CHAPTER II

DIFFERENTIAL RINGS, FIELDS AND IDEALS

The materials of this chapter are drawn from reference [1] .

In this chapter we shall present the definitions of differential rings, fields and ideals and some of their basic properties which are somewhat different from rings, fields and ideals in abstract algebra since it is a three-operational system; these operations are called addition, multiplication and derivation. The analysis of differential algebra will follow the same pattern already laid out for abstract algebra.

Definition 2-1 A differential ring is a commutative algebraic ring together with an operation D (called derivation) such that

i) Da is defined and is an element of the ring for every element a in the ring,

ii) $D(a + b) = Da + Db$ for all a,b in the ring,

iii) $D(ab) = Da.b + a.Db$ for all a,b in the ring.

Definition 2-2 A differential field is an algebraic field together with an operation D satisfying properties i), ii) and iii, above.

Definition 2-3 If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two differential rings (fields) with operation D, $\mathcal{R}_1$ is said to be a differential subring (subfield) of $\mathcal{R}_2$, provided $\mathcal{R}_1$ is a subring (subfield) of $\mathcal{R}_2$ and the D operator for $\mathcal{R}_2$ coincides with the D operator for $\mathcal{R}_1$ whenever $a \in \mathcal{R}_1$.

Before proceeding furthur, let us look at several examples.

Example 2-1 Let R be a commutative ring with identity 1, R[x] is a differential ring with operator D defined by Dx = 1 and Da = 0 for all a in R.

Example 2-2 Let R be a ring as above R[x, e^x] is a differential ring with the operator D defined by Dx = 1, $De^x = e^x$ and Da = 0 for all a in R. We then have R[x] is a differential subring of R [x, e^x] .

Example 2-3 Let R be the same ring as in Example 1

$$R[x^2] = \left\{ \sum_{k=o}^{n} a_k x^{2k} \mid a_k \in R, k \in Z^+ \right\}$$

R [x^2] is not differential ring if we use the operator D of Example 1, since $Dx^2 = 2x$ which is not element in R [x^2]. If we defined D by $Dx^2 = 1$, Da = 0 for all a in R, then R [x^2] is a differential ring but not a differential subring of R [x^2], since Dx^2 for R[x^2] does not coincide with Dx^2 for R[x].

Example 2-4 Let S be a subset of $\mathbb{R}$. Let $\mathcal{R}$ be the set of all infinitely differentiable functions from S into $\mathbb{R}$. The pointwise sum and product of f and g, denoted by f + g and f.g, respectively, are the functions which satisfy.

$$(f + g)(x) = f(x) + g(x)$$

$$(f.g)(x) = f(x) . g(x)$$

Then $\mathcal{R}$ is a differential ring with an operator D (differentiation in Calculus)

<u>Example 2-5</u> Let F be a field. Then $F(x)$ and $F(x, e^x)$ are differential fields with the operator D as before.

Throughout this chapter $\mathcal{R}$ and $\mathcal{F}$ will denote a differential ring and field respectively and D will denote the derivation operator of a differential ring or field.

<u>Lemma 2-1</u> Let 0 be the zero element of $\mathcal{R}$ and $-a$ be additive inverse of a in $\mathcal{R}$. Then

i) $D0 = 0$

ii) $D(-a) = -Da$.

Furthermore, if $\mathcal{R}$ is a differential ring with identity 1, then $D1 = 0$.

<u>Proof</u> By definition 2-1 (ii),

$$D0 = D(0 + 0) = D0 + D0,$$

hence $D0 = 0$. Since

$$0 = D0 = D(a - a) = Da + D(-a),$$

therefore $D(-a) = -Da$. By defi ition 2-1 (iii)

$$D1 = D(1 \cdot 1) = D1.1 + 1D1,$$

implies that $D1 = 0$.

<u>Lemma 2-2</u> Let $a, b \in \mathcal{F}$, $b \neq 0$ and denote $\frac{1}{b}$ the inverse under multiplication of b. Then

$$D\frac{a}{b} = \frac{bDa - aDb}{b^2} .$$

<u>Proof</u> Let $b \neq 0$, first we find $D\frac{1}{b}$. Since $D1 = 0$, $0 = D1 = D(b.\frac{1}{b})$
$= Db.\frac{1}{b} + b.D\frac{1}{b}$,

hence $D\frac{1}{b} = \frac{-Db}{b^2}$. We now find $D\frac{a}{b}$.

$$D\,\frac{a}{b} = D(a.\,\frac{1}{b}) = Da.\,\frac{1}{b} + a.D\,\frac{1}{b}$$

$$= \frac{Da}{b} - a\,\frac{Db}{b^2}$$

$$= \frac{bDa - aDb}{b^2}\,.$$

Definition 2-4 An element c of a differential ring or field with an operator D is said to be a constant if $Dc = 0$.

If a and b are constants of a differential ring or field, then

$$D(a + b) = Da + Db = 0$$

$$D(ab) = Da.b + a.Db = 0.$$

Hence a + b and ab are also constants of the differential ring or field respectively. If $c \neq 0$ is a constant of a differential field, then by lemma 2-2

$$D\,\frac{1}{c} = \frac{-Dc}{c^2} = 0.$$

This implies that the set of all constants is a differential subring or subfield, respectively, of the original differential ring or field and is called the subring or subfield of constants.

Definition 2-5 A differential ideal in $\mathcal{R}$ is an ideal in $\mathcal{R}$, when $\mathcal{R}$ is considered as a ring, and which is closed with respect to differentiation.

Example 2-6 Let I be an ideal in R, the ring as in Example 2-1. It is clear that $I[x]$ forms an ideal in $R[x]$. We shall show that $I[x]$ is also a differential ideal in $R[x]$. What we must prove is that $I[x]$ is closed with respect to differentiation. Let $A = a_o + a_1x + \ldots + a_nx^n \in I[x]$, i.e. $a_o, a_1, \ldots, a_n \in I$.

$$DA = a_1 + 2a\ x + \ldots + nax^{n-1} .$$

Since a_1, 2a $, \ldots, na_n \in I$, hence $DA \in I[x]$. Thus $I[x]$ is a differential ideal in $R[x]$.

Example 2-7 Let $\mathbb{R}[x]$ be a differential ring with operator D as before. Let

$$I = \{ f(x) \in \mathbb{R}[x] \mid f(1) = 0 \}.$$

Then I is an ideal in $\mathbb{R}[x]$ but I is not a differential ideal in $\mathbb{R}[x]$ since $f(x) = x^2 - 1 \in I$, but

$$(Df)(x) = D(x^2 - 1) = 2x$$

$$(Df)(1) = 2 \neq 0.$$

Thus I is not closed with respect to the operator D.

Definition 2-6 A differential ideal Φ in $\mathcal{R}$ is said to be a prime differential ideal if $ab \in \Phi$, a,b $\mathcal{R}$ implies that $a \in \Phi$ or $b \in \Phi$.

Example 2-8 Let $R[x]$ be a differential ring as before. We claim that $I[x]$ is a prime differential ideal of $R[x]$ if I is a prime ideal of R. We already know that $I[x]$ is a differential ideal of $R[x]$. It remains to show that $I[x]$ is prime. To show this, let $a(x) = a_o + a_1x + \ldots + a_mx^m$ and $b(x) = b_o + b_1x + \ldots + b_nx^n$ be two elements in $R[x]$ such that $a(x)b(x) \in I[x]$. By definition 1-6

$$a(x)b(x) = c_o + c_1x + \ldots + c_kx^k ,$$

where $k \leqslant m + n$ and

$$c_t = a_tb_o + a_{t-1}b_1 + \ldots + a_ob_t .$$

Since $a(x)b(x) \in I[x]$, $c_o, c_1, \ldots, c_k \in I$. Suppose that $b(x) \notin I[x]$, then there is at least one $b_i \notin I$ for some i. Let j be the smallest value of i such that $b_j \notin I$. Consider

$$c_j = a_j b_o + a_{j-1} b_1 + \ldots + a_1 b_{j-1} + a_o b_j \in I.$$

Since I is an ideal, $a_j b_o + a_{j-1} b_1 + \ldots a_1 b_{j-1} \in I$, hence $a_o b_j \in I$. Since I is a prime ideal and $b_j \notin I$, therefore $a_o \in I$. We now consider

$$c_{j+1} = a_{j+1} b_o + a_j b_1 + \ldots a_2 b_{j-1} + a_1 b_j + a_o b_{j+1} \in I$$

and $a_{j+1} b_o + \ldots a_2 b_{j-1} + a_o b_{j+1} \in I$, then $a_1 b_j \in I$ and hence $a_1 \in I$. Continuing in this manner, we obtain $a_o, a_1, \ldots, a_m \in I$. Thus $a(x) \in I[x]$. This implies that $I[x]$ is a prime differential ideal of $R[x]$.

<u>Definition 2-7</u> A differential ideal Π in $\mathcal{R}$ is called <u>perfect differential ideal</u> if Π contains an element of $\mathcal{R}$ whenever it contains some power of that element i.e. $a^t \in \Pi$ implies $a \in \Pi$.

<u>Theorem 2-1</u> A prime (differential) ideal is perfect (differential) ideal.

<u>Proof</u> Let Π be an arbitrary prime (differential) ideal in a (differential) ring $\mathcal{R}$. Let $a^t \in \Pi$, $a^t = a.a^{t-1}$, if $a \in \Pi$ we are done, so assume that $a \notin \Pi$, therefore $a^{t-1} \in \Pi$ since Π is prime. Using this property again $a^{t-1} = a.a^{t-2}$, implying $a^{t-2} \in \Pi$. Continuing in this way, we have $a \in \Pi$. That is Π is perfect.

<u>Remarks</u> The converse of above theorem is not true.

For an illustration of a differential ideal which is perfect but not prime. Let I = (6) be ideal generated by 6 in $\mathbb{Z}$. We claim that the differential ideal $I[x]$ is perfect in $\mathbb{Z}[x]$ but not prime. First we shall show that $I[x]$ is not prime. Since $6 = 2.3 \in I[x]$, 2 and 3 $\notin I[x]$ so we can conclude that $I[x]$ is not prime. Before showing $I[x]$ is perfect, we shall show that if $a \in \mathbb{Z}$ and $a^t \in I$, then $a \in I$. Since $a^t \in I$, there exists $m \in \mathbb{Z}$ such that $a^t = 6m$ i.e. $a^t = 2.3.m$. Since 2 and 3 are prime, 2 and 3 divide a. Thus a is a multiple of 6 and hence is in I. We now prove that $I[x]$ is perfect. Let $a(x) = a_o + a_1x + \dots + a_nx^n$, $a_n \neq 0$ be an element in $\mathbb{Z}[x]$ such that $(a(x))^t \in I[x]$. In order to prove that $I[x]$ is perfect we must prove that $a(x) \in I$, i.e. $a_o, a_1, \dots, a_n \in I$. By theorem 1 in the Appendix

$$(a(x))^t = \sum_{k_o+k_1+\dots+k_n=t} \binom{t}{k_o, k_1, \dots, k_n} a_o^{k_o}(a_1x)^{k_1}\dots(a_nx^n)^{k_n}.$$

It is clear that the constant term of $(a(x))^t$ is a_o^t and hence $a_o \in I$. We now proceed by induction on n, the number of terms, we assume that $a_o, a_1, \dots, a_m \in I$ where $m < n$ and shall show that $a_{m+1} \in I$. Since $(a(x))^t$ contains the term $(a_{m+1}x^{m+1})^t = a_{m+1}^t x^{(m+1)t}$ and it can be easily seen that the coefficient of $x^{(m+1)t}$ in $(a(x))^t$ is

$$a_{m+1}^t + \sum_{k_o+k_1+\dots+k_n=t} \binom{t}{k_o, k_1, \dots, k_n} a_o^{k_o} a_1^{k_1} \dots a_{m+1}^{k_{m+1}} \dots a_n^{k_n}, \quad (1)$$

where $k_{m+1} \neq t$ and

$$(2) \quad k_1 + 2k_2 + \dots + (m+1)k_{m+1} + \dots + nk_n = (m+1)t.$$

We multiply both sides of equation (1) by m+1 and subtract equation (2), this gives us

$(m+1)k_o + mk_1 + (m-1)k_2 + \ldots + k_m - k_{m+2} - 2k_{m+3} - \ldots - (n-m-1)k_n = 0$

If $k_o, k_1, \ldots, k_m$ are all zero, then $k_{m+2}, \ldots, k_n$ are also zero, contrary to the fact that

$$t \neq k_{m+1} = k_1 + k_2 + \ldots + k_{m+1} + \ldots + k_n = t.$$

Therefore one of $k_o, k_1, \ldots, k_m$ is not zero and then each term of

$$\sum_{\substack{k_o+\ldots+k_n=t \\ \text{and } k_m \neq t}} \binom{t}{k_o, k_1, \ldots, k_n} a_o^{k_o} \ldots a_n^{k_n}, \text{ contains some } a_j^{k_j}, \ j = 1,2,\ldots,m$$

which is divisible by 6, by induction hypothesis. Since the coefficient of $x^{(m+1)t}$ is an element in I and $\Sigma\binom{t}{k_o,\ldots,k_n} a_o^{k_o} \ldots a_n^{k_n}$ can be divide by 6, it is necessary that a_{m+1}^t is divisible by 6 and hence $a_{m+1} \in I$. This completes the proof that I[x] is perfect.

<u>Theorem 2-2</u> Let $\{\Phi_\alpha\}$ be arbitrary collection of differential ideals of the differential ring $\mathcal{R}$ where α ranges over some index set I. Then $\bigcap\Phi_\alpha$ is also a differential ideal of $\mathcal{R}$.

Moreover, if Φ_α is perfect differential ideal, then $\bigcap\Phi_\alpha$ is also a perfect differential ideal.

<u>Proof</u> It follows from theorem 1-1 that $\bigcap\Phi_\alpha$ is an ideal so it remains to prove that $\bigcap\Phi_\alpha$ is closed with respect to differentiation. Let a be an arbitrary element in $\bigcap\Phi_\alpha$, therefore $a \in \Phi_\alpha \ \forall \ \alpha \in I$. Since Φ_α's are differential ideals, $Da \in \Phi_\alpha \ \forall \ \alpha \in I$. Hence $Da \in \bigcap\Phi_\alpha$. This proves the first part of the theorem.

We next prove that $\bigcap\Phi_\alpha$ is a perfect differential ideal whenever $\{\Phi_\alpha\}_{\alpha \in I}$ are perfect differential ideals. Let $a^p \in \bigcap\Phi_\alpha$, then $a^p \in \Phi_\alpha \; \forall \alpha \in I$. Since Φ_α's are perfect differential ideals, this implies that $a \in \Phi_\alpha \; \forall \alpha \in I$ and hence $a \in \bigcap\Phi_\alpha$. Thus $\bigcap\Phi_\alpha$ is a perfect differential ideal.

Definition 2-8 Let σ be any subset of $\mathcal{R}$

a) The intersection of all differential ideals in $\mathcal{R}$ containing σ will be called the differential ideal generated by σ ; in symbols, $[\sigma]$.

b) The intersection of all perfect differential ideals in containing σ is itself a perfect differential ideal containing σ and is called the perfect differential ideal generated by σ ; in symbols, $<\sigma>$.

Note Definition 2-8 (b) is always well-defined since $\mathcal{R}$ itself is perfect differential ideal containing σ .

Notation Let σ be any set of elements of $\mathcal{R}$. σ' will denote the set containing of all elements of $\mathcal{R}$ which have integral power in σ :

$$\sigma' = \{a \in \mathcal{R} \mid \text{there exists a } k \in \mathbb{Z}^+ \text{ such that } a^k \in \sigma\}.$$

We now define σ_n recursively as follows:

$$\sigma_1 = <\sigma>$$

$$\sigma_n = \langle \sigma'_{n-1} \rangle \qquad (n = 2,3,4,\ldots)$$

Remark The notation $\sigma_n = \langle \sigma_{n-1} \rangle$ is used only in this chapter.

<u>Lemma 2-3</u> Let β be the union of the set $\sigma_n (n = 1,2,\ldots)$. Then β is a perfect differential ideal. Furthermore, $\beta = \langle\sigma\rangle$.

<u>Proof</u> For a proof of the first statement, it is clear from the definition of σ'_n that $\sigma_n \subseteq \sigma'_n$. $\forall n = 1,2,\ldots$. Hence

$$\sigma_1 \subseteq \sigma_2 \subseteq \ldots \subseteq \sigma_n \subseteq \sigma_{n+1} \subseteq \ldots$$

and then follows from theorem 1-3 that β is an ideal. We now prove that β is a differential ideal. Let a be an arbitrary element of $\bigcup_{n=1}^{\infty} \sigma_n$, then there exists some k such that $a \in \sigma_k$. Since σ_k is a differential ideal, $Da \in \sigma_k$. Thus $\bigcup_{n=1}^{\infty} \sigma_n$ is closed with respect to the operator D, that is $\beta = \bigcup_{n=1}^{\infty} \sigma_n$ is a differential ideal. To see that β is a perfect differential ideal, let $a^t \in \beta$, then there is some r such that $a^t \in \sigma_r$ and hence $a \in \sigma'_r$. It follows directly from the definition of $[\sigma'_r]$ that $a \in [\sigma'_r] = \sigma_{r+1} \subseteq \beta$. This proves that β is a perfect differential ideal.

As regards the second assertion, since $\beta = \sigma_1 \cup \sigma_2 \cup \ldots$ and $\sigma \subseteq \sigma_1 \subseteq \beta$, β is a perfect differential ideal containing σ and $\langle\sigma\rangle$ is the intersection of all perfect differential ideals containing σ we can conclude that $\langle\sigma\rangle \subseteq \beta$. On the other hand to prove that $\beta \subseteq \langle\sigma\rangle$, it is sufficient to show that

$$\sigma_1 \subsetneq \sigma_2 \subsetneq \ldots \subseteq \sigma_n \subseteq \ldots \subseteq \langle\sigma\rangle .$$

From the fact that $[\sigma]$ is the smallest differential ideal containing σ and since $\langle\sigma\rangle$ is a differential ideal containing σ , then

$$\sigma_1 = [\sigma] \subsetneq \langle\sigma\rangle .$$

We now proceed by induction on n. Assume that for all $m < n$, $\sigma_m \subseteq \langle\sigma\rangle$. From this, we shall prove that $\sigma_n \subseteq \langle\sigma\rangle$. Let a be an arbitrary element in σ'_{n-1}. It implies that there exists some integer k such that $a^k \in \sigma_{n-1}$, by our assumption $a^k \in \langle\sigma\rangle$. Since $\langle\sigma\rangle$ is a perfect differential ideal, $a \in \langle\sigma\rangle$. Hence $\sigma'_{n-1} \subseteq \langle\sigma\rangle$. Thus $[\sigma'_{n-1}] = \sigma_n \subseteq \langle\sigma\rangle$ and this completes the proof of the lemma.

<u>Notation</u> $D^k y$ will denote the k th-derivative of y i.e. the element obtained by differentiating y k-times.

<u>Definition 2-9</u> Let $\mathcal{F}$ be a differential field. A <u>form</u> in n indeterminates $y_1, y_2, \ldots, y_n$ is a polynomial in $y_1, y_2, \ldots, y_n$ and a finite number of their derivatives with coefficients in $\mathcal{F}$. That is, a finite sum of the form

$$\sum a_{i_1 i_2 \cdots i_n j_1 j_2 \cdots j_n} (D^{i_1} y_1)^{j_1} (D^{i_2} y_2)^{j_2} \cdots (D^{i_n} y_n)^{j_n}$$

where the i_r, j_s are non-negative integers and $a_{i_1 \ldots i_n j_1 \ldots j_n} \in \mathcal{F}$ $(D^0 y_i = y_i)$.

<u>Notation</u> $\mathcal{F}\{y_1, y_2, \ldots, y_n\}$ denotes the set of all forms in n indeterminates $y_1, y_2, \ldots, y_n$ with coefficients in $\mathcal{F}$.

<u>Remark</u> $\mathcal{F}\{y_1, y_2, \ldots, y_n\}$ forms a differential ring.

It follows directly from the definition of $\mathcal{F}\{y_1, y_2, \ldots, y_n\}$ that $\mathcal{F}\{y_1, y_2, \ldots, y_n\}$ is closed with respect to differentiation and it is not difficult to verify the commutative, associative, and distributive laws. So we shall omit the proof of this remark.

<u>Definition 2-10</u> <u>The partial derivative of a form A</u> in $\mathcal{F}\{y_1, y_2, \ldots, y_n\}$ with respect to $D^k y_j$, denoted by $\frac{\partial A}{\partial D^k y_j}$, is the form obtained by

differentiating A assuming that the other variables in A are constants.

From now on $\mathcal{R}$ will denote the differential ring $\mathcal{F}\{y_1, \ldots, y_n\}$ where $\mathcal{F}$ is a fixed differential field of characteristic zero.

<u>Lemma 2-4</u> If a differential ideal α in $\mathcal{R}$ contains a positive integral power a^p of an element a in $\mathcal{R}$ it contains the positive integral power $(Da)^{2p-1}$.

<u>Proof</u> Since α is a differential ideal, it is closed with respect to the operator D. So $a^p \in \alpha$ implies that $Da^p = pa^{p-1} Da \in \alpha$. From the fact that $\mathcal{F}$ is a subset of $\mathcal{R}$ and $\mathcal{F}$ is a differential field of characteristic zero, by theorem 1-11 $\mathcal{F}$ contains $\frac{1}{p}$. Hence using the fact that α is an ideal, we see that $a^{p-1}.Da \in \alpha$. (If $\mathcal{R}$ were of characteristic p, we could not draw this conclusion) Consider the element $a^{p-r}.(Da)^s$ where $0 \leqslant r < p$ and $0 \leqslant s$. If α contains $a^{p-r}.(Da)^s$, then α contains

$$D(a^{p-r}.(Da)^s) = (p-r)a^{p-r-1}(Da)^{s+1} + s.a^{p-r}(Da)^{s-1}.D^2a.$$

Multiplying both sides of this equation by Da, this gives

$$Da.(D(a^{p-r}(Da)^s) = (p-r)a^{p-r-1}(Da)^{s+2} + s.a^{p-r}(Da)^s.D^2a.$$

Since α contains $Da.D(a^{p-r}(Da)^s)$ and $sa^{p-r}(Da)^s.D^2a$, it contains $(p-r)a^{p-r-1}(Da)^{s+2}$, hence α contains $a^{p-r-1}(Da)^{s+2}$. We already know that $a^{p-1}Da \in \alpha$, so we apply this result p-1 times to the element $a^{p-1}Da$, thus obtaining $(Da)^{2p-1} \in \alpha$. This proves the lemma.

<u>Theorem 2-3</u> Let σ be any subset of a differential ring $\mathcal{R}$ and let a be an arbitrary element of the perfect differential ideal $<\sigma>$ in $\mathcal{R}$, then there exists a positive integer k such that $a^k \in [\sigma]$.

<u>Proof</u> Let a be an arbitrary element in $<\sigma>$, if $a \in [\sigma]$ we are done. Now suppose that $a \in <\sigma> - [\sigma]$. By lemma 2-3, $<\sigma> = \sigma_1 \cup \sigma_2 \cup \ldots$, therefore there exists a smallest postive integer n such that $a \in \sigma_n$.

At this point we shall show that some power of a is in σ_{n-1} . Since σ_n is a differential ideal, we know that the derivatives of all elements in σ_n belong to σ_n and thus it follows from theorem 1-2 and the fact that $\sigma_n = [\sigma'_{n-1}]$, that a can be written as a linear combination of a finite number of elements of σ'_{n-1} and a finite number of the derivatives of elements of σ'_{n-1} with coefficients in $\mathcal{R}$. It is convenient to suppose that a can be written in the form

$$a = \sum_{i=1}^{r} c_i a_i ,$$

where $c_i \in \mathcal{R}$ and $a_i \in \sigma'_{n-1}$ or a_i are the derivatives of elements of σ'_{n-1} .

We next claim that each a_i has some power in σ_{n-1} . In case a_i is an element of σ'_{n-1} , it is clear from the definition that a_i has a power in σ_{n-1} . Consider the case that a_i is the derivative of an element in σ'_{n-1} which is not an element in σ'_{n-1} . (The derivatives of elements in σ_{n-1} are not necessarily elements in σ'_{n-1} since σ'_{n-1} may not be a differential ideal). To prove this case, let $b \in \sigma'_{n-1}$, then there exists an $m \in \mathbb{Z}^+$ such that $b^m \in \sigma_{n-1}$. According to lemma 2-4,

$(Db)^{2m-1} \in \sigma_{n-1}$. Continuing in this manner, for any $\ell \in Z^+$, we get $D^{\ell}b$ must have some power in σ_{n-1}. This establishes the claim. We then assume that each a_i has a power s_i in σ_{n-1}. Let $s = \max\{s_1, s_2, \ldots, s_r\}$.

If $s_i < s$, then there exist $m_i \in Z^+$, $i = 1,2,\ldots, r$ such that $s_i + m_i = s$ and $a_i^s = a_i^{s_i} a_i^{m_i} \in \sigma_{n-1}$. Claim that a^{rs-r+1} belongs to σ_{n-1}. To prove this, consider

$$a^{rs-r+1} = (c_1a_1 + c_2a_2 + \ldots + c_ra_r)^{rs-r+1} .$$

By theorem 1 in the Appendix

$$a^{rs-r+1} = \sum_{k_1+\ldots+k_r=rs-r+1} \binom{rs-r+1}{k_1,k_2,\ldots,k_r} (c_1a_1)^{k_1}(c_2a_2)^{k_2}\ldots(c_ra_r)^{k_r}.$$

Now we shall prove that there exists some $k_i \geq s$. Assume that all k_i, $i = 1,2,\ldots, r$ are less than s, then $k_i \leq s-1$, therefore, $rs-r+1 = k_1 + k_2 + \ldots + k_r \leq r(s-1) < r(s-1) + 1 = rs-r+1$ which is impossible. Hence we obtain the result that there are some k_i, $i = 1,2,\ldots, r$ such that $k_i \geq s$. Since each term of the expansion contains some power of c_ia_i greater than or equal to s, we see that each term must belong to σ_{n-1} implying that a^{rs-r+1} is in σ_{n-1}. By the same reasoning we treat the element a^{rs-r+1} as a was treated. Thus a^{rs-r+1} has a power in σ_{n-2}. Continuing in this manner a finite number of times we find that there exists some $k \in Z^+$ such that $a^k \in \sigma_1 = [\sigma]$. This completes the proof of the theorem.

Remark The theorem above is not true for differential rings of non-zero characteristic.

Let Z_p = the ring of integers modulo p.

$Z_p\{y\}$ is the differential ring in y of characteristic p

Let $\sigma = \{y^p, (Dy)^p, (D^2_y)^p, \ldots, (D^n_y)^p\}$.

Since $D(D^i_y)^p = p(D^i_y)^{p-1} \cdot D^{i+1}_y = 0$ for all $0 \leq i \leq n$, therefore $[\sigma] = (\sigma)$. Consider $<\sigma>$, the perfect differential ideal generated by σ. Since $(D^n_y)^p \in <\sigma>$, and $<\sigma>$ is a perfect differential ideal, then $D^n_y \in <\sigma>$, hence $D^{n+1}_y \in <\sigma>$. But it is clear that D^{n+1}_y has no power in $[\sigma]$. Thus we can conclude that the theorem is not true if the differential ring has non-zero characteristic.

The results of the next lemma is independent of theorem above and is true for differential rings of non-zero characteristic.

Lemma 2-5 If a perfect differential ideal Π contains the product ab of any two elements a and b then it contains the product $D^m_a . D^n_b$ for any non-negative integers n, m.

Proof We shall first prove that $(D^m_a)b$ belongs to Π for any m. This proof is by induction on m. Assume that $(D^m_a)b$ is in Π, by the closure of the operator D

$$D((D^m_a).b) = (D^{m+1}_a).b + D^m_a.Db$$

is in Π. Multiply both sides of this equation by $(D^{m+1}_a)b$, we obtain that

$$(D^{m+1}_a).b.D((D^m_a).b) = ((D^{m+1}_a).b)^2 + (D^m_a).b.D^{m+1}_a.Db$$

is in Π. By induction hypothesis $(D^m_a)b$ is in Π, then $(D^m_a).b.D^{m+1}_a.Db$ is also in Π. This implies that $((D^{m+1}_a)b)^2$ belongs to Π and then $(D^{m+1}_a)b$ is in Π, since Π is a perfect differential ideal. We now furthur prove that $D^m a.D^n b$ is in Π for any n. Fix m, assume that $D^m aD^n b \in \Pi$. Then Π contains

$$D(D^m a.D^n b) = D^{m+1}a.D^n b + D^m a.D^{n+1}b.$$

Hence Π contains

$$D^m a\, D^{n+1}b.D(D^m aD^n b) = D^m aD^n b.D^{m+1}aD^{n+1}b + (D^m aD^{n+1}b)^2 .$$

This implies that $(D^m aD^{n+1}b)^2 \in \Pi$. Since Π is a perfect differential ideal, Π contains $D^m a.D^{n+1}b$. This proves the lemma.

<u>Theorem 2-4</u> Let σ be any subset of $\mathcal{R}$. Let $<\sigma , a >$, $<\sigma , b >$ and $<\sigma, ab >$ be the perfect differential ideals generated by the sets $\sigma \cup\{a\}$, $\sigma \cup\{b\}$ and $\sigma \cup\{ab\}$ respectively, where $a,b \in \mathcal{R}$. Then

$$<\sigma , a > \cap <\sigma , b > \;=\; <\sigma , ab > .$$

<u>Proof</u> Since ab belongs to $<\sigma, a>$ and $<\sigma, b>$, every element of $<\sigma, ab >$ is in $<\sigma, a>$ and $<\sigma, b >$. Thus $<\sigma, ab > \subseteq <\sigma, a> \cap <\sigma , b>$. It remains only to show that $<\sigma, a > \cap <\sigma, b > \subseteq <\sigma, ab >$. Let d be an arbitrary element in $<\sigma, a > \cap <\sigma, b>$, in other words $d \in <\sigma, a >$ and $d \in <\sigma, b >$. By theorem 2-3, there exist positive integers r and s such that $d^r \in [\sigma, a]$ and $d^s \in [\sigma , b]$. Being elements of $[\sigma, a]$ and $[\sigma, b]$, both d^r and d^s can be expressed as finite linear expressions, for d^r of the elements of σ and a and their derivatives, and for d^s of the elements of σ and b and their derivatives with coefficients in $\mathcal{R}$. Consider d^{r+s} , the product of d^r and d^s. Each term of the expression for d^{r+s} contains elements of σ or a product $D^m aD^n b$, for any positive integers m, n. Since $<\sigma, ab >$ contains ab, it then follows from lemma 2-5 that $< \sigma, ab >$ also contains $D^m aD^n b$ for all m, n. Thus d^{r+s} is in $<\sigma, ab >$. Since $<\sigma, ab >$ is a perfect differential ideal, $<\sigma, ab>$ contains d. This completes the proof of the theorem.

The goal of the remainder of this chapter is to study a special kind of differential ring which we shall call Notherian perfect differential ring defined below.

Definition 2-11 Let σ be any subset of a differential ring $\mathcal{R}$. We say that a perfect differential ideal Π has σ as a basis whenever Π is generated by the set σ, i.e. $\Pi = \langle\sigma\rangle$.

Definition 2-12 A differential ring $\mathcal{R}$ is said to be Noetherian perfect differential ring if every perfect differential ideal of $\mathcal{R}$ has a finite basis.

The existence of Noetherian perfect differential rings will be proved in the next chapter but we shall first study some properties of this kind of differential ring.

Theorem 2-5 Let

$$\Pi_1 \subseteq \Pi_2 \subseteq \dots \subseteq \Pi_n \subseteq \dots$$

be an infinite sequence of perfect differential ideals of Noetherian perfect differential ring. Then there exists an integer n such that

$$\Pi_n = \Pi_{n+1} = \dots$$

Proof Set $$\Pi = \bigcup_{n=1}^{\infty} \Pi_n .$$

By the same argument of the proof of lemma 2-3, we already know that Π is a differential ideal. Let a^k be an arbitrary element in Π, then there exists a sufficiently high subscript m such that a^k is in Π_m. Since Π_m is a perfect differential ideal, $a \in \Pi_m$ and hence $a \in \Pi$.

This implies that Π is a perfect differential ideal. By hypothesis, Π is Noetherian perfect differential ring, Π has a finite basis which must be contained in one ideal Π_n of the sequence if n is sufficiently large; hence $\Pi \subseteq \Pi_n$ and since

$$\Pi = \Pi_1 \cup \Pi_2 \cup \ldots ,$$

we see that $\Pi = \Pi_n$. Hence

$$\Pi = \Pi_n = \Pi_{n+1} = \ldots .$$

<u>Definition 2-13</u> A perfect differential ideal Π in any differential ring will be called <u>reducible</u> if there exist perfect differential ideals α and β such that $\Pi \subset \alpha$, $\Pi \subset \beta$ and $\Pi = \alpha \cap \beta$.

<u>Definition 2-14</u> A perfect differential ideal is said to be <u>irreducible</u> if it is not reducible.

<u>Theorem 2-6</u> A perfect differential ideal which is irreducible is prime.

<u>Proof</u> We shall show that a perfect differential ideal which is not prime is reducible. Suppose that Π is a perfect differential ideal which is not prime. Then there are two elements a and b in $\mathcal{R}$ such that $ab \in \Pi$ and a,b do not belong to Π . Consider the two perfect differential ideals $< \Pi, a >$ and $< \Pi, b >$ generated by the sets obtained by adjoining elements a and b, respectively, to the perfect differential ideal Π . Each of $< \Pi, a >$ and $< \Pi, b >$ contains Π as a proper subset. It then follows from theorem 2-4 that $< \Pi, a > \cap < \Pi, b > = < \Pi, ab >$. Since $ab \in \Pi$, therefore, $< \Pi, ab > = \Pi = < \Pi , a > \cap < \Pi, b>$. Hence Π satisfies definition 2-13, implying that Π is reducible. This proof is complete.

The following theorem provides an important result with which to close this chapter.

<u>Theorem 2-7</u> In Noetherian perfect differential ring $\mathcal{R}$, any perfect differential ideal in $\mathcal{R}$ is the intersection of a finite set of irreducible perfect differential ideals. Furthermore, if Π is any perfect differential ideal in $\mathcal{R}$ and there exist irreducible perfect differential ideals $\Pi_1, \Pi_2, \ldots, \Pi_r$ such that

$$\Pi = \Pi_1 \cap \Pi_2 \cap \ldots \cap \Pi_r ,$$

where $\Pi_i \not\subseteq \Pi_j$ for $i \neq j$, then the set $\Pi_1, \Pi_2, \ldots, \Pi_r$ is unique.

<u>Proof</u> The theorem as stated actually consists of two distinct subtheorems; the first asserts the existence of a decomposition of the given perfect differential ideal into a finite intersection of irreducible perfect differential ideals, the second assures us that this decomposition is unique. We shall prove each of these subtheorems seperately.

We first prove the existence assertion. To begin with, suppose that this assertion is not true. Then there exists a perfect differential ideal Π which is not the intersection of a finite number of irreducible perfect differential ideals. There are two cases to consider. In case one Π can be written as itself, then Π is reducible (otherwise, it contradicts the assumption that Π is not the intersection of a finite number of irreducible perfect differential ideals.). The other case is $\Pi = \bigcap_{i=1}^{\infty} \beta_i$ where β_i are perfect differential ideals, but we can rewrite Π as $\Pi = \alpha \cap \beta$ where $\alpha = \beta_1$, and $\beta = \bigcap_{i=2}^{\infty} \beta_i$. Thus, in either case, Π is reducible. So Π can be written as $\Pi = \gamma \cap \eta$ where $\Pi \subset \gamma$ and $\Pi \subset \eta$. At least one of γ or η is not the intersection of a finite number of irreducible perfect differential ideals, else Π is the intersection of a finite number of irreducible perfect differential ideals which contradicts to the assumption.

Let Π_1 denote such a perfect differential ideal. By the same reasoning as before, Π_1 is reducible and $\Pi_1 = \gamma_2 \cap \eta_2$ where $\Pi_1 \subset \gamma_2$ and $\Pi_1 \subset \eta_2$. Let Π_2 be one of these two differential ideals which is not the intersection of a finite number of irreducible perfect differential ideals, hence $\Pi_1 \subset \Pi_2$. By this process, we obtain an infinite sequence of perfect differential ideals $\Pi_1, \Pi_2, . \Pi_n, \quad \ldots$ such that

$$\Pi_1 \subset \Pi_2 \subset \ \ldots \ \subset \Pi_n \subset \ \ldots$$

and there is no positive integer n such that

$$\Pi_n = \Pi_{n+1} = \ldots$$

This contradiction of theorem 2-5 proves the existence assertion.

As regards to the uniquencess assertion, suppose that $\Pi = \alpha_1 \cap \alpha_2 \cap \ldots \cap \alpha_r$ and $\Pi = \beta_1 \cap \beta_2 \cap \ldots \cap \beta_s$ where $\alpha_i \not\subseteq \alpha_j$, $\beta_i \not\subseteq \beta_j$ for $i \neq j$. We shall prove that $r = s$ and the α's coincide with the β's after a suitable rearrangement.

First, we claim that $\alpha_1 \subseteq \beta_i$ for some $i = 1,2,\ldots, s$. To prove this claim, assume that α_1 is not contained in β_i for all $i = 1,2,\ldots, s$, then there exists $b_i \in \beta_i$ $i = 1,2,\ldots, s$ not in α_1. Since β_i is an ideal, $b_1 b_2 \ldots b_s$ is in β_i for all $i = 1,2,\ldots, s$. Hence $b_1 b_2 \ldots b_s$ is in Π and then $b_1 b_2 \ldots b_s$ belongs to α_1. Since b_i is not in α_1 for each $i = 1,2,\ldots, s$ we get a contradiction since the irreducible perfect differential ideal α_1 must be prime, by theorem 2-6. Now we have that α_1 is contained in some β_i, we may suppose that $\alpha_1 \subseteq \beta_1$ after rearrangement of the β's. Similary we can show that $\beta_1 \subseteq \alpha_k$ for some $k = 1,2,\ldots, r$. Then β_1 must be contained in α_1, otherwise $\beta_1 \subseteq \alpha_k$ some $k \neq 1$, hence $\alpha_1 \subseteq \beta_1 \subseteq \alpha_k$, contradicting the hypothesis that $\alpha_1 \not\subseteq \alpha_k$ $1 \neq k$. That is $\alpha_1 = \beta_1$. By the same reasoning, α_2 is contained in some β_k which $k \neq 1$, else $\alpha_2 \subseteq \alpha_1$, suppose it is β_2 and then β_2 is contained in α_2. This implies that $\alpha_2 = \beta_2$.

Continuing in this manner we obtain $r = s$ and the α's coincide with the β's after a suitable rearrangement. This completes the proof of theorem.

As an immediate consequence of this theorem we have the following.

Corollary In Noetherian perfect differential ring $\mathcal{R}$, any perfect differential ideal in $\mathcal{R}$ is the intersection of a finite set of prime differential ideals.

Furthermore, if Π is any perfect differential ideal in $\mathcal{R}$ and there exist prime differential ideals $\Pi_1, \Pi_2, \ldots, \Pi_r$ such that

$$\Pi = \Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_r ,$$

where $\Pi_i \not\subseteq \Pi_j$ for $i \neq j$ then the set $\Pi_1, \ldots, \Pi_r$ is unique.

We shall call the prime differential ideals $\Pi_1, \Pi_2, \ldots, \Pi_r$ the prime components of Π .