CHAPTER III

DEGREE SEQUENCES OF PARTITE HYPERGRAPHS

3.1 Introduction.

Let $H = (V_1, V_2, V_3; \mathcal{E})$ be a (3,3)-hypergraph in which the vertices are labelled such that

$$V_1 = \{v_1, \ldots, v_{n_1}\},$$

$$V_2 = \{v_{n_1+1}, \ldots, v_{n_1+n_2}\},$$

$$V_3 = \{v_{n_1+n_2+1}, \ldots, v_{n_1+n_2+n_3}\}.$$

The 3-partite finite sequence

$$(d_H(v_1), \ldots, d_H(v_{n_1}); d_H(v_{n_1+1}), \ldots, d_H(v_{n_1+n_2});$$

$$d_H(v_{n_1+n_2+1}), \ldots, d_H(v_{n_1+n_2+n_3}))$$

will be called the degree sequence of H, and will be denoted by δ_H. For example, let

$$V_1 = \{v_1, v_2\},$$

$$V_2 = \{v_3, v_4\},$$

$$V_3 = \{v_5, v_6, v_7\},$$

and

$$\mathcal{E} = \{\{v_1,v_3,v_5\},\{v_1,v_3,v_6\},\{v_1,v_3,v_7\},\{v_2,v_3,v_6\},\{v_2,v_4,v_7\}\}.$$

Then $\delta_H = (3,2;4,1;1,2,2)$.

Now, let

$$\delta = (d_1,\ldots,d_{n_1};d_{n_1+1},\ldots,d_{n_1+n_2};d_{n_1+n_2+1},\ldots,d_{n_1+n_2+n_3})$$

be a 3-partite finite sequence in $\mathbb{P}$. We shall say that δ is <u>realizable</u> if there exists a (3,3)-hypergraph $H = (V_1,V_2,V_3;\mathcal{E})$ in which

$$V_1 = \{v_1,\ldots,v_{n_1}\},$$

$$V_2 = \{v_{n_1+1},\ldots,v_{n_1+n_2}\},$$

$$V_3 = \{v_{n_1+n_2+1},\ldots, v_{n_1+n_2+n_3}\},$$

and

$$d_H(v_i) = d_i \quad \text{for} \quad 1 \le i \le n_1+n_2+n_3.$$

Such a (3,3)-hypergraph is called a <u>realization</u> of δ.

A question arises. How can one determine whether a given 3-partite finite sequence is realizable ? To answer this question, we shall find conditions which are necessary or sufficient for the existence of a realization of any given 3-partite finite sequence.

Throughout this chapter, by δ_t we mean a finite sequence having n_t terms in $\mathbb{P}$.

3.2 <u>The Main Theorem</u>.

Before giving the theorem, let us make a remark.

3.2.1 Remark. Let

$$\delta = (\delta_1(1),\ldots,\delta_1(n_1);\delta_2(1),\ldots,\delta_2(n_2);\delta_3(1),\ldots,\delta_3(n_3))$$

be a 3-partite finite sequence in $\mathbb{P}$. If δ is realizable, then we have

$$(3.2.1.1) \qquad \sum_{i=1}^{n_1} \delta_1(i) = \sum_{j=1}^{n_2} \delta_2(j) = \sum_{k=1}^{n_3} \delta_3(k).$$

Proof : Assume that δ is realizable. Let $H = (V_1,V_2,V_3;\mathcal{E})$ be a realization of δ. Assume that

$$V_1 = \{x_1,\ldots,x_{n_1}\},$$

$$V_2 = \{y_1,\ldots,y_{n_2}\},$$

$$V_3 = \{z_1,\ldots,z_{n_3}\}.$$

Then

$$d_H(x_i) = \delta_1(i) \text{ for } 1 \le i \le n_1,$$

$$d_H(y_j) = \delta_2(j) \text{ for } 1 \le j \le n_2,$$

$$d_H(z_k) = \delta_3(k) \text{ for } 1 \le k \le n_3.$$

Since $\bigcup_{x_i \in V_1} \mathcal{E}(x_i) = \mathcal{E}$ and $\mathcal{E}(x_i) \cap \mathcal{E}(x_{i'}) = \phi$ for $i \ne i'$,

we have

$$\sum_{x_i \in V_1} d_H(x_i) = \sum_{x_i \in V_1} |\mathcal{E}(x_i)|$$

$$= \Big|\bigcup_{x_i \in V_1} \mathcal{E}(x_i)\Big|$$

$$= |\mathcal{E}|.$$

Hence

$$\sum_{i=1}^{n_1} \delta_1(i) = |\mathcal{E}|.$$

Similarly,

$$\sum_{j=1}^{n_2} \delta_2(j) = |\mathcal{E}| \quad \text{and} \quad \sum_{k=1}^{n_3} \delta_3(k) = |\mathcal{E}|.$$

Hence we obtain the equations (3.2.1.1). #

3.2.2 Theorem. Let

$$\delta = (\delta_1(1),\ldots,\delta_1(n_1);\delta_2(1),\ldots,\delta_2(n_2);\delta_3(1),\ldots,\delta_3(n_3))$$

be a 3-partite finite sequence in $\mathbb{P}$ that satisfies the equations (3.2.1.1). Then the following statements are equivalent :

(i) δ is realizable.

(ii) There exists a $n_1 \times n_2$-matrix μ_{12} over $\mathbb{N}$ such that

(3.2.2.1) $$\delta_1(i) = \sum_{j=1}^{n_2} \mu_{12}(i,j) \quad \text{for } 1 \leq i \leq n_1,$$

(3.2.2.2) $$\delta_2(j) = \sum_{i=1}^{n_1} \mu_{12}(i,j) \quad \text{for } 1 \leq j \leq n_2,$$

(3.2.2.3) $$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \min\{\mu_{12}(i,j), |B|\} \geq \sum_{k\in B} \delta_3(k)$$

for all $B \subseteq \{1,\dots,n_3\}$.

(iii) There exists a $n_2 \times n_3$-matrix μ_{23} over $\mathbb{N}$ such that

$$\delta_2(j) = \sum_{k=1}^{n_3} \mu_{23}(j,k) \quad \text{for } 1 \le j \le n_2 \tag{3.2.2.4}$$

$$\delta_3(k) = \sum_{j=1}^{n_2} \mu_{23}(j,k) \quad \text{for } 1 \le k \le n_3, \tag{3.2.2.5}$$

$$\sum_{j=1}^{n_2} \sum_{k=1}^{n_3} \min\{\mu_{23}(j,k), |B|\} \ge \sum_{i \in B} \delta_1(i) \tag{3.2.2.6}$$

for all $B \subseteq \{1,\dots,n_1\}$.

(iv) There exists a $n_1 \times n_3$-matrix μ_{13} over $\mathbb{N}$ such that

$$\delta_1(i) = \sum_{k=1}^{n_3} \mu_{13}(i,k) \quad \text{for } 1 \le i \le n_1, \tag{3.2.2.7}$$

$$\delta_3(k) = \sum_{i=1}^{n_1} \mu_{13}(i,k) \quad \text{for } 1 \le k \le n_3, \tag{3.2.2.8}$$

$$\sum_{i=1}^{n_1} \sum_{k=1}^{n_3} \min\{\mu_{13}(i,k), |B|\} \ge \sum_{j \in B} \delta_2(j) \tag{3.2.2.9}$$

for all $B \subseteq \{1,\dots,n_2\}$.

Proof : To show that the statement (i) implies the statement (ii), assume that δ is realizable. Let $H = (V_1,V_2,V_3;\mathcal{E})$ be a realization of δ . Assume that

$$V_1 = \{x_1,\ldots,x_{n_1}\},$$

$$V_2 = \{y_1,\ldots,y_{n_2}\},$$

$$V_3 = \{z_1,\ldots,z_{n_3}\}.$$

Then

$$d_H(x_i) = \delta_1(i) \quad \text{for} \quad 1 \le i \le n_1,$$

$$d_H(y_j) = \delta_2(j) \quad \text{for} \quad 1 \le j \le n_2,$$

$$d_H(z_k) = \delta_3(k) \quad \text{for} \quad 1 \le k \le n_3.$$

Define a $n_1 \times n_2$-matrix μ_{12} over $\mathbb{N}$ by

$$\mu_{12}(i,j) = d_H(\{x_i,y_j\}) \quad \text{for } 1 \le i \le n_1, \quad 1 \le j \le n_2.$$

To show that μ_{12} satisfies the equation (3.2.2.1), let $x_i \in V_1$. Since $\mathcal{E}(x_i) = \bigcup_{y_j \in V_2} \mathcal{E}(\{x_i,y_j\})$ and $\mathcal{E}(\{x_i,y_{j'}\}) \cap \mathcal{E}(\{x_i,y_j\}) = \phi$ for $j \ne j'$, we have

$$(1) \qquad |\mathcal{E}(x_i)| = \Big|\bigcup_{y_j \in V_2} \mathcal{E}(\{x_i,y_j\})\Big|$$

$$= \sum_{y_j \in V_2} |\mathcal{E}(\{x_i,y_j\})|.$$

By (1), the properties of H and the definition of μ_{12}, we have

$$\delta_1(i) = d_H(x_i)$$

$$= |\mathcal{E}(x_i)|$$

$$= \sum_{y_j \in V_2} |\mathcal{E}(\{x_i, y_j\})|$$

$$= \sum_{y_j \in V_2} d_H(\{x_i, y_j\})$$

$$= \sum_{j=1}^{n_2} \mu_{12}(i,j).$$

Hence we obtain the equation (3.2.2.1) Similarly, μ_{12} satisfies the equation (3.2.2.2).

To show that μ_{12} satisfies the inequality (3.2.2.3), let $N = (\{u\}, X, Y, \{v\};\alpha)$ be a bipartite transportation network in which $X = V_1 \times V_2$, $Y = \{1,\ldots,n_3\}$, and

$$\alpha((x_i, y_j), k) = 1 \text{ for } (x_i,y_j) \in X, \quad k \in Y;$$

$$\alpha(u,(x_i,y_j)) = \mu_{12}(i,j) \quad \text{for} \quad (x_i,y_j) \in X;$$

$$\alpha(k, v) = \delta_3(k) \quad \text{for} \quad k \in Y;$$

$$\alpha(v,u) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j).$$

Define a function ψ on the set of arcs in N as follows :

For each intermediate arc $((x_i, y_j), k)$, let

$$\psi((x_i,y_j),k) = \begin{cases} 1 & \text{if } \{x_i,y_j, z_k\} \in \mathcal{E}, \\ 0 & \text{if } \{x_i,y_j,z_k\} \notin \mathcal{E}, \end{cases}$$

and for all other arcs a, let $\psi(a) = \alpha(a)$. By the definitions of ψ and α, we have

$$\psi(v,u) = \alpha(v,u) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j) = \sum_{(x_i,y_j) \in X} \alpha(u,(x_i,y_j)) = \sum_{(x_i,y_j) \in X} \psi(u,(x_i,y_j)).$$

Hence ψ is conservative at u. By the definitions of ψ, α and the equations (3.2.2.1), (3.2.1.1), we have

$$\psi(v,u) = \alpha(v,u) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j) = \sum_{i=1}^{n_1} \delta_1(i) = \sum_{k=1}^{n_3} \delta_3(k) = \sum_{k \in Y} \alpha(k,v) = \sum_{k \in Y} \psi(k,v).$$

Hence ψ is conservative at v. For any vertex $(x_i,y_j) \in X$, by the definitions of ψ, α and μ_{12}, we have

$$\begin{aligned}\psi(u,(x_i,y_j)) &= \alpha(u,(x_i,y_j)) \\ &= \mu_{12}(i,j) \\ &= d_H(\{x_i,y_j\}) \\ &= |\{E \in \mathcal{E} / \{x_i,y_j\} \subsetneq E\}| \\ &= |\{z_k \in V_3/\{x_i,y_j,z_k\} \in \mathcal{E}\}|.\end{aligned}$$

Observe, from the definition of $\psi((x_i,y_j), k)$, that

$$\begin{aligned}\sum_{k\in Y} \psi((x_i,y_j),k) &= |\{k\in Y/\psi((x_i,y_j),k) = 1\}| \\ &= |\{z_k\in V_3/\{x_i,y_j,z_k\} \in \mathcal{E}\}|.\end{aligned}$$

Hence

$$\psi(u,(x_i,y_j)) = \sum_{k\in Y} \psi((x_i,y_j),k).$$

i.e. ψ is conservative at every vertex $(x_i,y_j) \in X$.

For any vertex $k \in Y$, by the definitions of ψ, α and the properties of H, we have

$$\begin{aligned}\psi(k,v) &= \alpha(k,v) \\ &= \delta_3(k) \\ &= d_H(z_k) \\ &= |\{E \in \mathcal{E}/ z_k \in E\}| \\ &= |\{(x_i,y_j) \in X / \{x_i,y_j,z_k\} \in \mathcal{E}\}|\end{aligned}$$

$$= \quad |\{(x_i,y_j) \in X \ / \ \psi((x_i,y_j),k) = 1\}|$$

$$= \sum_{(x_i,y_j) \in X} \psi((x_i,y_j),k).$$

Hence ψ is conservative at every vertex $k \in Y$. Therefore ψ is a flow in N . From the definition of ψ , we see that ψ is compatible and saturates all the sink arcs in N . Therefore, by Theorem 2.2.3, we have

(2) $$F_N(B) \geq d_N(B) \quad \text{for all } B \subseteq Y.$$

By the equations (2.2.1) and (2.2.2), we have

$$F_N(B) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \min \{\mu_{12}(i,j), |B|\}$$

and

$$d_N(B) = \sum_{k \in B} \delta_3(k).$$

Hence (2) becomes the inequality (3.2.2.3).

To show that the statement (ii) implies the statement (i), assume that there exists a $n_1 \times n_2$-matrix μ_{12} over $\mathbb{N}$ that satisfies the equations (3.2.2.1), (3.2.2.2) and the inequality (3.2.2.3). Let

$$V_1 = \{x_1,\ldots,x_{n_1}\},$$

$$V_2 = \{y_1,\ldots,y_{n_2}\},$$

$$V_3 = \{1,\ldots,n_3\},$$

be disjoint sets. Let $N = (\{u\}, X, Y, \{v\}; \alpha)$ be a bipartite transportation network where $X = V_1 \times V_2$, $Y = V_3$, and

$$\alpha((x_i,y_j),k) = 1 \text{ for } (x_i,y_j) \in X, \quad k \in Y;$$

$$\alpha(u,(x_i,y_j)) = \mu_{12}(i,j) \text{ for } (x_i,y_j) \in X ;$$

$$\alpha(k, v) = \delta_3(k) \text{ for } k \in Y;$$

$$\alpha(v,u) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j).$$

For $B \subsetneq Y$, by the equations (2.2.1) and (2.2.2), we have

$$(3) \qquad F_N(B) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \min \{\mu_{12}(i,j), |B|\},$$

and

$$(4) \qquad d_N(B) = \sum_{k \in B} \delta_3(k).$$

Hence, by(3) and (4), the inequality (3.2.2.3) becomes

$$F_N(B) \geq d_N(B) \quad \text{for all } B \subseteq Y.$$

Therefore, by Theorem 2.2.3, N has a compatible flow f that saturates all the sink arcs. By the definition of α , the equation (3.2.2.1) and the assumption that δ satisfies the equations (3.2.1.1), we get

$$\sum_{(x_i,y_j) \in X} \alpha(u,(x_i,y_j)) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j)$$

$$= \sum_{i=1}^{n_1} \delta_1(i)$$

$$= \sum_{k=1}^{n_3} \delta_3(k)$$

$$= \sum_{k \in Y} \alpha(k,v).$$

Hence, by Remark 2.2.4, f also saturates all the source arcs. Now, let $V = V_1 \cup V_2 \cup V_3$, and

$$\mathcal{E} = \{\{x_i,y_j,k\} / \ x_i \varepsilon V_1,\ y_j \ \varepsilon V_2,\ k \ \varepsilon V_3,\ f((x_i,y_j),k) = 1\}$$

Observe that for each $E = \{x_i,y_j,k\}$ in $\mathcal{E}$, we have

$$|E \cap V_1| = |\{x_i\}| = 1 ;$$

$$|E \cap V_2| = |\{y_j\}| = 1 ;$$

$$|E \cap V_3| = |\{k\}| = 1.$$

Hence $H = (V,\mathcal{E})$ is a (3,3)-hypergraph with 3-partition (V_1,V_2,V_3). For each $x_i \ \varepsilon \ V_1$ and $y_j \ \varepsilon \ V_2$, we have

$$(5) \qquad d_H(\{x_i,y_j\}) = |\{E \ \varepsilon \ \mathcal{E} / \ \{x_i,y_j\} \subseteq E\}|$$

$$= |\{k \ \varepsilon \ V_3 / \ \{x_i,y_j,k\} \ \varepsilon \ \mathcal{E}\}|$$

$$= |\{k \ \varepsilon \ V_3 \ / \ f((x_i,y_j),k) = 1\}|.$$

Since the capacities of all intermediate arcs is 1, the value of f on these arcs are either 0 or 1. Hence

$$(6) \qquad \sum_{k \ \varepsilon \ V_3} f((x_i,y_j),k) = |\{k \ \varepsilon \ V_3 / \ f((x_i,y_j),k) = 1\}|.$$

Therefore, by (5) and (6), we get

(7) $$d_H(\{x_i,y_j\}) = \sum_{k \in V_3} f((x_i,y_j),k).$$

Since f is conservative at (x_i,y_j) and saturates all the source arcs, we have

(8) $$\sum_{k \in V_3} f((x_i,y_j),k) = f(u,(x_i,y_j))$$

$$= \alpha(u,(x_i,y_j)).$$

Therefore, by (7), (8) and the definition of α, we have

(9) $$d_H(\{x_i,y_j\}) = \mu_{12}(i,j).$$

Hence, by the properties of H and the equation (3.2.2.1), we get

$$d_H(x_i) = |\mathcal{E}(x_i)|$$

$$= \Big|\bigcup_{y_j \in V_2} \mathcal{E}(\{x_i,y_j\})\Big|$$

$$= \sum_{y_j \in V_2} |\mathcal{E}(\{x_i,y_j\})|$$

$$= \sum_{y_j \in V_2} d_H(\{x_i,y_j\})$$

$$= \sum_{j=1}^{n_2} \mu_{12}(i,j)$$

$$= \delta_1(i).$$

This shows that $d_H(x_i) = \delta_1(i)$ for $1 \leq i \leq n_1$.

Similarly, $d_H(y_j) = \delta_2(j)$ for $1 \leq j \leq n_2$.

To show that $d_H(k) = \delta_3(k)$ for $1 \leq k \leq n_3$, let $k \in V_3$.

Observe, from the definition of H, that

$$(10) \qquad d_H(k) = |\{E \in \mathcal{E} / k \in E\}|$$

$$= |\{(x_i,y_j) \in X \,|\, \{x_i,y_j,k\} \in \mathcal{E}\}|$$

$$= |\{(x_i,y_j) \in X \,/\, f((x_i,y_j),k) = 1\}|$$

$$= \sum_{(x_i,y_j) \in X} f((x_i,y_j),k).$$

Since f is conservative at k and saturates all the sink arcs, we have

$$(11) \qquad \sum_{(x_i,y_j) \in X} f((x_i,y_j),k) = f(k,v)$$

$$= \alpha(k,v).$$

Therefore, by (10), (11) and the definition of α, we have

$$d_H(k) = \delta_3(k).$$

This shows that H is a realization of δ.

Similarly, we can show that the statement (iii) and (iv) are equivalent to the statement (i). #

3.3 Some Illustrations.

In this section, we illustrate how our main theorem (Theorem 3.2.2) can be applied. First, we need the following :

3.3.1 Remark. Let μ_{12} be a $n_1 \times n_2$ - matrix over $\mathbb{N}$ that satisfies

(3.3.1.1) $$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j) = \sum_{k=1}^{n_3} \delta_3(k),$$

and

(3.3.1.2) $$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \min \{\mu_{12}(i,j), |B|\} \geq \sum_{k \varepsilon B} \delta_3(k)$$

for all $B \subseteq \{1,\ldots,n_3\}$,

which is the same as the inequality (3.2.2.3). Then we have

(3.3.1.3) $$\mu_{12}(i,j) \leq n_3 \text{ for } 1 \leq i \leq n_1, \quad 1 \leq j \leq n_2;$$

(3.3.1.4) $$|\{(i,j)/\mu_{12}(i,j) = n_3\}| \leq \min_{1 \leq k \leq n_3} \delta_3(k).$$

Proof : Let μ_{12} be a $n_1 \times n_2$- matrix over $\mathbb{N}$ that satisfies the equation (3.3.1.1) and the inequality (3.3.1.2). Then, by Remark 2.2.5, we have

(1) $$\sum_{k=1}^{n_3} \min \{\delta_3(k), |B|\} \geq \sum_{(i,j) \varepsilon B} \mu_{12}(i,j)$$

for all $B \subseteq \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$. Let i,j be given, where $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. For $B = \{(i,j)\}$, (1) becomes

(2) $$\sum_{k=1}^{n_3} \min \{\delta_3(k), 1\} \geq \mu_{12}(i,j).$$

Note that

$$1 \geq \min\{\delta_3(k),\ 1\} \quad \text{for } 1 \leq k \leq n_3.$$

Hence

(3) $$n_3 \geq \sum_{k=1}^{n_3} \min\{\delta_3(k), 1\}.$$

Therefore, by (3) and (2), we get

$$n_3 \geq \mu_{12}(i,j).$$

Hence we obtain the inequality (3.3.1.3).

To show the inequality (3.3.1.4), let

$$S = \{(i,j) / \mu_{12}(i,j) = n_3\},$$

and

$$\delta_3(k_o) = \min_{1 \leq k \leq n_3} \delta_3(k).$$

Suppose that

(4) $$|S| > \delta_3(k_o).$$

Consider the bipartite transportation network $N = (\{u\}, X, Y, \{v\}; \alpha)$, where $X = \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$, $Y = \{1,\ldots, n_3\}$, and

$$\alpha((i,j),k) = 1 \text{ for } (i,j) \in X,\ k \in Y;$$

$$\alpha(u,(i,j)) = \mu_{12}(i,j) \text{ for } (i,j) \in X;$$

$$\alpha(k,v) = \delta_3(k) \text{ for } k \in Y;$$

$$\alpha(v,u) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j).$$

For $B \subseteq Y$, by the equations (2.2.1) and (2.2.2), we have

$$F_N(B) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \min\{\mu_{12}(i,j), |B|\},$$

and

$$d_N(B) = \sum_{k \in B} \delta_3(k).$$

Hence the inequality (3.3.1.2) becomes

$$F_N(B) \geq d_N(B) \quad \text{for all } B \subseteq Y.$$

Therefore, by Theorem 2.2.3, N has a compatible flow ψ that saturates all the sink arcs. By the definition of α and the equation (3.3.1.1), we have

$$\sum_{(i,j) \in X} \alpha(u,(i,j)) = \sum_{k \in Y} \alpha(k,v).$$

Hence, by Remark 2.2.4, ψ also saturates all the source arcs. For each vertex $(i,j) \in S$, we have

$$\sum_{k=1}^{n_3} \psi((i,j),k) = \psi(u,(i,j))$$

$$= \alpha(u,(i,j))$$

$$= \mu_{12}(i,j)$$

$$= n_3.$$

Hence, since the value of ψ at each intermediate arc in N is either 0 or 1, it follows that

$$\psi((i,j),k) = 1 \text{ for } (i,j) \in S, \quad 1 \leq k \leq n_3.$$

Therefore

(5) $$\sum_{(i,j) \in S} \psi((i,j),k_o) = |S|.$$

Since $S \subseteq X$, we have

(6) $$\sum_{(i,j) \in S} \psi((i,j),k_o) \leq \sum_{(i,j) \in X} \psi((i,j),k_o).$$

Hence, by (4), (5), (6) and the conservation property of ψ at k_o, we get

$$\delta_3(k_o) < |S|$$

$$= \sum_{(i,j) \in S} \psi((i,j),k_o)$$

$$\leq \sum_{(i,j) \in X} \psi((i,j),k_o)$$

$$= \psi(k_o,v),$$

Therefore ψ does not saturate the sink arc (k_o,v), which is a contradiction. This proves that $|S| \leq \delta_3(k_o)$. Hence we obtain the inequality (3.3.1.4). #

3.3.2 Example. Let

$$\delta = (11, 9, 8, 4; 8, 8, 7, 6, 3; 17, 14, 1).$$

If δ is realizable, then by Theorem 3.2.2, we must be able to find a 4×5-matrix $\mu = (a_{ij})$ satisfying (3.2.2.1), (3.2.2.2), and (3.2.2.3). So, we shall look for such a matrix μ. Note that the matrix μ must also satisfies (3.3.1.3) and (3.3.1.4). By (3.2.2.1) and (3.2.2.2), the matrix μ must have row sums and column sums as indicated in the following table :

(1)

						row sum
	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	11
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	9
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	8
	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	4
column sum	8	8	7	6	3	

By (3.3.1.3) and (3.3.1.4), we have

(2) $$a_{ij} \leq 3 \text{ for } i = 1,2,3,4 \text{ ; } j = 1,2,3,4,5 \text{ ;}$$

and

(3) $$|\{(i,j) \ / \ a_{ij} = 3\}| \leq 1.$$

From the table (1), the non-negative integers $a_{11}, \ldots, a_{15}$ must be such that

$$a_{11} + \ldots + a_{15} = 11,$$

hence at least one of them must be larger than 2. Then, by (2), we can conclude that such a_{ij} must be 3. By (3), at most one of the a_{ij}'s can be 3, hence exactly one of $a_{11}, \ldots, a_{15}$ is 3. Hence all the remaining a_{ij}'s must be 2, for each term must be less than 3 and they must have the sum that equals to 11-3 = 8.

Since the first row contains a 3 and, by (3), the whole table (1) can contain at most one 3, hence we have

(2′) $$a_{ij} \leq 2 \text{ for } i = 2,3,4 \text{ ; } j = 1,2,3,4,5.$$

Since each term in the second row can be at most 2 and the row sum must be 9, hence this row must consist of four 2's and one 1.

Since the sum of the fifth column is 3, we are forced to have

$$a_{15} = 2 \text{ ; } a_{25} = 1 \text{ ; } a_{35} = a_{45} = 0 \text{ ;}$$

$$a_{21} = a_{22} = a_{23} = a_{24} = 2 .$$

Hence the table (1) must be as follows :

(1′)

						row sum
	*	*	*	*	2	11
	2	2	2	2	1	9
	a_{31}	a_{32}	a_{33}	a_{34}	0	8
	a_{41}	a_{42}	a_{43}	a_{44}	0	4
column sum	8	8	7	6	3	

where one of the entries indicated by * is 3, and the others are 2.

Since the sum of the third row is 8 and each of the terms a_{31}, a_{32}, a_{33}, a_{34} must be less than 3. Hence each must be 2. Hence (1′) becomes

(1″)

						row sum
	*	*	*	*	2	11
	2	2	2	2	1	9
	2	2	2	2	0	8
	a_{41}	a_{42}	a_{43}	a_{44}	0	4
column sum	8	8	7	6	3	

Since the sum of the forth column is 6, hence $a_{14} = 2$ and $a_{44} = 0$. There are three cases for the first row :

$$a_{11} = 3 \text{ or } a_{12} = 3 \text{ or } a_{13} = 3.$$

For each case, the remaining a_{ij}'s are completely determined. We have the following :

Case 1. $a_{11} = 3$,

$$\mu = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Case 2. $a_{12} = 3$,

$$\mu = \begin{bmatrix} 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Case 3. $a_{13} = 3$,

$$\mu = \begin{bmatrix} 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Hence, if δ is realizable, then the required μ must be among those in the three cases. In fact, a straightforward verification shows that μ in Case 1 satisfies (3.2.2.1), (3.2.2.2), and (3.2.2.3) of Theorem 3.2.2. Hence δ is realizable.

3.3.3 Example. Let

$$\delta = (11,9,8,4;\ 8,8,7,6,3;\ 18,13,1).$$

If δ is realizable, then by Theorem 3.2.2, we must be able to find a 4×5 - matrix $\mu = (a_{ij})$ that satisfies (3.2.2.1), (3.2.2.2), and (3.2.2.3). By comparing the given δ with the one in the previous example (Example 3.3.2), we see that the required μ must be among those in the three cases of the previous example. However, by choosing $B = \{1\}$, it turns out that none of these μ's satisfies (3.2.2.3). Hence there does not exist any 4×5 - matrix μ that satisfies (3.2.2.1), (3.2.2.2), and (3.2.2.3). Therefore the given δ is not realizable.

3.4 Some Simple Necessary Conditions.

In this section, we provide some necessary conditions for a given 3-partite finite sequence to be realizable. These conditions can be checked by inspection. In cases where any of the conditions fail, we can conclude that the given sequence is not realizable. Thus we can avoid the more complicated test given in the previous section.

3.4.1 Theorem. Let

$$\delta = (\delta_1(1),\ldots,\delta_1(n_1);\ \delta_2(1),\ldots,\delta_2(n_2)\ ;\ \delta_3(1),\ldots,\delta_3(n_3))$$

be a 3 - partite finite sequence in $\mathbb{P}$. If δ is realizable, then we have the following :

(3.4.1.1) For all $B \subseteq \{1,\ldots,n_2\}$,

$$\sum_{i=1}^{n_1} \min\{\delta_1(i),\ n_3|B|\} \geq \sum_{j \in B} \delta_2(j).$$

(3.4.1.2) For all $B \subseteq \{1,\ldots,n_3\}$,

$$\sum_{j=1}^{n_2} \min\{\delta_2(j),\ n_1|B|\} \geq \sum_{k \in B} \delta_3(k).$$

(3.4.1.3) For all $B \subseteq \{1,\ldots,n_1\}$,

$$\sum_{k=1}^{n_3} \min\{\delta_3(k),\ n_2|B|\} \geq \sum_{i \in B} \delta_1(i).$$

(3.4.1.4) $n_1 n_3 \geq \delta_2(j)$ for $1 \leq j \leq n_2$.

(3.4.1.5) $n_1 n_2 \geq \delta_3(k)$ for $1 \leq k \leq n_3$

(3.4.1.6) $n_2 n_3 \geq \delta_1(i)$ for $1 \leq i \leq n_1$

Proof : Assume that δ is realizable. Then, by Remark 3.2.1, δ satisfies the equations (3.2.1.1). Hence, by Theorem 3.2.2, there exists a $n_1 \times n_2$ - matrix μ_{12} over $\mathbb{N}$ that satisfies the equations (3.2.2.1), (3.2.2.2) and the inequality (3.2.2.3). Consider a bipartite transportation network $N = (\{u\}, X, Y, \{v\}; \alpha)$ where $X = \{x_1, \ldots, x_{n_1}\}$, $Y = \{1, \ldots, n_2\}$, and

$$\alpha(x_i,j) = n_3 \quad \text{for } 1 \le i \le n_1 , \quad 1 \le j \le n_2 ;$$

$$\alpha(u,x_i) = \delta_1(i) \quad \text{for } 1 \le i \le n_1 ;$$

$$\alpha(j,v) = \delta_2(j) \quad \text{for } 1 \le j \le n_2 ;$$

$$\alpha(v,u) = \sum_{i=1}^{n_1} \delta_1(i).$$

Define a function ψ on the set of arcs in N as follows :

$$\psi(x_i,j) = \mu_{12}(i,j) \quad \text{for } 1 \le i \le n_1, \quad 1 \le j \le n_2;$$

$$\psi(a) = \alpha(a) \quad \text{for all other arcs a.}$$

By the definitions of ψ and α , we have

$$\psi(v,u) = \alpha(v,u)$$

$$= \sum_{i=1}^{n_1} \delta_1(i)$$

$$= \sum_{x_i \varepsilon X} \alpha(u,x_i)$$

$$= \sum_{x_i \varepsilon X} \psi(u,x_i).$$

Hence ψ is conservative at u. By the definitions of ψ , α and the equations (3.2.1.1), we have

$$\psi(v,u) = \alpha(v,u)$$

$$= \sum_{i=1}^{n_1} \delta_1(i)$$

$$= \sum_{j=1}^{n_2} \delta_2(j)$$

$$= \sum_{j \in Y} \alpha(j,v)$$

$$= \sum_{j \in Y} \psi(j,v).$$

Hence ψ is conservative at v. For any vertex $x_i \in X$, by the definitions of ψ, α and the equation (3.2.2.1), we have

$$\psi(u,x_i) = \alpha(u,x_i)$$

$$= \delta_1(i)$$

$$= \sum_{j=1}^{n_2} \mu_{12}(i,j)$$

$$= \sum_{j \in Y} \psi(x_i,j).$$

Hence ψ is conservative at every vertex $x_i \in X$. For any vertex $j \in Y$, by the definitions of ψ, α and the equation (3.2.2.2),

we have

$$\psi(j,v) = \alpha(j,v)$$

$$= \delta_2(j)$$

$$= \sum_{i=1}^{n_1} \mu_{12}(i,j)$$

$$= \sum_{x_i \varepsilon X} \psi(x_i, j).$$

Hence ψ is conservative at every vertex $j \varepsilon Y$. Therefore ψ is a flow in N. By the equations (3.2.2.1) and (3.2.1.1), we get

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j) = \sum_{k=1}^{n_3} \delta_3(k).$$

Hence, since μ_{12} satisfies the inequality (3.2.2.3), it follows from Remark 3.3.1 that

$$\mu_{12}(i,j) \leq n_3 \quad \text{for} \quad 1 \leq i \leq n_1, \quad 1 \leq j \leq n_2.$$

Hence, by the definitions of ψ and α, we have

$$\psi(x_i,j) \leq \alpha(x_i,j) \quad \text{for } 1 \leq i \leq n_1, \quad 1 \leq j \leq n_2.$$

Therefore ψ is compatible. Since ψ saturates all the sink arcs in N, it follows from Theorem 2.2.3 that

(1) $$F_N(B) \geq d_N(B) \quad \text{for all } B \subseteq Y.$$

By the equations (2.2.1) and (2.2.2), we have

$$F_N(B) = \sum_{i=1}^{n_1} \min\{\delta_1(i), n_3|B|\}$$

and

$$d_N(B) = \sum_{j \varepsilon B} \delta_2(j).$$

Hence (1) becomes the inequality (3.4.1.1).

Let j be given, where $1 \le j \le n_2$. For $B = \{j\}$, the inequality (3.4.1.1) becomes

$$(2) \qquad \sum_{i=1}^{n_1} \min\{\delta_1(i), n_3\} \ \ge \ \delta_2(j).$$

Note that

$$n_3 \ge \min\{\delta_1(i), n_3\} \quad \text{for} \quad 1 \le i \le n_1.$$

Hence

$$(3) \qquad n_1 n_3 \ \ge \ \sum_{i=1}^{n_1} \min\{\delta_1(i), n_3\}.$$

Therefore, by (3) and (2), we get.

$$n_1 n_3 \ \ge \ \delta_2(j).$$

Hence we obtain the inequality (3.4.1.4).

Similarly, we obtain the inequalities (3.4.1.2), (3.4.1.3), (3.4.1.5) and (3.4.1.6). #

3.4.2 Example. Let

$$\delta \ = \ (5,3,2,2;\ 7,5;\ 7,5).$$

Since δ does not satisfy the inequality (3.4.1.6), hence it follows from Theorem 3.4.1 that δ is not realizable. #

3.4.3 Example. Let

$$\delta \ = \ (15,13,12,11;\ 14,10,8,7,6,6\ ;\ 23,21,4,3).$$

It can be verified that $B = \{1,2\}$ violates the inequality (3.4.1.2). Hence, by Theorem 3.4.1, δ is not realizable. #

3.4.4 Remark. In general, the necessary conditions given by Remark 3.2.1 and Theorem 3.4.1 are not sufficient for the existence of a realization of a 3-partite finite sequence in $\mathbb{P}$. The 3 - partite finite sequence δ given in Example 3.3.3 is a counterexample. However, for certain classes of 3 - partite finite sequences in $\mathbb{P}$, the above necessary conditions are also sufficient. We treat these in the next section.

3.5 The Special Cases.

The following theorem is a consequence of Theorem 3.2.2.

3.5.1 Theorem. Let

$$\delta = (\delta_1(1),\dots,\delta_1(n_1);\delta_2(1),\dots,\delta_2(n_2);\delta_3(1),\dots,\delta_3(n_3))$$

be a 3 - partite finite sequence in $\mathbb{P}$ such that

(3.5.1.1) $$\delta_3(1) = \dots = \delta_3(n_3) = c.$$

Then δ is realizable if and only if

(3.5.1.2) $$\sum_{i=1}^{n_1} \delta_1(i) = \sum_{j=1}^{n_2} \delta_2(j) = cn_3, \text{ and}$$

(3.5.1.3) $$\sum_{i=1}^{n_1} \min\{\delta_1(i), n_3|B|\} \geq \sum_{j\in B} \delta_2(j)$$

for all $B \subseteq \{1,\dots,n_2\}$.

Proof : If δ is realizable, then by Remark 3.2.1, δ must satisfy (3.2.1.1). Since δ satisfies (3.5.1.1), hence (3.2.1.1) becomes (3.5.1.2). By Theorem 3.4.1, δ must also satisfy (3.4.1.1), which is the same as (3.5.1.3). Therefore (3.5.1.2) and (3.5.1.3) are necessary.

To show the sufficiency part, assume that δ satisfies (3.5.1.2) and (3.5.1.3). Consider the bipartite transportation network $N = (\{u\}, X, Y, \{v\}; \alpha)$, where

$X = \{x_1, \ldots, x_{n_1}\}$, $Y = \{1, \ldots, n_2\}$, and

$$\alpha(x_i, j) = n_3 \quad \text{for } 1 \le i \le n_1, \ 1 \le j \le n_2;$$

$$\alpha(u, x_i) = \delta_1(i) \quad \text{for } 1 \le i \le n_1;$$

$$\alpha(j, v) = \delta_2(j) \quad \text{for } 1 \le j \le n_2;$$

$$\alpha(v, u) = \sum_{i=1}^{n_1} \delta_1(i).$$

For $B \subseteq Y$, by the equations (2.2.1) and (2.2.2), we have

$$F_N(B) = \sum_{i=1}^{n_1} \min\{\delta_1(i), n_3|B|\},$$

$$d_N(B) = \sum_{j \in B} \delta_2(j).$$

Hence the inequality (3.5.1.3) becomes

$$F_N(B) \ge d_N(B) \quad \text{for all } B \subseteq Y.$$

Therefore, by Theorem 2.2.3, N has a compatible flow ψ that saturates all the sink arcs. By the definition of α and the equations (3.5.1.2), we have

$$\sum_{x_i \in X} \alpha(u,x_i) = \sum_{j \in Y} \alpha(j,v).$$

Hence, by Remark 2.2.4, ψ also saturates all the source arcs. Define a $n_1 \times n_2$ - matrix μ_{12} over $\mathbb{N}$ as follows :

$$\mu_{12}(i,j) = \psi(x_i,j) \quad \text{for} \quad 1 \leq i \leq n_1, \quad 1 \leq j \leq n_2.$$

Then, since ψ saturates all the source arcs and is conservative at every vertex $x_i \in X$, we get

$$\begin{aligned} \delta_1(i) &= \alpha(u,x_i) \\ &= \psi(u,x_i) \\ &= \sum_{j \in Y} \psi(x_i,j) \\ &= \sum_{j=1}^{n_2} \mu_{12}(i,j). \end{aligned}$$

Hence μ_{12} satisfies the equation (3.2.2.1). Since ψ saturates all the sink arcs and is conservative at every vertex $j \in Y$, we get

$$\begin{aligned} \delta_2(j) &= \alpha(j,v) \\ &= \psi(j,v) \\ &= \sum_{x_i \in X} \psi(x_i,j) \end{aligned}$$

$$= \sum_{i=1}^{n_1} \mu_{12}(i,j).$$

Hence μ_{12} satisfies the equation (3.2.2.2). Next, we shall show that

(1) $$n_3 \cdot \min\{ c, |C|\} \geq \sum_{(i,j)\in C} \mu_{12}(i,j)$$

for all $C \subseteq \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$.

Let $C \subseteq \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$. Then, by the equations (3.2.2.1) and (3.5.1.2), we get

(2) $$\sum_{(i,j)\in C} \mu_{12}(i,j) \leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j)$$

$$= \sum_{i=1}^{n_1} \delta_1(i)$$

$$= cn_3.$$

Since the value of ψ at each intermediate arc (x_i,j) is at most n_3, hence it follows from the definition of μ_{12} that

$$\mu_{12}(i,j) \leq n_3 \text{ for } 1 \leq i \leq n_1, \ 1 \leq j \leq n_2.$$

So, we have

(3) $$\sum_{(i,j)\in C} \mu_{12}(i,j) \leq n_3|C|.$$

Therefore, by (2) and (3), we get

$$\sum_{(i,j)\in C} \mu_{12}(i,j) \le \min\{cn_3, n_3|C|\}$$

$$= n_3 \cdot \min\{c, |C|\}.$$

This shows that μ_{12} satisfies (1). Since δ satisfies the equation (3.5.1.2) and μ_{12} satisfies the equation (3.2.2.1), we have

(4) $$cn_3 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_{12}(i,j).$$

Therefore, by (4) and (1), it follows from Remark 2.2.5 that μ_{12} also satisfies the inequality (3.2.2.3). Hence, by Theorem 3.2.2, δ is realizable. #

3.5.2 <u>Corollary</u>. Let

$$\delta = (\delta_1(1),\ldots,\delta_1(n_1);\delta_2(1),\ldots,\delta_2(n_2);\delta_3(1),\ldots,\delta_3(n_3))$$

be a 3-partite finite sequence in $\mathbb{P}$ such that

(3.5.2.1) $$\begin{cases} \delta_1(1) = \ldots = \delta_1(n_1) = c_1; \\ \delta_2(1) = \ldots = \delta_2(n_2) = c_2; \\ \delta_3(1) = \ldots = \delta_3(n_3) = c_3. \end{cases}$$

Then δ is realizable if and only if

(3.5.2.2) $$c_1n_1 = c_2n_2 = c_3n_3, \text{ and}$$

(3.5.2.3) $$c_1 \le n_2n_3.$$

Proof : To show the necessity part, assume that δ is realizable. Then, by Remark 3.2.1 and Theorem 3.4.1, δ satisfies the equation (3.2.1.1) and the inequality (3.4.1.6). By (3.5.2.1), (3.2.1.1) becomes (3.5.2.2); and (3.4.1.6) becomes (3.5.2.3).

To show the sufficiency part, assume that δ satisfies (3.5.2.2) and (3.5.2.3). Since δ also satisfies (3.5.2.1), hence (3.5.2.2) becomes (3.5.1.2). To apply the theorem, we need to show that δ satisfies (3.5.1.3), which, for this case, is

(1) $$n_1 \cdot \min \{ c_1, n_3|B|\} \geq c_2|B|$$

$$\text{for all } B \subseteq \{1, \ldots, n_2\}.$$

Let $B \subseteq \{1,\ldots,n_2\}$. Then

$$|B| \leq n_2 ;$$

(2) $$c_2|B| \leq c_2 n_2 = c_1 n_1 .$$

The last equation follows from (3.5.2.2). Observe that

(3) $$\begin{aligned} c_2 &= \frac{c_1 n_1}{n_2} \\ &\leq \frac{(n_2 n_3) n_1}{n_2} \\ &= n_1 n_3 . \end{aligned}$$

The first equation follows from (3.5.2.2); and the second ineqaulity follows from (3.5.2.3). From (3), it follows that

(4) $$c_2|B| \leq n_1 n_3 |B| .$$

Hence, by (2) and (4), we get (1) as required. #