CHAPTER III

WEAKLY FACTORIZABLE INVERSE SEMIGROUPS

We introduce weakly factorizable inverse semigroups which gives a generalization of factorizable inverse semigroups. Weakly factorizable inverse semigroups are studied in detial in this chapter.

An inverse semigroup S is called a <u>weakly factorizable inverse semigroup</u> if there exist an inverse subsemigroup T of S which is a union of groups and a set of idempotents E of S such that S = T.E. Then every factorizable inverse semigroup is weakly factorizable.

Every group is factorizable, so it is weakly factorizable.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α . Then

$$E(S) = \{ e_\alpha \mid \alpha \in Y \}$$

where e_α denotes the identity of G_α for each $\alpha \in Y$. Because $G_\alpha e_\alpha = G_\alpha$ for all $\alpha \in Y$, its follows that $(\bigcup_{\alpha \in Y} G_\alpha).E(S) = S$. Hence S is weakly factorizable.

Because every semilattice S is a semilattice of groups, S is weakly factorizable.

Hence we have the following :

3.1 <u>Proposition</u>. The following inverse semigroups are weakly factorizable :

(i) Factorizable inverse semigroups .

(ii) Semilattices of groups.

Proposition 3.1 shows that weakly factorizable inverse semigroups give a generalization of factorizable inverse semigroups and semilattices of groups.

Any semilattice without identity is weakly factorizable but not factorizable.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α . Then Y has an identity if and only if S has an identity, so if Y has no identity, then S is weakly factorizable but not factorizable.

The next example shows that there exists a factorizable inverse semigroup but not a semilattice of groups. The following lemma is required first :

3.2 Lemma. [Introduction, page 8]. If S is a semilattice of groups, then $E(S) \subseteq C(S)$, where C(S) denotes the center of S.

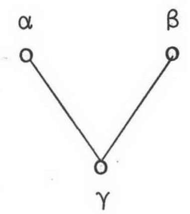
Example: Let $X = \{1,2\}$. Then the symmetric inverse semigroup on X, I_X, is factorizable [Corollary 1.4]. Next, we will show that $E(I_X) \not\subseteq C(I_X)$.

Let δ be the identity mapping on the set $\{1\}$. Then $\delta \in E(I_X)$. Now, let $\alpha \in I_X$ such that $\Delta\alpha = \{1\}$ and $\nabla\alpha = \{2\}$. Then $\alpha \notin E(I_X)$ and $\Delta(\delta\alpha) = \{1\}$ and $\Delta(\alpha\delta) = \phi$. Therefore $\delta\alpha \neq \alpha\delta$. Hence $\delta \notin C(I_X)$ which implies $E(I_X) \not\subseteq C(I_X)$. By Lemma 3.2, I_X is not a semilattice of groups. #

Because every factorizable inverse semigroup is weakly factorizable, the above example also shows that there exists a weakly factorizable inverse semigroup which is not a semilattice of groups.

Now, we still have a question whether a weakly factorizable inverse semigroup has to be either a factorizable inverse semigroup or a semilattice of groups. The following example shows that there is a weakly factorizable inverse semigroup which is neither a factorizable inverse semigroup nor a semilattice of groups.

<u>Example</u>: Let $Y = \{\alpha, \beta, \gamma\}$ be a semilattice with Hasse diagram :



Let A, B be finite disjoint sets such that $|A|>1$ and $|B|>1$, where $|X|$ denotes the cardinality of the set X. Since A and B are finite sets, I_A and I_B are both factorizable as $I_A = G_A.E(I_A)$ and $I_B = G_B.E(I_B)$ where G_A and G_B denote the permutation groups on A and B; respectively.

Let $S_\alpha = I_A$,

$S_\beta = I_B$,

and $S_\gamma = \{0\}$, a trivial group where 0 is a new symbol, $0 \notin I_A$ and $0 \notin I_B$. Let us consider the empty transformations of I_A and of I_B be distinct.

Set $S \doteq S_\alpha \cup S_\beta \cup S_\gamma$ and define the operation * on S as follows :

$$\delta * \delta' = \begin{cases} \delta\,\delta' & \text{if either } \delta, \delta' \in S_\alpha \text{ or } \delta, \delta' \in S_\beta. \\ 0 & \text{otherwise.} \end{cases}$$

Then (S,*) is a semilattice Y of inverse semigroups S_α, S_β and S_γ, so (S,*) is an inverse semigroup [Introduction, page 8]. It is clear that S has no identity, so it is not factorizable. Because $|A|>1$, there exist $a, a' \in A$ such that $a \neq a'$. Let $\delta_1, \delta_2 \in I_A \subseteq S$ such that

$$\Delta\delta_1 = \{a\} = \nabla\delta_1$$

and

$$\Delta\delta_2 = \{a\}, \quad \nabla\delta_2 = \{a'\}.$$

Then $\delta_1 \in E(I_A) \subseteq E(S)$ and $\delta_2 \notin E(I_A)$, so $\delta_2 \notin E(S)$. Thus, $\delta_1\delta_2 \neq \delta_2\delta_1$ because $\Delta\delta_1\delta_2 = \{a\}$ and $\Delta\delta_2\delta_1 = \phi$. Hence $\delta_1 \in E(S)$ and $\delta_1 \notin C(S)$, so $E(S) \nsubseteq C(S)$. Therefore S is not a semilattice of groups [Lemma 3.2].

Let $G_\alpha = G_A$, the permutation group on the set A, $G_\beta = G_B$ and $G_\gamma = S_\gamma = \{0\}$. Then $T = G_\alpha \cup G_\beta \cup G_\gamma$ is a semilattice Y of groups, so it is an inverse subsemigroup of S. Because $S = S_\alpha \cup S_\beta \cup S_\gamma$,

$$\begin{aligned} E(S) &= E(S_\alpha) \cup E(S_\beta) \cup E(S_\gamma) \\ &= E(I_A) \cup E(I_B) \cup \{0\}. \end{aligned}$$

But

$$\begin{aligned} S_\alpha &= I_A = G_A.E(I_A) = G_\alpha.E(S_\alpha), \\ S_\beta &= I_B = G_B.E(I_B) = G_\beta.E(S_\beta), \end{aligned}$$

and

$$S_\gamma = \{0\} = G_\gamma.$$

Then

$$\begin{aligned} S &= S_\alpha \cup S_\beta \cup S_\gamma \\ &= G_\alpha.E(S_\alpha) \cup G_\beta.E(S_\beta) \cup G_\gamma.\{0\} \\ &\subseteq (G_\alpha \cup G_\beta \cup G_\gamma).(E(S_\alpha) \cup E(S_\beta) \cup \{0\}) \subseteq S. \end{aligned}$$

Hence S = T.E(S), so S is a weakly factorizable inverse semigroup. #

If S is a weakly factorizable inverse semigroup as T.E, then T is a semilattice of groups. To show this, we need the following lemmas :

3.3 Lemma. Let S be an inverse semigroup. If S is a union of groups, then S is a disjoint union of groups.

Proof: Let $S = \bigcup_{i\in\Lambda} G_i$ be a union of groups G_i. In any group G, the identity of G is the only idempotent of G. Then $E(S) = \{e_i \mid i \in \Lambda\}$ where e_i is the identity of the group G_i for all $i \in \Lambda$. Let K be an index set such that

$$\{ e_k \mid k \in K \} = \{ e_i \mid i \in \Lambda \}$$

and $e_k \neq e_{k'}$ if $k \neq k'$. Then $E(S) = \{e_k \mid k \in K\}$. Claim that $S = \bigcup_{k\in K} H_{e_k}$ where H_{e_k} denotes the $\mathcal{H}$ - class of S containing e_k. Let $x \in S$. Then $x \in G_i$ for some $i \in \Lambda$. Because $e_i \in E(S)$, there exists $k \in K$ such that $e_k = e_i$. Since H_{e_k} is the greatest subgroup of S having e_k as its identity [Chapter I, page 11], $G_i \subseteq H_{e_k}$ and so $x \in \bigcup_{k\in K} H_{e_k}$. Hence $S = \bigcup_{k\in K} H_{e_k}$.

Since each $\mathcal{H}$ - class of a semigroup contains at most one idempotent [[1] , Lemma 2.15], it follows that $H_{e_k} \cap H_{e_{k'}} = \phi$ if $k \neq k'$. Hence $S = \bigcup_{k\in K} H_{e_k}$ is a disjoint union of groups. #

3.4 Lemma. Let S be an inverse semigroup and $S = \bigcup_{k\in K} G_k$ be a disjoint union of groups. Then S is a semilattice of groups.

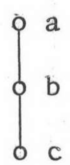
<u>Proof</u>: Let e_k denote the identity of the group G_k for all $k \in K$. Then $E(S) = \{e_k \mid k \in K\}$. Because S is an inverse semigroup, E(S) is a semilattice. Since for each $k \in K$, H_{e_k} is a maximum subgroup of S having e_k as its identity, $H_{e_k} = G_k$ for all $k \in K$. Hence $S = \bigcup_{k \in K} H_{e_k}$. Since S is an inverse semigroup, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent [[1], Corollary 2.19]. But each $\mathcal{L}$-class and each $\mathcal{R}$-class of S is a union of $\mathcal{H}$-class of S. Then for each $k \in K$, $L_{e_k} = H_{e_k} = R_{e_k}$. But $\mathcal{L}$ is right compatible and $\mathcal{R}$ is left compatible. Then $\mathcal{H} = \mathcal{L} = \mathcal{R}$ is a congruence.

Next, let $x \in H_{e_k}$ and $y \in H_{e_{k'}}$. Then $x \mathcal{H} e_k$ and $y \mathcal{H} e_{k'}$, so $xy \mathcal{H} e_k e_{k'}$. Hence $xy \in H_{e_k e_{k'}}$. This prove that $H_{e_k} \cdot H_{e_{k'}} \subseteq H_{e_k e_{k'}}$ for all $k, k' \in K$. Therefore, S is a semilattice E(S) of groups H_{e_k}. #

3.5 <u>Proposition</u>. Let S be a weakly factorizable inverse semigroup as T.E. Then T is a semilattice of groups.

We give a remark that if S is a weakly factorizable inverse semigroup as T.E, then $S = T.E \subseteq T.E(S) \subseteq S$ and so $S = T.E(S)$.

However, if S is a weakly factorizable inverse semigroup as T.E, then E is not necessarily to be E(S). For example, let S be a semilattice with Hasse diagram :



Let T = S. Then T is a semilattice of groups. Then E(S) = S, and S = T.{a} because a is the identity of S.

The following theorem shows various properties of weakly factorizable inverse semigroups :

3.6 Theorem. Let S be a weakly factorizable inverse semigroup as T.E. Then the following hold :

(i) S = E.T.

(ii) If e is the identity of T, then e is the identity of S.

(iii) For any $e \in E(T)$, $x \in S$, $xe = ex$; that is, $E(T) \subseteq C(S)$.

Proof: Let $T = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then $S = (\bigcup_{\alpha \in Y} G_\alpha) . E$.

(i) Let $x \in S = T.E$. Then $x^{-1} \in S$, so there exist $g \in G_\alpha$ for some $\alpha \in Y$ and $e \in E$ such that $x^{-1} = ge$. Therefore $x = eg^{-1} \in E.T$. Hence S = E.T.

(ii) Assume e is the identity of T. Let $x \in S$. Then there exist $k \in T$, $f \in E$ such that x = kf. Therefore

$$ex = e(kf) = (ek)f = kf = x,$$

and

$$xe = (kf)e = k(ef) = (ke)f = kf = x.$$

Hence e is the identity of S.

(iii) Let $e \in E(T)$ and $x \in S$. Then x = kf for some $k \in T$, $f \in E(S)$. Because T is a semilattice of groups, by Lemma 3.2, $E(T) \subseteq C(T)$. Then

$$ex = e(kf) = (ek)f = (ke)f = k(ef) = k(fe) = (kf)e = xe.$$

Thus $e \in C(S)$. Hence $E(T) \subseteq C(S)$, as required. #

Next, we show that if S is a weakly factorizable inverse semigroup as T.E, then the maximum group homomorphic image of S is a homomorphic image of the maximum group homomorphic image of T. The following lemma is required first :

3.7 Lemma. Let S be a weakly factorizable inverse semigroup as T.E. Then every σ - class of S intersects T.

Proof: Let $a\sigma$ be a σ - class of S. Then there exist $t \in T$ and $e \in E$ such that $a = te$ and so $ae = tee = te$. Hence $a\sigma = t\sigma$, so $t \in a\sigma$.#

3.8 Proposition. Let S be a weakly factorizable inverse semigroup as T.E. Then $S/\sigma(S)$ is a homomorphic image of $T/\sigma(T)$.

Proof: Let $\psi : T/\sigma(T) \rightarrow S/\sigma(S)$ be a map defined by

$$(t\sigma(T))\psi = t\sigma(S) \qquad (t \in T).$$

ψ is clearly well-defined because $E(T) \subseteq E(S)$, and it is easily seen that ψ is a homomorphism. To show ψ is onto, let $a\sigma(S) \in S/\sigma(S)$. By Lemma 3.7, there exists $t \in T$ such that $t \in a\sigma(S)$. Then $t\sigma(S) = a\sigma(S)$, so

$$(t\sigma(T))\psi = t\sigma(S) = a\sigma(S) . \quad \#$$

The homomorphism ψ in the proof of Proposition 3.8 is one-to-one if S is proper.

3.9 Theorem. Let S be a weakly factorizable inverse semigroup as T.E.

If S is proper, then $S/_{\sigma(S)}$ is isomorphic to $T/_{\sigma(T)}$ and hence S and T have the same maximum group homomorphic image.

<u>Proof</u>: Let $\psi : T/_{\sigma(T)} \rightarrow S/_{\sigma(S)}$ be a map defined by

$$(t\sigma(T))\psi = t\sigma(S) \qquad (t \in T).$$

From the proof of Proposition 3.8, ψ is an onto homomorphism.

To show ψ is one-to-one, let $t_1, t_2 \in T$ such that $t_1\sigma(S) = t_2\sigma(S)$. Then $t_1e = t_2e$ for some $e \in E(S)$. Then

$$t_2^{-1}t_1e = (t_2^{-1}t_2)e \quad .$$

But $t_2^{-1}t_2 \in E(S)$, so

$$\begin{aligned}(t_2^{-1}t_1)(t_2^{-1}t_2)e &= (t_2^{-1}t_1)e(t_2^{-1}t_2) \\ &= (t_2^{-1}t_2)e(t_2^{-1}t_2) \\ &= (t_2^{-1}t_2)e \quad .\end{aligned}$$

Since S is proper and $(t_2^{-1}t_2)e \in E(S)$, $t_2^{-1}t_1 \in E(S)$. But $t_2^{-1}t_1 \in T$, so $t_2^{-1}t_1 \in E(T)$. Then $t_2^{-1}t_1 = f$ for some $f \in E(T)$. Hence $t_2t_2^{-1}t_1 = t_2f$, so

$$\begin{aligned}t_1\sigma(T) &= (t_2t_2^{-1})\sigma(T)t_1\sigma(T) \\ &= (t_2t_2^{-1}t_1)\sigma(T) \\ &= (t_2f)\sigma(T) \\ &= (t_2\sigma(T))f\sigma(T) \\ &= t_2\sigma(T)\end{aligned}$$

since $t_2t_2^{-1}\sigma(T) = f\sigma(T)$ is the identity of the group $T/_{\sigma(T)}$.

Hence ψ is an onto isomorphism, so $S/_{\sigma(S)} \cong T/_{\sigma(T)}$ as required.#

The Green's relation $\mathcal{H}$ on a weakly factorizable inverse semigroup is studied, and the following proposition is obtained :

3.10 Proposition. Let S be a weakly factorizable inverse semigroup as T.E and let $T = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then for each $\alpha \in Y$, G_α is an $\mathcal{H}$ - class of S. Moreover; for $e \in E(S)$, if $H_e \cap T \neq \phi$, then $H_e = G_\alpha$ for some $\alpha \in Y$.

Proof: For each $\alpha \in Y$, let e_α denote the identity of G_α. Let $\alpha \in Y$. Since H_{e_α} is the maximum subgroup of S having e_α as its identity, $G_\alpha \subseteq H_{e_\alpha}$. Next, let $x \in H_{e_\alpha}$. Then $x^{-1}x = e_\alpha$ and $x = gf$ for some $\beta \in Y$ such that $g \in G_\beta$ and for some $f \in E$. Therefore

$$e_\alpha = x^{-1}x = (fg^{-1})(gf) = f(g^{-1}g)f = fe_\beta f = e_\beta f,$$

and so

$$x = (ge_\beta)f = g(e_\beta f) = ge_\alpha \in G_\beta G_\alpha \subseteq G_{\alpha\beta}.$$

Thus $x^{-1} \in G_{\alpha\beta}$ and so $e_\alpha = x^{-1}x \in G_{\alpha\beta}$. But $e_\alpha \in G_\alpha$. Hence $\alpha = \alpha\beta$ which implies $x \in G_\alpha$. Then $H_{e_\alpha} \subseteq G_\alpha$. Therefore $G_\alpha = H_{e_\alpha}$.

Next, let $e \in E(S)$ such that $H_e \cap T \neq \phi$. Then there exist $\alpha \in Y$ and $g \in G_\alpha$ such that $g \in H_e \cap T$. Claim that $H_e = G_\alpha$. Since $g \in H_e$, $g^{-1}g = e$. But $g^{-1}g = e_\alpha$, so $e = e_\alpha$. Hence, from the first part of the proof, we have $H_e = H_{e_\alpha} = G_\alpha$. #

Let S be a weakly factorizable inverse semigroup as T.E and $T = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice of groups G_α. Let A be an ideal of S. If $\alpha \in Y$ and $G_\alpha \cap A \neq \phi$, then as the proof of Lemma 2.2, $G_\alpha \subseteq A$. It is possible that $A \cap G_\alpha = \phi$ for all $\alpha \in Y$. For example, let $X = \{a,b\}$. Then the symmetric inverse semigroup on X is

$$I_X = \{ 0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}$$

where 0, 1, α_i (i = 1, 2, 3, 4, 5) are defined the same as in the example of Chapter II. That is, the table of multiplication is as follows :

o	1	α_5	0	α_1	α_2	α_3	α_4
1	1	α_5	0	α_1	α_2	α_3	α_4
α_5	α_5	1	0	α_4	α_3	α_2	α_1
0	0	0	0	0	0	0	0
α_1	α_1	α_3	0	α_1	0	α_3	0
α_2	α_2	α_4	0	0	α_2	0	α_4
α_3	α_3	α_1	0	0	α_3	0	α_1
α_4	α_4	α_2	0	α_4	0	α_2	0

$\Delta\alpha_1 = \{a\} = \nabla\alpha_1$,

$\Delta\alpha_2 = \{b\} = \nabla\alpha_2$,

$\Delta\alpha_3 = \{a\}, \nabla\alpha_3 = \{b\}$,

$\Delta\alpha_4 = \{b\}, \nabla\alpha_4 = \{a\}$,

$\Delta\alpha_5 = \{a,b\} = \nabla\alpha_5$,

such that $a\alpha_5 = b$,

$b\alpha_5 = a$.

Then I_X is factorizable as $G_X.E(I_X)$, $G_X = \{1,\alpha_5\}$, $E(I_X) = \{0, 1, \alpha_1, \alpha_2\}$, so I_X is weakly factorizable. From its table of multiplication, the set $K = \{0,\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is an ideal of I_X and $K \cap G_X = \phi$.

The above example also shows that an ideal of a weakly factorizable inverse semigroup is not necessarily weakly factorizable. To show the ideal K of I_X is not weakly factorizable, first we find all the inverse subsemigroups of K. It is easy to check that all the inverse subsemigroups of K are

$K_1 = \{0\}$, $K_2 = \{\alpha_1\}$, $K_3 = \{\alpha_2\}$, $K_4 = \{0, \alpha_1\}$

$K_5 = \{0, \alpha_2\}$, $K_6 = \{0, \alpha_1, \alpha_2\}$ and K .

Because K_1, K_2, K_3, K_4, K_5 and K_6 are semilattices, they are semilattices of groups. Since $\alpha_1 \in E(K)$ and $\alpha_1\alpha_3 = \alpha_3$ and $\alpha_3\alpha_1 = 0$, $\alpha_1\alpha_3 \neq \alpha_3\alpha_1$ so $\alpha_1 \notin C(K)$, the center of K. This shows that $E(K) \not\subseteq C(K)$. Hence K is not a semilattice of groups [Lemma 3.2]. Then all of the inverse subsemigroups of K which are semilattices of groups are K_1, K_2, K_3, K_4, K_5, K_6. Next, we show that $K_i.E(K) \neq K$ for all $i \in \{1,2,3,4,5,6\}$. Since $E(K) = \{0, \alpha_1, \alpha_2\}$ and $K_i \subseteq E(K)$ for all $i \in \{1,2,3,4,5,6\}$, $K_i.E(K) \subseteq E(K)$ for all $i \in \{1,2,3,4,5,6\}$. But $E(K) \neq K$. Hence K is not weakly factorizable.

We end this chapter by introducing a property of ideal A of a weakly factorizable inverse semigroup to let A be also weakly factorizable.

3.11 <u>Proposition</u>. Let S be a weakly factorizable inverse semigroup as T.E, A be an ideal of S and A has its identity. Then if T contains the identity of A, then A is weakly factorizable.

<u>Proof</u>: Let 1_A be the identity of the ideal A, and let $T = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Let

$$Y_A = \{\alpha \in Y \mid G_\alpha \cap A \neq \phi\}. \text{ Then } Y_A = \{\alpha \in Y \mid G_\alpha \subseteq A\}.$$

Since $1_A \in A \cap T$, $Y_A \neq \phi$. Claim that Y_A is an ideal of Y, let $\alpha \in Y_A$, $\beta \in Y$. Then $G_\alpha \subseteq A$ and so $G_\alpha G_\beta \subseteq A$. But $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. Then $G_{\alpha\beta} \cap A \neq \phi$, and hence $\alpha\beta \in Y_A$. Thus Y_A is an ideal of Y and then it is also a semilattice.

Set $T_A = \bigcup_{\alpha \in Y_A} G_\alpha$. Then it follows that $T_A \subseteq A$ and it is a semilattice Y_A of groups G_α. By assumption, 1_A is also the identity of T_A so $1_A \in E(T_A)$. But $E(T_A) = \{ e_\alpha | \alpha \in Y_A \}$. Then $1_A = e_\lambda$ for some $\lambda \in Y_A$.

Next, we show that $A = T_A.E(A)$. Let $a \in A$. Since $A \subseteq S$ and $S = T.E = (\bigcup_{\alpha \in Y} G_\alpha) . E$, $a = ge$ for some $\beta \in Y$, $g \in G_\beta$ and for some $e \in E$. Therefore

$$a = 1_A a 1_A = 1_A g e 1_A = (e_\lambda g)(e 1_A).$$

Because Y_A is an ideal and $\lambda \in Y_A$, $\lambda\beta \in Y_A$. Since $e_\lambda g \in G_{\lambda\beta}$, it follows that $e_\lambda g \in T_A$. Since $e1_A \in E(S)$ and $e1_A \in A$, $e1_A \in E(A)$. It then follows that $a \in T_A.E(A)$. Therefore $A \subseteq T_A.E(A)$. But $T_A \subseteq A$ and $E(A) \subseteq A$, so $T_A.E(A) = A$, completing the proof of the proposition. #