CHAPTER III

LOCALLY CYCLIC GROUPS

The materials of this chapter are drawn from reference [6].

The simplest groups are the cyclic groups and these are either the additive group $\mathbb{Z}$ of the integers or the additive group $\mathbb{Z}_m$ of $\mathbb{Z}$ modulo m for some integer $m > 0$.

In this chapter, we discuss another class of simple groups and characterize these as subgroups of the additive rationals $\mathbb{Q}$ or of the additive group $\mathbb{Q}/\mathbb{Z}$.

3.1 <u>Definition</u>. A group is said to be <u>locally cyclic</u> if every finitely generated subgroup of it is cyclic; or equivalently, every pair (or every finite number) of elements of the group is contained in a cyclic subgroup.

3.2 <u>Examples</u>. 1. Cyclic groups are locally cyclic.

2. $\mathbb{Q}$ is locally cyclic.

<u>Proof</u> : Let $\frac{m}{n}$, $\frac{s}{t}$ be in $\mathbb{Q}$. Then $\frac{m}{n}$ and $\frac{s}{t}$ are both in the cyclic subgroup $\left[\frac{1}{nt}\right]$ of $\mathbb{Q}$.

3.3 <u>Lemma</u>. An ascending union $\bigcup_{\alpha \in A} G_\alpha$ of locally cyclic subgroups G_α of a group G is locally cyclic.

<u>Proof</u> : Since an ascending union of subgroups is a subgroup, we only need to check the condition of 3.1 .

Let a, b be in $\bigcup_{\alpha \in A} G_\alpha$. Then a $\in G_\gamma$ and b $\in G_\beta$ for some γ , $\beta \in A$. We may assume that $G_\gamma \subset G_\beta$ (see 2.1). Then a, b $\in G_\beta$ so that a, b $\in [c]$ for some c $\in G_\beta$. Since $[c] \subset \bigcup_{\alpha \in A} G_\alpha$, the ascending union is locally cyclic.

3.4 <u>Lemma</u>. Locally cyclic groups are abelian.

<u>Proof</u> : This is an immediate consequence of the fact that cyclic groups are abelian.

3.5 <u>Lemma</u>. Subgroups of locally cyclic groups are locally cyclic.

<u>Proof</u> : Let A be a subgroup of locally cyclic group G. Let a, b $\in$ A; then a, b $\in [c]$ for some c $\in$ G. Therefore a, b $\in [c] \cap A$ which is a subgroup of cyclic subgroup of A. This proves that A is locally cyclic.

3.6 <u>Lemma</u>. Homomorphic images of locally cyclic group are locally cyclic.

<u>Proof</u> : Let H be a homomorphic image of locally cyclic group G, where θ is the given homomorphism. Let c_1, $c_2 \in$ H, then $c_1 = \theta(a)$, $c_2 = \theta(b)$ for some a, b $\in$ G. Since a,b $\in$ G, then there exists d $\in$ G such that a, b $\in [d]$. Let $c = \theta(d)$; then

$c \in H$. Hence $[c]$ is the homomorphic image of $[d]$ under θ such that $c_1, c_2 \in [c]$, that is, H is locally cyclic.

3.7 <u>Lemma</u>. A locally cyclic group is either torsion or torsion-free.

<u>Proof</u> : If a is a non-zero element of finite order, and b an element of infinite order, then the group generated by a and b is not cyclic, for if it is a cyclic group $[c]$, then $a = mc$, $b = nc$. Hence c is of both infinite and finite order, which is a contradiction. Therefore the lemma is proved.

Next we prove the main theorem of this chapter.

3.8 <u>Theorem</u>. A group is locally cyclic if and only if it is isomorphic to a subgroup of a homomorphic image of the group of additive rationals.

<u>Proof</u> : In view of the Lemmas 3.5, 3.6 and Example 3.2 a subgroup of a homomorphic image of the additive rationals is locally cyclic.

Conversely, suppose G is a locally cyclic groups. Then by Lemmas 3.4 and 3.7 G is abelian and is either torsion or torsion-free.

In the case that G is torsion, then by theorem 2.9 G is the direct sum of its p-components G_p. Hence none of its primary components can be decomposable, by Theorem 2.14 . Therefore G_p is p-cocyclic by Theorem 2.15.

It is resulted from Theorem 2.13 that G is isomorphic to a subgroup of a homomorphic image of the group of additive rationals.

If G is torsion-free, then for any element a of G and any n in $\mathbb{Z}$, the set of natural numbers, there is at most one element x in G such that nx = a ; for if nx = ny = a , then x = y, by 3.4 and 2. 6 (b) .

Now let $c \neq 0$ be a fixed element of G and define c_n to be the element of G such that $nc_n = c$ if there is such an element ; otherwise define c_n to be 0. Then $[\{c_n / n \in \mathbb{Z}\}] = G$; for if x is in G, since $[\{x,c\}]$ is cyclic, there is a generator a of $[\{x, c\}]$ and a natural number n with na = c ; it follows from the above remark that $a = c_n$ so that $x \in [c_n]$.

Now, for each integer $i \geqslant 1$ G_i is the cyclic group $[\{c_1, \ldots, c_i\}] = [a_i]$, then $G_i \lneq G_{i+1}$, and there are natural numbers m_i such that $a_i = m_i a_{i+1}$. If follows that there is an isomorphism between $G = \bigcup_{i=1}^{\infty} G_i$ and $\left[\left\{\prod_{k=0}^{i-1} m_k^{-1} \middle/ i = 1, 2, \ldots, m_0 = 1\right\}\right]$ of the additive rationals, where a_i corresponds to $\prod_{k=0}^{i-1} m_k^{-1}$. This proves the theorem.

At last, consider the additive group of real numbers. Since the additive group of real numbers is not isomorphic to any subgroups of the additive group of rationals, it then follows by Theorem 3.8 that the additive group of reals is not locally cyclic.