CHAPTER VI

TORSION LOCALLY CYCLIC DECOMPOSABLE ABELIAN GROUPS

The materials of this chapter are drawn from reference [2].

In this chapter, the problem as to which abelian torsion groups are locally cyclic decomposable is solved. Namely, we prove that, for an abelian torsion group to be locally cyclic decomposable it is either locally cyclic or else there is a prime p such that it is the set-theoretical union of a disjoint family of p-cocyclic subgroups, where a family of subgroups is disjoint if each pair of the family meets at 0.

For what follows, group means <u>abelian group</u>.

6.1 <u>Lemma</u>. If G is a p-group (p a prime), then G is locally cyclic decomposable if and only if G is a set-theoretical union of a disjoint family of p-cocyclic subgroups.

<u>Proof</u> : Since locally cyclic subgroups of G are indecomposable by Theorem 2.14, they are p-cocyclic by Theorem 2.15. Hence G is locally cyclic decomposable implies that G is the union of a disjoint family of p-cocyclic subgroups.

The converse is clear by Definition 2.11 and Remark 2.12 (a).

6.2 Lemma. A torsion group G is locally cyclic if and only if each of its primary components is indecomposable.

Proof : If G is locally cyclic, then none of its primary components can be decomposable by Theorem 2.14.

Conversely, the assumption implies that each p-primary component G_p of G is p-cocyclic by Theorem 2.15. Hence each G_p is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})_p$, the p-primary component of $\mathbb{Q}/\mathbb{Z}$, the rationals $\mathbb{Q}$ modulo the integers $\mathbb{Z}$ by Theorem 2.13. These isomorphisms then induce an isomorphism of G onto some subgroup of $\mathbb{Q}/\mathbb{Z}$. It then follows from Theorem 3.8 that G is locally cyclic.

6.3 Corollary. Let G be a torsion group. If G has at least two (non-zero) primary components, then G is locally cyclic decomposable if and only if G is locally cyclic.

Proof : Since every locally cyclic group is locally cyclic decomposable, we only need to prove the converse.

Suppose G is locally cyclic decomposable. Let $G = \sum_{p \in \mathbb{P}'} G_p$ be the direct sum decomposition of G into (non-zero) primary components. Suppose that one of the primary components, say G_q, is decomposable. Let $G_q = A \oplus B$, $0 \neq a \in A$ and $0 \neq b \in B$. For each p in $\mathbb{P}^* = \mathbb{P}' - \{q\}$. Choose $g_p \neq 0$ from G_p.

Let $M = [a] \oplus \sum_{p \in \mathbb{P}^*} [g_p]$

and $N = [b] \oplus \sum_{p \in \mathbb{P}^*} [g_p]$.

Since the p-primary components of M and N are cyclic groups of orders powers of p, they are locally cyclic, hence they are indecomposable by Theorem 2.14. Therefore, M and N are locally cyclic by Lemma 6.2. Let M* and N* be maximal locally cyclic subgroups of G containing M and N, respectively, that such M* and N* exist follows from Lemma 4.6. Since a is in M* and b is in N* and since a and b can not belong to the same cyclic subgroup by Theorem 2.14, $M^* \neq N^*$. But $M^* \cap N^*$ contains the non-void set $\sum_{p \in \mathbb{P}^*} [g_p]$, since $\mathbb{P}^*$ is not empty by assumption. It then follows from Theorem 4.7 that G is not locally cyclic decomposable.

Thus each primary component of G is indecomposable so that the conclusion of the corollary follows from Lemma 6.2.

6.4 Theorem. A torsion group G is locally cyclic decomposable if and only if either

a) G is locally cyclic,

or b) there exists a prime p such that G is the set-theoretical union of a disjoint family of p-cocyclic subgroups.

Proof: The theorem follows from Lemma 6.1 and Corollary 6.3