CHAPTER III

CLASSICAL HARMONIC ANALYSIS

1. Characters on the Torus

By $\mathbb{R}$ we denote the real axis; that is, the set of all real numbers with the usual topology derived from the metric $|x - y|$. This topological space has the algebraic structure of an additive abelian group. Moreover, the following functions are continuous :

$$f_1 \;:\; \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,\, y) \longmapsto x + y$$

and

$$f_2 \;:\; \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto -x.$$

We say that $\mathbb{R}$ is a <u>topological group</u>.

1.1 <u>Definition</u>. If x and y are real numbers, we say that x and y are <u>equal modulo one</u>, in symbol $x \equiv y \pmod 1$, if $x - y$ is an integer.

This relation is an equivalence relation on $\mathbb{R}$ and we shall denote the equivalence class containing x by $\dot{x}$. Let $\mathbb{T}$ be the set of all equivalence classes $\dot{x}$ $(x \in \mathbb{R})$.

For any $\dot{x}$ any $\dot{y}$ in $\mathbb{T}$, define

$$\dot{x} + \dot{y} = \widehat{x + y}^{\,\bullet}$$

and

$$-\dot{x} = \widehat{-x}^{\,\bullet} .$$

Then $\mathbb{T}$ with this operation is an additive abelian group. Algebraically speaking, we say that $\mathbb{T}$ is the quotient group $\mathbb{R}$ modulo the subgroup $\mathbb{Z}$ of integers.

Let Δ denote the unit circle in the complex plane $\mathbb{C}$. Then Δ is a multiplicative abelian group under the usual complex multiplication. Moreover, Δ inherits a compact Hausdorff topology from the usual metric topology of $\mathbb{C}$ in which the following mappings are continuous :

$$f_1 : \Delta \times \Delta \longrightarrow \Delta$$
$$(x, y) \longmapsto x \cdot y$$

and

$$f_2 : \Delta \longrightarrow \Delta$$
$$x \longmapsto x^{-1}.$$

That is , Δ is a compact Hausdorff topological multiplicative abelian group.

The assignment

$$f : \mathbb{T} \longrightarrow \Delta$$
$$\dot{x} \longmapsto e^{2\pi i x}$$

defines a group isomorphism. Thus we can endow $\mathbb{T}$ with a topology so that $\mathbb{T}$ and Δ are both isomorphic as well as homeomorphic. Note that in this topology for $\mathbb{T}$, a subset θ of $\mathbb{T}$ is open if and only if $f(\theta)$ is open. The symbol Ψ will be used to denote $\mathbb{T}$ together with this topology. Thus Ψ is a compact Hausdorff topological additive abelian group and is called the (1-dimensional) torus. Observe that the topology on Ψ is just the quotient topology of $\mathbb{R}$ by $\mathbb{Z}$ so that the quotient map

$$g : \mathbb{R} \longrightarrow \mathbb{T}$$
$$x \longmapsto \dot{x}$$

is continuous. Moreover a function $f : \mathbb{T} \longrightarrow \mathbb{C}$ is continuous if and only if $F = f \circ g$ is continuous on $\mathbb{R}$. We denote briefly $F(x) = f(\dot{x})$. It must be noticed that $f \longrightarrow f \circ g$ is a one-to-one correspondence between the set of all (continuous) functions on the torus and the set of all 1-periodic (continuous) functions on $\mathbb{R}$.

Our next goal is to find all continuous homomorphisms of $\mathbb{T}$ into Δ ; that is, we wish to find all continuous functions.

$$E : \mathbb{T} \longrightarrow \Delta$$

such that $E(\dot{x} + \dot{y}) = E(\dot{x}) . E(\dot{y})$.

Such functions are called <u>characters on</u> $\mathbb{T}$.

Let $E : \mathbb{T} \longrightarrow \Delta$ be a character. Consider the composition $F : \mathbb{R} \longrightarrow \Delta$ defined by

$$F(x) = E(\dot{x}) .$$

F is continuous and satisfies the following properties :

(1) $F(x + 1) = F(x)$

and

(2) $F(x + y) = F(x) F(y)$

for all x, y in $\mathbb{R}$. (Hence for each character E on $\mathbb{T}$, we obtained a 1-periodic continuous homomorphism $F : \mathbb{R} \longrightarrow \Delta$. Conversely, each 1-periodic continuous homomorphism F from $\mathbb{R}$ into Δ gives rise to a character E on $\mathbb{T}$ defined by :

$$E : \mathbb{T} \longrightarrow \Delta$$
$$\dot{x} \longmapsto F(x) .$$

And we clearly get a 1 - 1 correspondence between the set of all characters on Ψ and the set of all 1-periodic continuous homomorphisms of $\mathbb{R}$ into Δ.)

Let $E : \Psi \longrightarrow \Delta$ be a character and $F : \mathbb{R} \longrightarrow \Delta$ the associated 1-periodic continuous homomorphism. For any $p, q \in \mathbb{Z}$, $q \neq 0$,

$$\left[F\left(\frac{p}{q}x\right)\right]^q = F\left(\frac{p}{q}x.q\right) = F(px) = \left[F(x)\right]^p .$$

Letting $x = 1$,

$$\left[F\left(\frac{p}{q}\right)\right]^q = \left[F(1)\right]^p .$$

Since $F(1) \in \Delta$,

$$F(1) = e^{2\pi i \alpha}$$

for some $\alpha \in \mathbb{R}$ so that

$$\left[F\left(\frac{p}{q}\right)\right]^q = e^{2\pi i p \alpha} .$$

Hence

$$F\left(\frac{p}{q}\right) = e^{2\pi i \frac{p}{q}\alpha} \; e^{2\pi i \frac{k}{q}}$$

for some $k : 0 \leq k < q$, depending on p/q. But for $0 \neq t \in \mathbb{R}$,

$$F(p/q) = F(tp/tq) = e^{2\pi i \frac{p}{q}\alpha} \; e^{2\pi i \frac{k}{tq}} = e^{2\pi i \frac{p}{q}\alpha} .$$

Hence $k = 0$ and

$$F(x) = e^{2\pi i x \alpha}$$

for all $x \in \mathbb{Q}$, the rationals. Since F is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$,

$$F(x) = e^{2\pi i x \alpha} ,$$

for all $x \in \mathbb{R}$. Moreover,

$$e^{2\pi i\alpha} = F(0 + 1) = F(0) = 1$$

so that α is an integer. Conversely, if $n \in \mathbb{Z}$, then the assignment

$$\mathbb{R} \longrightarrow \Delta$$
$$x \longmapsto e^{2\pi inx}$$

defines a 1-periodic continuous homomorphism F_n from $\mathbb{R}$ into Δ which in turn induces a character

$$E_n : \mathbb{T} \longrightarrow \Delta$$

(3) $$\dot{x} \longmapsto F_n(x) = e^{2\pi inx} .$$

Finally, we have proved

1.2 <u>Proposition</u>. There exists a bijection between the set $\mathbb{Z}$ of all integers and the set of all characters on $\mathbb{T}$. The bijection is given by

$$n \longmapsto E_n,$$

where E_n is given by Equation (3).

It should be noted that these characters on $\mathbb{T}$ will play a fundamental part in the following chapters. Moreover, the additive abelian group structure on $\mathbb{Z}$ gives the following formula :

(4) $$E_{n+m}(\dot{y}) = E_n(\dot{y}) \; E_m(\dot{y})$$

for all $m,n \in \mathbb{Z}$ and all $\dot{y} \in \mathbb{T}$.

2. The Space $L^2(\mathbb{T})$

We have already constructed an infinite family of functions on the torus $\mathbb{T}$.

2.1 Definition. Any finite linear combinations of the functions E_n is a continuous function on $\mathbb{T}$. Such a function is called a trigonometric polynomial. Specifically, a trigonometric polynomial is a function of the form :

$$f(\dot{t}) = \sum_{n=-N}^{N} c_n e^{2\pi i n \dot{t}} \quad (\dot{t} \in \mathbb{T}) ,$$

where c_n are complex numbers, $n \in \{-N, \ldots, 0, \ldots, N\}$.

Let $C(\mathbb{T})$ denote the linear space of all complex-valued continuous functions defined on $\mathbb{T}$, with the uniform norm

$$\| f \|_\infty = \sup_{\dot{x} \in \mathbb{T}} | f(\dot{x}) |$$

Recall that due to the continuity of the homomorphism

$$\mathbb{R} \longrightarrow \mathbb{T}$$
$$x \longmapsto \dot{x} ,$$

we have a one-to-one correspondence between the set $C(\mathbb{T})$ and the set of all 1-periodic continuous functions from $\mathbb{R}$ into $\mathbb{C}$. Thus for any $f \in C(\mathbb{T})$, the associated 1-periodic continuous function from $\mathbb{R}$ into $\mathbb{C}$ will be denoted by F. We define the integral of f over $\mathbb{T}$, writing

$$\int_{\mathbb{T}} f(\dot{x}) \, d\dot{x} ,$$

as follows :

$$\int_{\mathbb{T}} f(\dot{x})\, d\dot{x} = \int_0^1 F(x)\, dx.$$

Now let $f, g \in C(\mathbb{T})$ and let F and G be the respective associated 1-periodic continuous functions. We define the scalar (or inner) product of f and g, denoted by (f,g), as follows :

$$(f,g) = \int_{\mathbb{T}} f(\dot{x})\, \overline{g(\dot{x})}\, d\dot{x} = \int_0^1 F(x)\, \overline{G(x)}\, dx.$$

2.2 Theorem. $(E_n, E_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$

Proof. $(E_n, E_m) = \int_0^1 e^{2\pi inx}\, e^{-2\pi imx}\, dx$

$$= \begin{cases} \left[\dfrac{e^{2\pi i(n-m)x}}{2\pi i(n-m)} \right]_0^1 = 0 & \text{if } n \neq m. \\ 1 & \text{if } n = m. \end{cases}$$

This completes the proof.

Recall that, $L^p(\mathbb{T})$ $(1 \leq p < \infty)$ is the set of all complex-valued, Lebesgue measurable functions f on $\mathbb{T}$ such that the L^p- norm of f,

$$\|f\|_p = \left\{ \int_{\mathbb{T}} |f(\dot{t})|^p\, d\dot{t} \right\}^{1/p}$$

is finite. Note that $\|f\|_2^2 = (f,f)$ for any $f \in C(\mathbb{T})$.

For convinience of notation, for any (continuous) function f on $\mathbb{T}$, the associated 1-periodic (continuous) function F on $\mathbb{R}$ will also be denoted by f, even though f is defined on $\mathbb{T}$.

2.3 Theorem. (Weierstrass Approximation Theorem). For any f in $C(\mathbb{T})$ and any $\varepsilon > 0$ there is a trigonometric polynomial P such that $\|f - P\|_\infty < \varepsilon$ or $|f(t) - P(t)| < \varepsilon$ for every real t.

Proof. Suppose we had trigonometric polynomials Q_1, Q_2, $Q_3, \ldots$, with the following properties :

(a) $Q_k(t) \geq 0$ for $t \in \mathbb{R}$.

(b) $$\int_{-1/2}^{1/2} Q_k(t)\, dt = 1.$$

(c) If $n_k(\delta) = \sup\left\{ Q_k(t) : \delta \leq |t| \leq \frac{1}{2}\right\}$, then

$$\lim_{k \to \infty} n_k(\delta) = 0 \text{ for every } \delta > 0.$$

Another way of stating (c) is to say that $Q_k(t) \longrightarrow 0$ uniformly on $\left[-\frac{1}{2}, -\delta\right] \cup \left[\delta, \frac{1}{2}\right]$, for every $\delta > 0$.

To each $f \in C(\mathbb{T})$ we associate the function P_k defined by

$$P_k(t) = \int_{-1/2}^{1/2} f(t - s)\, Q_k(s)\, ds \quad (k = 1, 2, 3, \ldots).$$

If we replace s by $-s$ and then by $s - t$, the periodicity of f and Q_k shows that the value of the integral does not change.

Hence

(1) $$P_k(t) = \int_{-1/2}^{1/2} f(s)\, Q_k(t - s)\, ds \quad (k = 1, 2, 3, \ldots).$$

Since each Q_k is a trigonometric polynomial, Q_k is of the form

(2) $$Q_k(t) = \sum_{n=-N_k}^{N_k} a_{n,k}\, e^{2\pi i n t},$$

and if we replace t by t - s in (2) and substitute the value of $Q_k(t - s)$ into (1), we see that each P_k is of the form

$$P_k(t) = \int_{-1/2}^{1/2} f(s)\, Q_k(t - s)\, ds$$

$$= \int_{-1/2}^{1/2} f(s) \sum_{n=-N_k}^{N_k} a_{n,k}\, e^{2\pi in(t - s)}\, ds$$

$$= \sum_{n=-N_k}^{N_k} a_{n,k}\, b_n\, e^{2\pi int}$$

where $b_n = \int_{-1/2}^{1/2} f(s)\, e^{-2\pi ins}\, ds = (f, E_n) \in \mathbb{C}$. Thus P_k is also a trigonometric polynomial.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous on Ψ, there exists a $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ wherever $|t - s| < \delta$. By (b), we have

$$P_k(t) - f(t) = \int_{-1/2}^{1/2} \{f(t - s) - f(t)\}\, Q_k(s)\, ds,$$

and then (a) implies, for all t, that

$$|P_k(t) - f(t)| \leq \int_{-1/2}^{1/2} |f(t - s) - f(t)|\, Q_k(s)\, ds = A_1 + A_2,$$

where A_1 is the integral over $[-\delta, \delta]$ and A_2 is the integral over $\left[-\frac{1}{2}, -\delta\right] \cup \left[\delta, \frac{1}{2}\right]$. In A_1, the integrand is less than $\varepsilon . Q_k(s)$, so $A_1 < \varepsilon$, by (b). In A_2, we have $Q_k(s) \leq n_k(\delta)$, hence

$$A_2 = \int_{-1/2}^{-\delta} |f(t - s) - f(t)|\, Q_k(s)\, ds + \int_{\delta}^{1/2} |f(t - s) - f(t)|\, Q_k(s)\, ds$$

$$\leq n_k(\delta) \int_{-1/2}^{-\delta} |f(t-s) - f(t)|\, ds + n_k(\delta) \int_{\delta}^{1/2} |f(t-s) - f(t)|\, ds$$

$$\leq \quad n_k(\delta) \int_{-1/2}^{1/2} |f(t-s) - f(t)| \; ds$$

$$\leq \quad n_k(\delta) \left(\int_{-1/2}^{1/2} |f(t-s)| \; ds \; + \int_{-1/2}^{1/2} |f(t)| \; ds \right)$$

$$\leq \quad n_k(\delta) \; (\; \|f\|_\infty + |f(t)| \,)$$

$$\leq \quad 2 \, \|f\|_\infty \, n_k(\delta) < \varepsilon$$

for sufficiently large k, by (c). Since these estimates are independent of t, we have proved that

$$\lim_{k \to \infty} \|f - P_k\|_\infty = 0.$$

It remains to construct the Q_k. Here is a simple one. Put

$$Q_k(t) = c_k \left(\frac{1 + \cos 2\pi t}{2} \right)^k ,$$

where c_k is chosen so that (b) holds; that is, choose c_k such that $\int_{-1/2}^{1/2} \left(\frac{1 + \cos 2\pi t}{2} \right)^k dt = \frac{1}{c_k}$. Then, for any $t \in \mathbb{R}$,

$$\left(\frac{1 + \cos 2\pi t}{2} \right)^k \geqslant 0 \text{ implies } \int_{-1/2}^{1/2} \left(\frac{1 + \cos 2\pi t}{2} \right)^k dt \geqslant 0.$$

Hence $c_k \geqslant 0$ and $Q_k = c_k \left(\frac{1 + \cos 2\pi t}{2} \right)^k \geqslant 0$, which proves (a). We proceed to show (c). Since Q_k is even. (b) show that

$$1 = 2c_k \int_0^{1/2} \left(\frac{1 + \cos 2\pi t}{2} \right)^k dt \geqslant 2c_k \int_0^{1/2} \left(\frac{1 + \cos 2\pi t}{2} \right)^k \sin 2\pi t dt$$

$$= -\frac{c_k}{2^k \pi} \int_0^{1/2} (1 + \cos 2\pi t)^k \, d \cos 2\pi t$$

$$= \frac{-c_k}{2^k\pi(k+1)}\left[(1+\cos\pi)^{k+1} - (1+\cos 0)^{k+1}\right]$$

$$= \frac{c_k\, 2^{k+1}}{2^k\pi\,(k+1)}$$

$$= \frac{2\,c_k}{\pi(k+1)} .$$

Q_k is decreasing on $\left[0, \frac{1}{2}\right]$, since $Q'_k(t) \leq 0$ for $t \in \left[0, \frac{1}{2}\right]$. It follows that

$$Q_k(t) \leq Q_k(\delta) \leq \frac{\pi\,(k+1)}{2}\left[\frac{1+\cos 2\pi\delta}{2}\right]^k$$

for $0 < \delta \leq |t| \leq \frac{1}{2}$.

This implies (c), since $1 + \cos 2\pi\delta < 2$ if $0 < \delta \leq \frac{1}{2}$.

The theorem is completely proved.

2.4 Theorem. The orthogonal family $\{E_n\}$ is total in $L^2(\Psi)$, where $\{E_n\}$ is total in $L^2(\Psi)$ means that if $f \in L^2(\Psi)$ with $(f, E_n) = 0$ for all n in $\mathbb{Z}$, then $f = 0$ a.e.. This is equivalent to the statement that the set of all trigonometric polynomials is dense in $L^2(\Psi)$.

Proof. Since $C(\Psi)$ is dense in $L^2(\Psi)$, for any $f \in L^2(\Psi)$, and any $\varepsilon > 0$ there is a $g \in C(\Psi)$ such that $\| f - g \|_2 < \frac{\varepsilon}{2}$.

By Theorem 2.3, there is a trigonometric P such that $\| g - P \|_\infty < \frac{\varepsilon}{2}$. But $\| g - P \|_2 \leq \| g - P \|_\infty < \frac{\varepsilon}{2}$, so that $\| f - P \|_2 < \varepsilon$. This completes the proof.

3. Fourier Series

3.1 Definition. For any $f \in L^1(\mathbb{T})$, we define the Fourier coefficients of f by the formula

$$(1) \quad \hat{f}(n) = \int_{-1/2}^{1/2} f(t)\, e^{-2\pi i n t}\, dt \quad (n = 0, \pm 1, \pm 2, \dots).$$

Since $|\hat{f}(n)| \leq \|f\|_1 < +\infty$, for all $n \in \mathbb{Z}$, the set of all integers, we thus associate with each $f \in L^1(\mathbb{T})$ a function $\hat{f}$, the Fourier transform of f, on $\mathbb{Z}$. The series

$$(2) \quad \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n t}$$

is called the Fourier series of f and its partial sums are given by

$$(3) \quad S_N(t) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{2\pi i n t} \quad (N = 0, 1, 2, \dots).$$

Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, for $1 < p \leq \infty$, (1) is applicable to every $f \in L^p(\mathbb{T})$.

3.2 Theorem. Let $1 \leq p < \infty$. Then $\hat{f}(n) \longrightarrow 0$ as $|n| \longrightarrow \infty$, for every $f \in L^p(\mathbb{T})$.

Proof. We have that $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, for $1 \leq p < \infty$, and that the trigonometric polynomials are dense in $C(\mathbb{T})$, by Theorem 2.3. If $\varepsilon > 0$ and $f \in L^p(\mathbb{T})$, then there is a $g \in C(\mathbb{T})$ and a trigonometric polynomial P such that $\|f - g\|_p < \frac{\varepsilon}{2}$ and $\|g - P\|_\infty < \frac{\varepsilon}{2}$. Since

$$\|g - P\|_p \leq \|g - P\|_\infty,$$

it follows that $\|f - P\|_p < \varepsilon$, and if $|n|$ is large enough (depending on P), then

$$|\hat{f}(n)| = \left| \int_{-1/2}^{1/2} \left\{ f(t) - P(t) \right\} e^{-2\pi int} \, dt \right|$$
$$\leq \|f - P\|_p < \varepsilon .$$

Thus $\hat{f}(n) \longrightarrow 0$ as $|n| \longrightarrow \infty$.

This completes the proof.

Of course the convergence of the Fourier coefficients of a function does not, in general, say anything about the convergence of the Fourier series. In fact, the central problem in the classical study of Fourier series is : "to determine whether, and in what sense, the Fourier series of a function f represents the function f". The most obvious way of interpreting this problem is to ask if the Fourier series of f always converges to f(x) for all x or for almost all x. In this interpretation, we immediately get into difficulties, except in the case where the functions are from $L^2(\mathbb{T})$. In this case we have a very elegant theory (see section 1 of chapter V).

4. There Are Functions Which Are Not Pointwise Limits of their Fourier Series.

4.1 Lemma. Let $\{f_n\}$ be any sequence in $(C(\mathbb{T}), \|\cdot\|_\infty)$ converging to a function f on $\mathbb{T}$. Then f is in $C(\mathbb{T})$.

Proof. Let $\varepsilon > 0$ be given. Let $\dot{x}_0$ be any point in $\mathbb{T}$. By hypothesis, we choose $N \in \mathbb{Z}(> 0)$ so that $\|f - f_n\|_\infty < \frac{\varepsilon}{3}$ for all $n \geq N$. Since f_n is continuous, there is a $\delta > 0$ such that

$$|f_n(\dot{x}) - f_n(\dot{x}_o)| < \frac{\varepsilon}{3} \text{ for all } \dot{x} \in \Psi \text{ satisfying } |\dot{x} - \dot{x}_o| < \delta.$$

Hence we have

$$\begin{aligned} |f(\dot{x}) - f(\dot{x}_o)| &\leq |f(\dot{x}) - f_N(\dot{x})| + |f_N(\dot{x}) - f_N(\dot{x}_o)| \\ &\quad + |f_N(\dot{x}_o) - f(\dot{x}_o)| \\ &\leq 2\, \| f - f_N \|_\infty + | f_N(\dot{x}) - f_N(\dot{x}_o)| \\ &< \varepsilon \end{aligned}$$

for all $\dot{x}$ in Ψ satisfying $|\dot{x} - \dot{x}_o| < \delta$. Hence f is continuous and the proof is complete.

4.2 Theorem. The space $(C(\Psi), \|\cdot\|_\infty)$ is a Banach space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C(\Psi)$ for the uniform norm and let $\varepsilon > 0$ be given. For each $\dot{x}$ in Ψ, $\{f_n(\dot{x})\}$ is a sequence of complex numbers such that for all $m, n \geq N$

$$|f_m(\dot{x}) - f_n(\dot{x})| \leq \|f_m - f_n\|_\infty < \varepsilon.$$

Since $\mathbb{C}$ is complete, the limit $\lim_{n \to \infty} f_n(\dot{x})$ exists. This defines a function $\dot{x} \longmapsto f(\dot{x}) = \lim_{n \to \infty} f_n(\dot{x})$. By letting m goes to ∞ in the above inequality, we get for all $\dot{x}$.

$$|f(\dot{x}) - f_n(\dot{x})| < \varepsilon$$

or

$$\|f - f_n\|_\infty < \varepsilon.$$

By Lemma 4.1, f is then continuous and the proof is complete.

4.3 A convergence problem. A natural question to ask : Is it true that, for every $f \in C(\mathbb{T})$, the Fourier series of f converges to f(x) at every point x ? The answer is negative as given below.

By Definition 3.1, the nth partial sum of the Fourier series of f at the point x is given by

$$(1)\quad S_n(f;x) = \int_{-1/2}^{1/2} f(t)\, D_n(x-t)\, dt \quad (n = 0, 1, 2, \dots),$$

where

$$(2)\quad D_n(t) = \sum_{k=-n}^{n} e^{2\pi ikt} .$$

The problem is to determine whether

$$(3)\quad \lim_{n\to\infty} S_n(f;x) = f(x)$$

for every $f \in C(\mathbb{T})$ and for every real x.

We shall see that the Banach-Steinhaus theorem answers the question negatively. Put

$$(4)\quad S^*(f;x) = \sup_n |S_n(f;x)| .$$

To begin with, take $x = 0$, and define

$$(5)\quad T_n f = S_n(f;0) \quad (f \in C(\mathbb{T}),\ n = 1,2,3,\dots) .$$

By Theorem 4.2, $C(\mathbb{T})$ is a Banach space, relative to the uniform norm $\|f\|_\infty$. It follows from (1) that each T_n is a bounded linear functional on $C(\mathbb{T})$, of norm

$$(5)\quad \|T_n\| = \sup_{0 \neq f \in C(\mathbb{T})} \frac{|T_n f|}{\|f\|_\infty} \leq \int_{-1/2}^{1/2} |D_n(t)|\, dt = \|D_n\|_1 .$$

We claim that

(6) $\|T_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

This will be proved by showing that equality holds in (5) and that

(7) $\|D_n\|_1 \longrightarrow \infty$ as $n \longrightarrow \infty$.

Multiply (2) by $e^{\pi it}$ and by $e^{-\pi it}$ and subtract one of the resulting two equations from the other, to obtain

$$(8) \quad D_n(t) = \frac{\sin 2\pi\,(n+\frac{1}{2})\,t}{\sin \pi\, t} .$$

Since $|\sin x| \leq |x|$ for all real x, (8) shows that

$$\|D_n\|_1 = \int_{-1/2}^{1/2} |D_n(t)|\, dt$$

$$= 2\int_0^{1/2} \left|\frac{\sin 2\pi\,(n+\frac{1}{2})\,t}{\sin \pi\, t}\right| dt$$

$$\geq \frac{2}{\pi}\int_0^{1/2} \left|\sin 2\pi\,(n+\frac{1}{2})\,t\right| \frac{dt}{t}$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} |\sin t'|\, \frac{d(t'/2\pi(n+1/2))}{(t'/2\pi(n+1/2))}$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} |\sin t'|\, \frac{dt'}{t'}$$

$$\geq \frac{2}{\pi}\int_0^{n\pi} |\sin t|\, \frac{dt}{t}$$

$$= \frac{2}{\pi}\sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} |\sin t|\, \frac{dt}{t}$$

$$\geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin t| \, dt$$

$$= \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \left(2 \int_{0}^{\pi/2} \sin t \, dt \right)$$

$$= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \longrightarrow \infty ,$$

which proves (7).

Next, fix n, and put $g(t) = 1$ if $D_n(t) \geq 0$, $g(t) = -1$ if $D_n(t) < 0$. There exist $f_j \in C(T)$ such that $-1 \leq f_j \leq 1$ and $f_j(t) \longrightarrow g(t)$ for every t, as $j \longrightarrow \infty$. By Dominated Convergence Theorem,

$$\lim_{j \to \infty} T_n(f_j) = \lim_{j \to \infty} \int_{-1/2}^{1/2} f_j(-t) \, D_n(t) \, dt$$

$$= \int_{-1/2}^{1/2} g(-t) \, D_n(t) \, dt$$

$$= \| D_n \|_1 .$$

Hence $\| T_n \| = \sup_{\|f\|_\infty \leq 1} \frac{|T_n(f)|}{\|f\|_\infty} \geq \frac{|T_n(f_j)|}{\|f_j\|_\infty}$ for all j, and

$$\lim_{j \to \infty} \| T_n \| = \| T_n \| \geq \lim_{j \to \infty} \frac{|T_n(f_j)|}{\|f_j\|_\infty} = \frac{\| D_n \|_1}{\| g \|_\infty} = \| D_n \|_1$$

so that $\| T_n \| = \| D_n \|_1$ and we have proved (6).

Since (6) holds, the Banach-Steinhaus theorem now asserts that $S^*(f \,;\, 0) = \infty$ for every f in some dense G_δ - set in $C(T)$.

We choose $x = 0$ for convenience. It is clear that the same result holds for every other x.

To each real number x there corresponds a set $E_x \in C(\mathbb{T})$ which is dense G_δ in $C(\mathbb{T})$, such that $S^*(f; x) = \infty$ for every $f \in E_x$.

In particular, the Fourier series of each $f \notin E_x$ diverges at x, and we have a negative answer to our question.

We can extend this result to $L^p(\mathbb{T})$ space for $1 \leq p < \infty$. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, for $1 \leq p < \infty$, and for each real number x there corresponds a set E_x which is a dense G_δ in $C(\mathbb{T})$. Then E_x is dense in $L^p(\mathbb{T})$ and the Fourier series of each $f \in E_x \subset L^p(\mathbb{T})$ diverges at x.

5. Analoque of Riesz-Fisher Is False for $L^1(\mathbb{T})$

5.1 Theorem. Let C_0 be the space of all complex functions $\hat{f}$ on $\mathbb{Z}$ such that $\hat{f}(n) \longrightarrow 0$ as $n \longrightarrow \pm\infty$, with the supremum norm

$$\|\hat{f}\|_\infty = \sup\left\{ |\hat{f}(n)| : n \in \mathbb{Z} \right\}.$$

Then C_0 is a Banach space.

Proof. Let $\{\hat{f}_n\}$ be any Cauchy sequence in C_0. Let $\varepsilon > 0$ be given. There is an $n_0 \in \mathbb{Z} (> 0)$ such that for all $m, n \geq n_0$, $\|\hat{f}_m - \hat{f}_n\|_\infty < \frac{\varepsilon}{2}$.

For each i in $\mathbb{Z}$, $|\hat{f}_m(i) - \hat{f}_n(i)| \leq \|\hat{f}_m - \hat{f}_n\|_\infty < \frac{\varepsilon}{2}$, this implies that $\{\hat{f}_n(i)\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete. Then the limit $\lim_{n\to\infty} \hat{f}_n(i)$ exists and defines a function $i \longmapsto \hat{f}(i) = \lim_{n\to\infty} \hat{f}_n(i)$. Moreover, for all $i \in \mathbb{Z}$,

$$|\hat{f}(i) - \hat{f}_n(i)| < \frac{\varepsilon}{2} < \varepsilon$$

or $\|\hat{f} - \hat{f}_n\|_\infty < \varepsilon$.

It remains to show that $\hat{f} \in C_o$; that is, $\hat{f}(n) \longrightarrow 0$ as $n \longrightarrow \pm\infty$. For sufficiently large m such that $\|\hat{f}_m - \hat{f}\|_\infty < \frac{\varepsilon}{2}$ we have $\hat{f}_m(n) \longrightarrow 0$ as $n \longrightarrow \pm\infty$. Then there is an $n' \in \mathbb{Z}(>0)$ such that for all $|n| \geq n'$,

$$|\hat{f}(n)| \leq |\hat{f}(n) - \hat{f}_m(x)| + |\hat{f}_m(x)| < \varepsilon .$$

This completes the proof.

If $\{a_n\}$ is a sequence of complex number such that $a_n \longrightarrow 0$ as $n \longrightarrow \pm\infty$, does it follow that there is an $f \in L^1(\mathbb{T})$ such that $\hat{f}(n) = a_n$ for all $n \in \mathbb{Z}$? In other words is something like the Riesz-Fisher theorem holds in this situation ?

This will be answered negatively with the aid of the Open Mapping Theorem.

5.2 <u>Theorem</u>. The mapping : $f \longmapsto \hat{f}$ is a one-to-one bounded linear transformation of $L^1(\mathbb{T})$ into (but not onto) C_o.

<u>Proof</u>. Define T by $Tf = \hat{f}$. Then T is linear. By Theorem 5.1, T maps $L^1(\mathbb{T})$ into C_o, and since $|\hat{f}(n)| \leq \|f\|_1$ for all n, so that $\|T\| \leq 1$. Let us now prove that T is one-to-one, that is, $Tf = 0$ implies $f = 0$ in $L^1(\mathbb{T})$.

Suppose then $f \in L^1(\mathbb{T})$ and $\hat{f}(n) = 0$ for every $n \in \mathbb{Z}$.

Then

$$(1) \qquad \int_{-1/2}^{1/2} f(t)\, g(t)\, dt = 0$$

for any trigonometric polynomial g. By Theorem 2.3, we know that the polynomials are dense in $C(\mathbb{T})$, therefore for any $g \in C(\mathbb{T})$, there is a sequence of trigonometric polynomials

$\{g_n\}$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ for every real x. Since every convergence sequence is bounded, there is an $M > 0$ such that $|f(x)\, g_n(x)| \leq M\,|f(x)|$ for all n, all real x, and $M\,|f(x)| \in L^1(\Phi)$. By Lebesque's Dominated Convergence Theorem, we have $\lim_{n\to\infty} \int_{-1/2}^{1/2} f(x)\, g_n(x)\, dx = \int_{-1/2}^{1/2} f(x)\, g(x)\, dx$. Hence (1) holds for every $g \in C(\Phi)$.

If g is the characteristic function of any measurable set in Φ. By Corollary to Lusin's theorem, there is a sequence $\{g_n\}$ in $C(\Phi)$ such that $|g_n| \leq 1$ and

$$g(x) = \lim_{n\to\infty} g_n(x) \quad \text{a.e.} \quad .$$

Apply Lebesque's Dominated Convergence Theorem once, more, then (1) holds if g is the characteristic function of any measurable set E in Φ. Since $f \in L^1(\Phi)$ and $\int_{\Phi} f(x)\, \chi_E(x)\, dx = \int_E f(x)\, dx = 0$ for every measurable set E in Φ, then we have $f = 0$ a.e. on Φ.

If the range of T were all of C_o, by Corollary 2.3.2, there exists a $\delta > 0$ such that

$$\| \hat{f} \|_\infty \geq \delta \| f \|_1 \quad \text{for every } f \in L^1(\Phi).$$

But if $D_n(t)$ is defined as in 4.3, then $D_n \in L^1(\Phi)$, $\| \hat{D}_n \|_\infty = 1$ for $n = 1,2,\ldots$, and $\| D_n \|_1 \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence there is no $\delta > 0$ such that the inequality

$$\| \hat{D}_n \|_\infty \geq \delta \| D_n \|_1$$

holds for every n.

This completes the proof.

Now $L^p(\mathbb{T}) \subsetneq L^1(\mathbb{T})$ for all $1 < p \leq \infty$. For any sequence of complex number $\{a_n\}$ such that $a_n \longrightarrow 0$ as $n \longrightarrow \pm\infty$, it does not follow that there is an $f \in L^p(\mathbb{T})$, for $1 \leq p \leq \infty$, such that $\hat{f}(n) = a_n$ for all $n \in \mathbb{Z}$.

Remark. As we have seen some alternative interpretation of the meaning of "representation of a function" is desirable. This we shall do in the next chapter.