artinian ring R, every nonnilpotent right ideal I contains a nonzero idempotent element.

Proof : Since I is a nonnilpotent right ideal of R, the collection K of nonnilpotent right ideals of R which are contained in I is not empty. By the minimum condition on right ideals, there exists a minimal element I_1 of K. It is clear that I_1^2 is also a right ideal of R and since I_1 is nonnilpotent, I_1^2 is a nonnilpotent right ideal of R. Therefore $I_1^2 = I_1$, by the minimality property of I_1.

Next consider the collection M of all right ideals L of R such that

(1) $$LI_1 \neq 0 \quad \text{and}$$

(2) $$L \subseteq I_1 .$$

M is nonempty, since I_1 is in M, and hence M has a minimal member, say L_1.

By (1) $x \neq 0$ in L_1 such that $xI_1 \neq \{0\}$. Since L_1 is a right ideal, $xI_1 \subseteq L_1$. But xI_1 is a right ideal contained in I_1 such that

$$(xI_1)I_1 = xI_1^2 = xI_1 \neq \{0\} .$$

Therefore, by the minimality of L_1, $xI_1 = L_1$. Hence $\exists$ an $a \in I_1 \subseteq I$ such that $x = xa$ and so

$$x = xa = (xa)a = xa^2 = (xa)a^2 = xa^3 = \ldots.$$

or $x = xa^n$ for all n.

Therefore, a is a nonnilpotent element, and

$$xa = xa^2$$

that is $$x(a-a)^2 = 0.$$

Now, set

$$N = \{ u \in I_1 \mid xu = 0 \}.$$

We can see that N is a right ideal of R which is contained in I_1, and since $xI_1 = L_1 \neq \{0\}$, N is properly contained in I_1. Since I_1 is a minimal member of K, N is nilpotent. From the above, $x(a^2-a) = 0$ which implies that $a^2 - a$ is in N, hence $a^2 - a$ is a nilpotent element of I. Then we can apply Lemma 4.1 to get a polynomial q(x) with integer coefficients such that $e = aq(a)$ is a nonzero idempotent element in I, and this completes the proof of the theorem

Q.E.D.

<u>Corollary 4.3</u> : Let R be a right Artinian ring. Then a right ideal I of R is nilpotent if and only if every element of I is nilpotent.

<u>Proof</u> : First suppose that I is a nilpotent ideal of R. Then there exists $k > 0$ such that $I^k = \{0\}$. For each $a \in I$, $a^k \in I^k$. Therefore $a^k = 0$, that is, a is a nilpotent element.

Next, suppose that every element in I is nilpotent. If I is not a nilpotent ideal, then by Theorem 4.2 I contains a nonzero idempotent element, say e. Since $e \in I$, e is a nilpotent element, that is, there exists $k > 0$ such that $e^k = 0$. But $e = e^2 = e^2.e = e^3 = e^2.e^2 = e^4 = \ldots = e^k = 0$, which is a contradiction. Therefore I is a nilpotent ideal of R.

<u>Q.E.D.</u>

<u>Lemma 4.4</u> : Let R be a ring which has no nilpotent 2-sided ideals, except the zero ideal. Then R possesses no nonzero nilpotent right (left) ideals.

<u>Proof</u> : Let I be any nilpotent right ideal of R, say $I^n = \{0\}$. Since I is a right ideal, we can see that RI is also a right ideal of R, and since R is a left ideal of R., RI is a left ideal of R. Therefore RI is a two-sided ideal of R. Consider $(RI)^n$

$$
\begin{aligned}
(RI)^n &= RIRI\ldots RI \\
&= R(IR)(IR)\ldots(IR)I \\
& R(I)(I)\ldots(I)I \\
& RI^n = \{0\}.
\end{aligned}
$$

But R has no nonzero nilpotent two sided ideal, hence $RI = \{0\}$. Since $RI \subseteq I$, I is also a left ideal of R.

Therefore I is a nilpotent two-sided ideal of R, making $I = \{0\}$. That is, R has no nonzero nilpotent right ideals.

Q.E.D.

Now, we come to the theorem which show that the idempotent elements occur as an unavoidable part of our theory.

Theorem 4.5 : Ler R be a semisimple right Artinian ring and let $I \neq 0$ be a right ideal of R. Then $I = eR$ for some idempotent element e in R.

Proof : Since R is a semisimple ring, I cannot be nilpotent, hence by Theorem 4.2 it must have a nonzero idempotent element. If e is an idempotent element in I, let $A(e) = \{x \in I \mid ex = 0\}$. The set of right ideals $\{A(e) \mid e^2 = e \neq 0 \in I\}$ is a nonempty set so it has a minimal element $A(e_0)$. If $A(e_0) = 0$, then, since for any $x \in I$ we have $e_0(x-e_0x) = 0$, then $x-e_0x$ is in $A(e_0)$. Hence $x-e_0x = 0$ i.e. $x = e_0x$ for all x in I, and consequently, $I = e_0I$. Since $e_0 \in I$, $e_0R \subsetneq I$. Therefore, we have $I = e_0I \subsetneq e_0R \subseteq I$. This implies that $I = e_0R$.

If $A(e_0) \neq 0$ then, since $A(e_0)$ is a nonzero right ideal of R, $A(e_0)$ must have a nonzero idempotent element, say e_1. By the definition of $A(e_0)$, e_1 is in I and $e_0e_1 = 0$. Set $e_2 = e_0 + e_1 - e_1e_0$. Then e_2 is in I and

$$e_2^2 = (e_0 + e_1 - e_1e_0)(e_0 + e_1 - e_1e_0)$$

$$= e_0 + e_1 - e_1e_0 = e_2 .$$

That is, e_2 is an idempotent element. Moreover,

$$e_2e_1 = (e_0 + e_1 - e_1e_0)e_1 = e_1 \neq 0.$$

Hence in particular, $e_2 \neq 0$. Now, if $e_2x = 0$, then

$$(e_0 + e_1 - e_1e_0)x = 0,$$

hence,

$$e_0(e_0 + e_1 - e_1e_0)x = 0 .$$

Thus

$$(e_0^2 + e_0e_1 - e_0e_1e_0)\, x = 0,$$

That is, $e_0x = 0$. Therefore $A(e_2) \subseteq A(e_0)$. But $e_0e_1 = 0$ and $e_2e_1 \neq 0$. then $e_1 \in A(e_0)$ and $e_1 \notin A(e_2)$. That is, $A(e_2) \neq A(e_0)$ and $A(e_2)$ is property contained in $A(e_0)$. This contradicts the minimality of $A(e_0)$. Therefore the case $A(e_0) \neq 0$ is impossible.

Q.E.D.

Remark : Notice that the idempotent e acts as a left identity for the right ideal $I = eR$. Indeed, if $x \in I$, then $x = ey$ for some $y \in R$; Therefore,

$$ex = e^2y = ey = x .$$

Definition 4.6 : The center C of a ring R is the set

$$C = \left\{ x \in R \;\middle|\; xy = yx \text{ for all } y \in R \right\}.$$

Theorem 4.5 has two interesting corollaries.

Corollary 1 : If R is a semisimple right Artinian ring and A is an ideal of R, then $A = eR = Re$ where e is a unique idempotent element in the center of R.

Proof : Since A is a right ideal of R, Theorem 4.5 implies that $A = eR$ for some nonzero idempotent element e in R. Let $B = \{x-xe \mid x \in A\}$. Since $ex = x$ for all x in A and $(x-xe)e = 0$ for all x in A, we have $Be = \{0\}$ so $BA = BeA = \{0\}$.

However, as A is also a left ideal of R, B must be a left ideal of of R. Moreover, $B^2 \subsetneq BA = \{0\}$, implies that $B^2 = \{0\}$. But R is a semisimple ring, thus $B = \{0\}$. That is $x = xe$ for all x in A, and $A = Ae \subsetneq Re \subseteq A$. Therefore $A = Re$.

Now, we are going to show that e is in the center of R. Let $y \in R$, then $ye \in A$. Hence $ye = e(ye)$ because e is a left identity for A. Moreover, $ey \in A$, hence $ey = (ey)e$ by virtue of the fact that e is a right identity for A. Therefore $ye = eye = ey$ for all y in R, that is, e is in the center of R.

To prove uniqueness of e. If e' is any other idempotent generator of A, then $e = ee' = e'$. This completes the proof of the corollary.

Q.E.D.

Corollary 2 : A semisimple right Artinian ring has a two-sided identity element.

Proof : R is an ideal of R so the result comes directly from the Corollary 1.

Before going to the Wedderburn structure theorem we first proof a lemma and corollary.

Lemma 4.7 : Let R be a semisimple right Artinian ring and $I = eR = Re$ be an ideal of R, e an idempotent. Then any right (left, 2-sided) ideal of I is also a right (left, 2-sided) ideal of R.

Proof : Suppose that J is arbitrary right ideal of I, considered as a ring. Since $I = Re$, $J \subseteq Re$. Therefore, each $a \in J$ can be written in the form $a = re$, for some $r \in R$. Since e is an idempotent element, $e^2 = e$ and we get

$$a = re = (re)e = ae \in Je$$

That is, $J = Je$ and hence

$$JR = (Je)R = J(eR) = JI \subseteq J$$

This makes J a right ideal of R.

Q.E.D.

For left and two-sided ideals of I, the same argument proves that they are also left and two-sided ideal of R respectively.

Corollary : Let R be a semisimple right Artinian ring. Viewed as rings

(1) each ideal of R is itself a semisimple right Artinian ring

and

(2) any minimal ideal of R is a simple ring.

Proof : For the first part of the theorem, let I be an ideal of R. Viewed as a ring, suppose that I is not semisimple, then there exists a nonzero ideal J of I such that J is a nilpotent ideal. Then from Lemma 4.6, we can conclude that J is also a nonzero nilpotent ideal of R. This contradicts the fact that R is semisimple ring. Therefore, viewed as a ring, I is a semisimple ring.

For the second part, let I be a minimal ideal of R. Suppose that I is not simple, viewed as a ring. Then there exists an ideal J of I such that $\{0\} \subset J \subset I$. From Lemma 4.6, J is also an ideal of R. Therefore I is not a minimal ideal of R which is a contradiction. That is, I must be simple, viewed as a ring.

Q.E.D.

Theorem 4.8 (Wedderburn) : Let R be a semisimple right Artinian ring. Then R is the (finite) direct sum of its minimal 2-sided ideals, each of which is a simple right Artinian.

Proof : Consider the collection of all nonzero 2-sided ideals of R. We may assume that this collection is not empty, since, if this collection is empty, then R itself is a simple ring and the theorem is already proved. Then by the minimum condition on right ideals, this collection possesses a minimal member, $I_1 \neq \{0\}$ say. From Corollary 1 of Theorem 4.5, $I_1 = e_1R = Re_1$ for some unique idempotent e_1 in the center of R. Since $(1-e_1)\,r = r - e_1r = r - re_1 = r(1-e_1)$, $\forall\, r \in R$, $(1-e_1)$ is in the center of R. Therefore, $J = (1-e_1)R$ forms an ideal of R. Now for any x in R, we may write

$$x = e_1x + (1-e_1)x.$$

That is, $R = I_1 + J_1$. Next, we have to prove that $I_1 \cap J_1 = \{0\}$. Let $x \in I_1 \cap J_1$, since $x \in J_1$, $x = (1-e_1)r$ for some $r \in R$. Since e_1 is an idempotent element, we get

(1) $\qquad e_1x = e\,(1-e_1)r = 0$

for $x \in I_1$, $x = e_1s$ for some $s \in R$. Therefore, we get

(2) $$e_1x = e_1(e_1s) = e_1^2s = e_1s = x.$$

Thus, from (1) and (2) we have $x = 0$, or equivalently, $I_1 \cap J_1 = \{0\}$.

Hence, $R = I_1 \oplus J_1$. By the minimality of I_1 and from the preceding Corollary, I_1 is simple when regarded as a ring.

Next, we shall consider J_1. Since R is a semisimple right Artinian ring, the preceding Corollary allows us to conclude that : J_1 is also semisimple and right Artinian when regarded as a ring. If $J_1 = \{0\}$, then the theorem is proved. If $J_1 \neq \{0\}$, then repeat the above process replacing R by J_1. This yields $J_1 = I_2 \oplus J_2$, with J_2 an ideal contained in J_1. Repeating the process we obtain

$$R = I_1 \oplus I_2 \oplus I_3 \oplus \ldots \oplus I_n \oplus J_n ,$$

where each $I_i = e_iR$ is a simple, idempotent-generated minimal ideal of R. Since $J_1 \supsetneqq J_2 \supsetneqq J_3 \supsetneqq \ldots$, then the minimum condition on right ideals of R implies that $J_n = \{0\}$ for some n. That is,

$$R = I_1 \oplus I_2 \oplus \ldots \oplus I_n .$$

All that left to prove is that I_i include all the minimal two-sided ideals of R. To prove this, let $I \neq \{0\}$ be any minimal ideal of R. Since $RI \subseteq I$, then the minimality of I implies that $RI = I$. Thus,

$$I = RI = I_1I \oplus I_2I \oplus \ldots \oplus I_nI.$$

Each I_iI is an ideal of R which is contained in I_i. Therefore the minimality of I_i implies that $I_iI = \{0\}$ or else $I_iI = I$. If $I_iI = \{0\}$

for all $i = 1,2,\ldots, n$, $I = \{0\}$, which is a contradiction. If $I_i I = I_i$ for some i, then $I_i = I_i I \subseteq I$ implies that $I_i = I$ for some i.

Q.E.D.

Knowing that any semisimple right Artinian ring can be represented as a direct sum of simple right Artinian rings, we are left to determine a satisfactory structure theory for simple rings in which the descending chain condition of right ideals holds. It will be found in due course that such rings are isomorphic to the matrix rings. The way to prove this is very long and can be found in [1], therefore the proof will be omitted.

Theorem 4.9 : Let R be a simple right Artinian ring with identity. Then there exist division ring D and suitable intebers n such that

$$R \cong M_n(D)$$

where $M_n(D)$ denotes the ring of $n \times n$ matrices over the division ring D.

Corollary : Let R be a semisimple right Artinian ring. Then there exist division rings D_i and suitable integers n_i $(i = 1,2,\ldots,r)$ such that

$$R \cong M_{n_i}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_r}(D_r).$$

where $M_{n_i}(D_i)$ denotes the ring of $n_i \times n_i$ matrices over division ring D_i.