INTRODUCTION

Let S be a semigroup. An element a of S is called an idempotent of S if $a^2 = a$. For a semigroup S, we denote E(S) to be the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a semilattice if for all $a,b \in S$, $a^2 = a$ and $ab = ba$. An element z of a semigroup S is a zero of S if $xz = zx = z$ for all $x \in S$. An element e of a semigroup S is an identity of S if $ex = xe = x$ for all $x \in S$. A zero and an identity of a semigroup are unique.

Let S be a semigroup, and let 1 be a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to one by defining $11 = 1$ and $1a = a1 = a$ for every $a \in S$. Throughout this thesis we will adhere to the following notation :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{otherwise.} \end{cases}$$

Let S be a semigroup. A subgroup of S is a subsemigroup of S which is also a group under the same operation.

Let S be a semigroup with identity 1. An element a of S is called a unit of S if there exists $a' \in S$ such that $aa' = a'a = 1$. Let G be the set of all units of S. Then

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}$$

and G is the greatest subgroup of S which has 1 as its identity.

Let S be a semigroup. An element a of S is <u>regular</u> if a = axa for some $x \in S$, and S is called a <u>regular semigroup</u> if every element of S is regular.

In any semigroup S, if $a, x \in S$ such that $a = axa$, then ax and xa are idempotents of S. Hence if S is a regular semigroup, then $E(S) \neq \phi$.

Let a and x be elements of a semigroup S such that $a = axa$ then

(i) $aS = aS^1$ and $S^1a = Sa$

and (ii) $aS = axS$ and $Sxa = Sa$.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if $a = axa$, $x = xax$. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [[1], Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. For any elements a,b of an inverse semigroup S and $e \in E(S)$, we have

$$(a^{-1})^{-1} = a, \quad (ab)^{-1} = b^{-1}a^{-1} \quad \text{and } e^{-1} = e$$

[[1] , Lemma 1.18]. Let T be a subset of a semigroup S. The <u>centralizer</u> of T in S is

$$C(T) = \{ a \in S \mid at = ta \text{ for all } t \in T \} .$$

The centralizer of S in S is the <u>center</u> of S. It then follows that if S is an inverse semigroup, then $E(S) \subseteq C(E(S))$.

Let P be a nonempty set and $\leq$ be a relation on P. If the relation $\leq$ is reflexive, antisymmetric and transitive, then $\leq$ is called a <u>partial order</u> on P, and (P, $\leq$), or P, is called a <u>partially ordered set</u>.

Let S be an inverse semigroup, $a, b \in S$. Then the following are equivalent :

$$\text{(i)} \quad aa^{-1} = ab^{-1},$$

$$\text{(ii)} \quad aa^{-1} = ba^{-1},$$

$$\text{(iii)} \quad a^{-1}a = a^{-1}b,$$

$$\text{(iv)} \quad a^{-1}a = b^{-1}a,$$

$$\text{(v)} \quad ab^{-1}a = a,$$

and

$$\text{(vi)} \quad a^{-1}ba^{-1} = a^{-1}$$

[[2], Lemma 7.1].

The relation $\leq$ defined on an inverse semigroup S by

$$a \leq b \text{ if and only if } aa^{-1} = ab^{-1}$$

is a partial order on S [[2] , Lemma 7.2], and this partial order is called the <u>natural partial order</u> on the inverse semigroup S. In this thesis, whenever we mention about a partial order on an inverse semigroup, we always mean the natural partial order.

In any inverse semigroup S, we have the following :

(i) $a \leq b$ if and only if $a = be$ for some $e \in E(S)$.

(ii) $a \leq b$ if and only if $a = fb$ for some $f \in E(S)$.

We note that the restriction of the natural partial order $\leq$ on an inverse semigroup S to E(S) is as follows : For $e, f \in E(S)$,

$$e \leq f \iff e = ef \ (= fe).$$

Then if S is a semilattice, $a \leq b$ in S if and only if $a = ab$ (= ba).

A reflexive, symmetric and transitive relation on a nonempty set X is an *equivalence relation* on X.

Let S be a semigroup. A relation ρ on S is *left compatible* if for all $a,b,c \in S$, $a \rho b$ implys $ca \rho cb$. *Right compatibility is defined* dually. By a *congruence* on S we mean an equivalence relation on S which is both right and left compatible. Then the equivalence relation ρ on S is a congruence on S if and only if for $a,b,c \in S$, $a \rho b$ imply $ca \rho cb$ and $ac \rho bc$.

If ρ is a congruence on a semigroup S, then the set

$$S/\rho \quad = \quad \{a\rho \ / \ a \in S\}$$

with the operation defined by

$$(a\rho)(b\rho) \quad = \quad (ab)\rho \qquad\qquad (a,b \in S)$$

is a semigroup, and is called the *quotient semigroup relative to the congruence* ρ.

Let ρ be a congruence on a semigroup S. Then the mapping $\psi : S \to S/\rho$ defined by

$$a\psi \quad = \quad a\rho \qquad\qquad (a \in S)$$

is an onto homomorphism and ψ will be denoted by $\rho^{\natural}$, and call it the *natural homomorphism of* S *onto* S/ρ.

Conversely, if $\psi : S \to T$ is a homomorphism from a semigroup S into a semigroup T, then the relation ρ on S defined by

$$a \rho b \iff a\psi = b\psi \qquad (a,b \in S)$$

is a congruence on S and $S/\rho \cong S\psi$, and ρ is called the *congruence on* S *induced by* ψ.

Let ρ be a congruence on an inverse semigroup S. Then

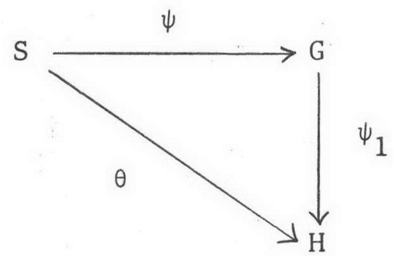
$$E(S/\rho) = \{ e\rho \mid e \in E(S) \}$$

[[2] , Theorem 7.36], S/ρ is an inverse semigroup and for $a \in S$, $(a\rho)^{-1} = a^{-1}\rho$ [[2] , Lemma 7.34].

Hence for all $a,b \in S$,

$$a \, \rho \, b \iff a^{-1} \, \rho \, b^{-1} .$$

A group G is called the <u>maximum group homomorphic image of a semigroup</u> S if there exists a homomorphism ψ from S onto G such that the following hold : For any group H and for any homomorphism θ from S onto H, there exists a unique group homomorphism ψ_1 from G onto H such that the diagram :



commutes, that is, $\psi\psi_1 = \theta$.

A congruence ρ on a semigroup S is called a <u>group congruence</u> if S/ρ is a group. If ρ is a group congruence on a semigroup S, then E(S) is contained in the ρ - class which represents the identity of the group S/ρ and hence $E(S) \subseteq e\rho$ for all $e \in E(S)$.

Let σ be a group congruence on a semigroup S such that for any group congruence ρ on S, $\sigma \subseteq \rho$. Then σ is called the <u>minimum group congruence on</u> S.

If σ is the minimum group congruence on a semigroup S, then S/σ

is the maximum group homomorphic image of S.

Munn [5] has shown that any inverse semigroup S has a minimum group congruence σ, and

$$\sigma = \{(a,b) \in S \times S \mid ae = be \text{ for some } e \in E(S)\} ;$$

equivalently,

$$\sigma = \{(a,b) \in S \times S \mid ea = eb \text{ for some } e \in E(S)\} .$$

Hence any inverse semigroup S has the maximum group homomorphic image, that is, S/σ. Throughout this thesis $\sigma(S)$, or σ if there is no danger of ambiguity, will be denoted for the minimum group congruence of the inverse semigroup S.

Let S be a semigroup. A nonempty subset A of S is a <u>left ideal</u> of S if $sa \in A$ for all $s \in S$, $a \in A$. A <u>right ideal</u> of S is defined dually. A nonempty subset of S is an <u>ideal</u> (or two-sided ideal) of S if it is both a left ideal and a right ideal of S. An arbitary intersection of left ideals, of right ideals and of ideals of a semigroup S is left ideal, a right ideal and an ideal of S; respectively.

An ideal of an inverse semigroup S is an inverse subsemigroup of S.

Let A be an ideal of a semigroup S. Then the relation ρ defined on S by $a\rho b$ if and only if either $a,b \in A$ or $a = b$, is a congruence and it is called the <u>Rees congruence induced by</u> the ideal A and S/ρ is the <u>Rees quotient semigroup induced by</u> the ideal A and it is denoted by S/A. Hence

$$a\rho = \begin{cases} \{a\} & \text{if } a \notin A, \\ A & \text{if } a \in A. \end{cases}$$

Let A be an ideal of a semigroup S. Then S/A is a homomorphic image of S. Since a homomorphic image of an inverse semigroup is an inverse semigroup, S/A is an inverse semigroup if S is an inverse semigroup.

Let A be a nonempty subset of a semigroup S. _The left ideal of S generated by_ A is the intersection of all left ideals of S containing A. The _right ideal of_ S _generated by_ A is defined dually. _The ideal of_ S _generated by_ A is the intersection of all ideals of S containing A. If A contains only one element, say a, the left ideal of S generated by A is called the _principal left ideal of_ S _generated by_ a, the _principal right ideal of_ S _generated by_ a and _the principal ideal of_ S _generated by_ a are defined similarly.

Let a be an element of a semigroup S. Then we have S^1a, aS^1 and S^1aS^1 are the principal left ideal of S generated by a, the principal right ideal of S generated by a and the principal ideal of S generated by a; respectively.

If S is a regular semigroup, then

$$S^1a = Sa, \quad aS^1 = aS \quad \text{and} \quad S^1aS^1 = SaS$$

for all $a \in S$. If S is a semilattice, then an ideal I of S is principal if and only if $I = aS = Sa = SaS$ for some $a \in S$.

An inverse semigroup S is _proper_ if for all $a \in S, e \in E(S)$, $ae = e$ imply $a \in E(S)$; equivalently, if for all $a \in S$, $e \in E(S)$, $ea = e$ imply $a \in E(S)$. An inverse subsemigroup of a proper inverse semigroup is clearly proper. Every group is proper, also every semilattice is proper.

Let S be an inverse semigroup. S is called an <u>F - inverse semigroup</u> if every σ - class of S has a maximum element (under the natural partial order on S).

McFadden [4] has shown that any F - inverse semigroup is proper and has an identity. But the converse is not generally true.

Let Y be a semilattice and a semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a disjoint union of subsemigroups S_α of S. S is called a <u>semilattice Y of semigroup</u> S_α if $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$; or equivalently, for all $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ imply $ab \in S_{\alpha\beta}$.

A semilattice of inverse semigroups is an inverse semigroup [[2], Theorem 7.52]. Then a semilattice of groups is also an inverse semigroup.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For each $\alpha \in Y$, let e_α denote the identity of the group G_α. Then

$$E(S) = \{ e_\alpha \mid \alpha \in Y \} .$$

Because S is an inverse semigroup, $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha}$ for all $\alpha, \beta \in Y$. Hence $E(S) \cong Y$ by the isomorphism $e_\alpha \longmapsto \alpha$ $(\alpha \in Y)$, and so S has an identity if and only if Y has an identity.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then E(S) is contained in the center of S [[1], Lemma 4.8]. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, define the mapping $\psi_{\alpha,\beta} : G_\alpha \to G_\beta$ by

$$a\psi_{\alpha,\beta} = ae_\beta \qquad (a \in G_\alpha).$$

Then the mappings $\psi_{\alpha,\beta}$ $(\alpha \geq \beta)$ are homomorphisms and for every $\alpha \in Y$,

$\psi_{\alpha,\alpha}$ is the identity mapping on G_α, Furthermore,

$$\psi_{\alpha,\beta}\,\psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$$

if $\alpha \geq \beta \geq \gamma$, and if $\alpha, \beta \in Y$, $a \in G_\alpha$, $b \in G_\beta$, then

$$ab = (a\psi_{\alpha,\alpha\beta})\,(b\psi_{\beta,\alpha\beta})$$

[[1] , Theorem 4.11]. For convenience, we will call the homomorphisms $\psi_{\alpha,\beta}$ ($\alpha \geq \beta$), defined as above, the corresponding homomorphisms of the semilattice Y of groups G_α.

A semigroup S is said to be factorizable if there exist a subgroup G of S and a set E of idempotents of S such that S = G.E.

We introduce the work of Chen and Hsieh [6] on factorizable inverse semigroups in the first chapter. Different studies of properties of factorizable inverse semigroup relating to minimum group congruences, maximum group homomorphic images, the property of being proper and the property of being F - inverse are also obtained in this chapter. It is shown that the maximum group homomorphic image of a proper factorizable inverse semigroup S is the group of units of S. Any proper factorizable inverse semigroup is F - inverse.

If is shown in the second chapter that an ideal A with its identity of a factorizable inverse semigroup is factorizable. Any Rees quotient semigroup of a factorizable inverse semigroup is factorizable.

In the last chapter, we introduce a generalization of a factorizable inverse semigroup which is called a weakly factorizable inverse semigroup. Every semilattice of groups is a weakly factorizable inverse semigroup. An example of a weakly factorizable inverse semigroup which is neither a semilattice of groups nor a factorizable inverse semigroup is given. It is proved that if S is a weakly factorizable inverse semigroup as T.E and S is proper, then S and T have the same maximum group homomorphic image. In general, an ideal of a weakly factorizable inverse semigroup need not be weakly factorizable. But, if an ideal A with its identity of a weakly factorizable inverse semigroup which factors as T.E and T contains the identity of A, then A is weakly factorizable.