CHAPTER I

PROPER INVERSE SEMIGROUPS

L. O' Carroll has proved that any proper inverse semigroup can be fully σ-embedded into an F-inverse semigroup.

In this chapter, we show that any semilattice of groups which is proper can be fully σ-embedded into an F-inverse semigroup which is also a semilattice of groups. Moreover, we show in general that this is true for the case of semilattices of inverse semigroups.

Let S be an proper inverse semigroup, and $\sigma(S)$ be the minimum group congruence on S. Let

$$M(S) = \{\phi \subset X \subseteq S \mid E(S).X = X \subseteq a\sigma(S) \text{ for some } a \in S\}.$$

L. O' Carroll has shown in [8] that the set M(S) under the usual set multiplication is an inverse semigroup, and for $X \in M(S)$ and $X \subseteq a\sigma(S)$, we have $X^{-1} = \{x^{-1} \mid x \in X\} \subseteq a^{-1}\sigma(S)$. Then he has given the following theorem :

1.1 <u>Theorem</u> [8]. Let S be a proper inverse semigroup. Then M(S) is an F-inverse semigroup with the semilattice of idempotents M(E(S)) and with $\{a\sigma(S) \mid a \in S\}$ as its set of maximum elements. The natural partial order on M(S) is that of inclusion, and $\psi : S \longrightarrow M(S)$ defined by $a\psi = aE(S)$ is a full σ-embedding of S into M(S). Moreover,

$$S|\sigma(S) \cong M(S)|\sigma(M(S)).$$

The next theorem shows that if a proper inverse semigroup S is a semilattice of groups, then $M(S)$ is also a semilattice of groups. The following lemmas are proved first :

1.2 Lemma. If S is a regular semigroup, then every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contains an idempotent.

Proof : Let L_a be the $\mathcal{L}$-class of S containing $a \in S$. Since S is regular, $a = axa$ for some $x \in S$. Then $Sa = Sxa$ and hence $xa \in L_a$. Since xa is an idempotent of S, L_a contains an idempotent.

Similarly, every $\mathcal{R}$-class of S contains an idempotent by using the fact : $a = axa$ in S implies $ax \in E(S)$ and $aS = axS$. #

1.3 Lemma. Let S be a regular semigroup whose idempotents are in the center of S. Then S is a semilattice Y of groups and $Y = E(S)$.

Proof : First, we show that $\mathcal{L} = \mathcal{R}$ which implies that $\mathcal{H}$ is a congruence on S. Let $(a, b) \in \mathcal{L}$. Then $Sa = Sb$. Since S is regular, there exist $x, y \in S$ such that $a = axa$ and $b = byb$ and hence $Sxa = Sa = Sb = Syb$. Then $a = syb$ and $b = txa$ for some $s, t \in S$. Since $E(S) \subseteq C(S)$, it follows that

$$a = s(yb) = (yb)s = (ybyb)s = b(yyb)s = b(yybs)$$

and

$$b = t(xa) = (xa)t = (xaxa)t = a(xxa)t = a(xxat).$$

Therefore $(a, b) \in \mathcal{R}$. This proves that $\mathcal{L} \subseteq \mathcal{R}$. Similarly, we can show that $\mathcal{R} \subseteq \mathcal{L}$, so $\mathcal{L} = \mathcal{R}$. Hence

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} = \mathcal{R} .$$

Since $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence, $\mathcal{H}$ is a congruence on S. Let $a \in S$. Then $H_a = L_a$ and by Lemma 1.2,

$L_a = L_e$ for some $e \in E(S)$, so $e \in H_a$. This proves that every $\mathcal{H}$-class contains an idempotent. Hence every $\mathcal{H}$-class is a group [[1], Theorem 2.16]. Therefore $S = \bigcup_{e \in E(S)} H_e$ is a disjoin union of groups [[1], Lemma 2.15]. For e, $f \in E(S)$ and x, $y \in S$, $x \mathcal{H} e$ and $y \mathcal{H} f$ imply $xy \mathcal{H} ef$, so $H_e H_f \subseteq H_{ef}$. This completes the proof of the lemma. #

1.4 Lemma. The set of all ideals of a semilattice is a semilattice under the usual set multiplication.

The proof of Lemma 1.4 is obvious.

1.5 Proposition. Let a proper inverse semigroup $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then M(S) is a semilattice $\bar{Y}$ of groups, where $\bar{Y}$ is the set of ideals of Y.

Proof : First, to show that $E(M(S)) \subseteq C(M(S))$, let $F \in E(M(S))$ and $X \in M(S)$. Because $E(M(S)) = M(E(S))$ by Theorem 1.1, $F \subseteq E(S)$. Since S is a semilattice of groups, $fx = xf$ for all $x \in S$, $f \in E(S)$, [Introduction page 11], so it follows that $FX = XF$. Thus $E(M(S)) \subseteq C(M(S))$.

Since M(S) is an inverse semigroup, it is regular, so by Lemma 1.3, M(S) is a semilattice $\bar{Y}$ of groups and $\bar{Y} = E(M(S)) = M(E(S))$. Next, we show that M(E(S)) is the set of all ideals of E(S). Because

$$M(E(S)) = \{\phi \subset X \subseteq E(S) \mid E(S).X = X \subseteq e\sigma(E(S)) \text{ for some } e \in E(S)\}$$

or

$$M(E(S)) = \{\phi \subset X \subseteq E(S) \mid E(S).X = X\},$$

every member of M(E(S)) is an ideal of E(S). Let I be an ideal of

$E(S)$. Then $I \neq \phi$, $E(S).I \subseteq I$ and $I \subseteq I^2 \subseteq E(S).I$, so $E(S).I = I$. Thus $I \in M(E(S))$. Therefore $\bar{Y} = M(E(S))$ is the set of ideals of $E(S)$. Since S is a semilattice Y of groups, $Y \cong E(S)$, so that $\bar{Y}$ is isomorphic to the set of ideals of Y. Hence the proposition is proved. #

The following theorem follows directly from Theorem 1.1 and Proposition 1.5 :

1.6 <u>Theorem</u>. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. If S is proper, then S can be fully σ-embedded into a semilattice $\bar{Y}$ of groups which is F-inverse, where $\bar{Y}$ is the set of all ideals of Y.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. It is easy to see that for each subsemigroup T of Y, $\bigcup_{\alpha \in T} S_\alpha$ is an inverse subsemigroup of S and

$$E(\bigcup_{\alpha \in T} S_\alpha) \quad = \quad \{e \in E(S_\alpha) \mid \alpha \in T\} \quad = \quad \bigcup_{\alpha \in T} E(S_\alpha).$$

Next, we show in more general that if S is a semilattice of inverse semigroups, then so is $M(S)$. The following lemma is given first :

1.7 <u>Lemma</u>. Let a proper inverse semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. For each ideal I of Y, let $S_I = \bigcup_{\alpha \in I} S_\alpha$,

$$A_I \quad = \quad \{X \in M(S_I) \mid X \cap S_\alpha \neq \phi \text{ for all } \alpha \in I\},$$

$$E_I \quad = \quad E(S_I),$$

and σ_I denote the minimum group congruence of S_I. If I and J be ideals of $E(S)$, then the following hold :

(1) E_I is an ideal of $E(S)$ and

$$E(S).E_I = E_I.E(S) = E_I.$$

(2) $A_I \subseteq M(S_I) \subseteq M(S)$.

(3) $A_I A_J \subseteq A_{IJ}$.

(4) A_I is an inverse subsemigroup of $M(S)$.

(5) $A_I \cap A_J = \phi$ if $I \neq J$.

Proof : First, we recall that

$M(S_I) = \{\phi \subset X \subseteq S_I \mid E_I X = X \subseteq a\sigma_I$ for some $a \in S_I\}$.

(1) Since $E(S_\alpha) \neq \phi$ for all $\alpha \in I$, $E_I = \bigcup_{\alpha \in I} E(S_\alpha) \neq \phi$. Let $e \in E(S)$ and $f \in E_I$. Then $e \in S_\gamma$, $f \in S_\alpha$ for some $\gamma \in Y$, $\alpha \in I$, so that $ef \in E(S)$ and $ef \in S_{\gamma\alpha} \subseteq S_I$ since I is an ideal of Y. Hence $ef \in E_I$. This proves E_I is an ideal of $E(S)$ and so

$$E(S).E_I = E_I.E(S) \subseteq E_I.$$

But $\quad E_I = E_I E_I \subseteq E(S).E_I.$

Hence $\quad E(S).E_I \subseteq E_I.E(S) = E_I.$

(2) Let $X \in M(S_I)$. Then $\phi \subset X \subseteq S_I \subseteq S$ and

$$E_I X = X \subseteq a\sigma_I \text{ for some } a \in S_I.$$

Hence by (1),

$$E(S).X = E(S).E_I X = E_I X = X \subseteq a\sigma_I \subseteq a\sigma(S).$$

Hence $X \in M(S)$. Therefore $M(S_I) \subseteq M(S)$. But $A_I \subseteq M(S_I)$. Then we obtain $A_I \subseteq M(S_I) \subseteq M(S)$.

(3) Let $X_1 \in A_I$ and $X_2 \in A_J$. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then $x_1 \in S_\alpha$ and $x_2 \in S_\beta$ for some $\alpha \in I$, $\beta \in J$, so that $x_1 x_2 \in S_{\alpha\beta} \subseteq S_{IJ}$. Hence $X_1 X_2 \subseteq S_{IJ}$. Because $X_1 \in A_I$ and $X_2 \in A_J$, we have

$$E_IX_1 = X_1 \subseteq a\sigma_I$$

for some $a \in S_I$, and

$$E_JX_2 = X_2 \subseteq b\sigma_J$$

for some $b \in S_J$, and

$$X_1 \cap S_\gamma \neq \phi,\ X_2 \cap S_\delta \neq \phi$$

for all $\gamma \in I$, $\delta \in J$.

Let $\gamma \in I$, $\delta \in J$. Then $X_1 \cap S_\gamma \neq \phi$ and $X_2 \cap S_\delta \neq \phi$ so that there exist $a, b \in S$ such that $a \in X_1 \cap S_\gamma$ and $b \in X_2 \cap S_\delta$. Hence $ab \in X_1X_2 \cap S_{\gamma\delta}$. This proves that $X_1X_2 \cap S_\alpha \neq \phi$ for all $\alpha \in IJ$. From

$$E_{IJ}X_1X_2 \subseteq E_IX_1X_2 = X_1X_2$$

and for $x_1 \in X_1$, $x_2 \in X_2$, say $x_1 \in S_\alpha$, $x_2 \in S_\beta$,

$$x_1x_2 = ((x_1x_2)(x_1x_2)^{-1})(x_1x_2) \in E(S_{\alpha\beta})X_1X_2 \subseteq E_{IJ}X_1X_2,$$

we have $E_{IJ}(X_1X_2) = X_1X_2$.

Let $x_1 \in X_1$ and $x_2 \in X_2$. Since $X_1 \subseteq a\sigma_I$ and $X_2 \subseteq b\sigma_J$, we have $x_1\sigma_Ia$ and $x_2\sigma_Jb$ so that $ex_1 = ea$ and $x_2f = bf$ for some $e \in E_I$, $f \in E_J$. Hence $fex_1x_2fe = feabfe$. Since $(fex_1x_2)(fe) = (feab)(fe)$ and fex_1x_2, $feab \in S_{IJ}$ and $fe \in E_{IJ}$, we have $(fex_1x_2, feab) \in \sigma_{IJ}$ and hence

$$(x_1x_2)\sigma_{IJ} = (fex_1x_2)\sigma_{IJ} = (feab)\sigma_{IJ} = (ab)\sigma_{IJ}.$$

Therefore $X_1X_2 \subseteq (ab)\sigma_{IJ}$.

Hence we prove $A_IA_J \subseteq A_{IJ}$.

(4) Since $E(S_\alpha) \subseteq E_I$ for all $\alpha \in I$, $E_I \cap S_\alpha \neq \phi$ for all $\alpha \in I$. Then $E_I \in A_I$ because $E_I \subseteq e\sigma_I$ for all $e \in E_I = E(S_I)$. Hence $A_I \neq \phi$. By (3), $A_IA_I \subseteq A_{II} = A_I$ and hence A_I is a subsemigroup of $M(S)$. Next, let $X \in A_I$. Then $X \in M(S_I)$ so that $X^{-1} = \{x^{-1} | x \in X\} \in M(S\ )$. Since $X \in A_I$, $X \cap S_\alpha \neq \phi$ for all $\alpha \in I$. Let $\alpha \in I$. Then there exists $a \in S$

such that $a \in X \cap S_\alpha$. Hence $a^{-1} \in X^{-1} \cap S_\alpha$, so $X^{-1} \cap S_\alpha \neq \phi$. This proves $X^{-1} \in A_I$. Hence A_I is an inverse subsemigroup of $M(S)$.

(5) Assume $I \neq J$. Then there exists $\alpha \in Y$ such that either $\alpha \in I \setminus J$ or $\alpha \in J \setminus I$, say $\alpha \in I \setminus J$. Since $S_J = \bigcup_{\beta \in J} S_\beta$, $S_\alpha \cap S_J = \phi$. Then for each $X \in A_J$, $X \subseteq S_J$ so that $X \cap S_\alpha = \phi$ and hence $X \notin A_I$. Thus $A_I \cap A_J = \phi$. #

1.8 <u>Proposition</u>. Let a proper inverse semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then $M(S)$ is a semilattice $\bar{Y}$ of inverse semigroups where $\bar{Y}$ is the set of all ideals of Y.

<u>Proof</u> : Assume that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice Y of inverse semigroups S_α and $\bar{Y}$ be the set of all ideals of Y. For each $I \in \bar{Y}$, let A_I be defined as above, that is,

$$A_I = \{X \in M(S_I) \mid X \cap S_\alpha \neq \phi \text{ for all } \alpha \in I\},$$

where $S_I = \bigcup_{\alpha \in I} S_\alpha$.

We claim that $M(S) = \bigcup_{I \in \bar{Y}} A_I$. By Lemma 1.7 (2), $A_I \subseteq M(S)$ for all $I \in \bar{Y}$, so that $\bigcup_{I \in \bar{Y}} A_I \subseteq M(S)$.

Conversely, let $X \in M(S)$. Let $I \subseteq Y$ be such that

$$\alpha \in I \text{ if and only if } X \cap S_\alpha \neq \phi.$$

Since $X \in M(S)$, $E(S).X = X$. We show I is an ideal of Y. Since $X \neq \phi$, $I \neq \phi$. Let $\alpha \in I$, $\beta \in Y$. Since $X \cap S_\alpha \neq \phi$, there exists $a \in S$ such that $a \in X \cap S_\alpha$. Let $e \in E(S_\beta)$. Then $ea \in E(S).X = X$ and $ea \in S_\beta S_\alpha \subseteq S_{\alpha\beta}$, so that $X \cap S_{\alpha\beta} \neq \phi$ which implies $\alpha\beta \in I$. Therefore I is an ideal of Y and $X \subseteq S_I$. For $x \in X$, $xx^{-1} \in E(S_I) = E_I$ so that $x = (xx^{-1})x \in E_I X$. Hence $X \subseteq E_I X$. Because $X \in M(S)$,

$$E(S).X = X \subseteq b\sigma(S)$$

for some $b \in S$. Thus,

$$X \subseteq E_I X \subseteq E(S).X = X$$

so that $E_I X = X$. Next, let $x \in X$. Since $x \in b\sigma(S)$, there exists $e \in E(S)$ such that $xe = be$. Pick $f \in E_I = E(S_I)$. Then $fe \in E_I$ and $bf \in S_I$ and

$$x(fe) = (xe)f = (be)f = (bf)(fe)$$

so that $x \in (bf)\sigma_I$. Thus $X \subseteq (bf)\sigma_I$. Hence $X \in A_I$.

Therefore $M(S) = \bigcup_{I \in \bar{Y}} A_I$. By Lemma 1.7 (5), this union is disjoint. By Lemma 1.7 (3) and (4), M(S) is a semilattice $\bar{Y}$ of inverse semigroups A_I. #

The next theorem follows directly from Theorem 1.1 and Proposition 1.8.

1.9 <u>Theorem</u>. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If S is proper, then S can be fully σ-embedded into an F-inverse semigroup which is a semilattice $\bar{Y}$ of inverse semigroups, where $\bar{Y}$ is the set of all ideals of Y.