## CHAPTER II

## PRELIMINARIES

In this chapter we shall collect some definitions and theorems which will be used in our investigation. We shall assume that the reader is familiar with common terms used in the set theory.

Let notations $\mathbb{N}$, $\mathbb{P}$, $\mathbb{R}^+$, $\mathbb{Q}^+$, $\mathbb{C}$ be the set of all non - negative integers, set of all positive integers, set of all positive real numbers, set of all positive rational numbers, and set of all complex numbers respectively.

DEFINITION 2.1 By a semigroup we mean an ordered pair $(S, \cdot)$, where S is a non-empty set and $\cdot$ is a binary operation on S satisfying the associative law, that is, for all x, y, z of S

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad .$$

For convenience, sometimes we shall denote the semigroup $(S, \cdot)$ simply by S.

DEFINITION 2.2 A semigroup $(S, \cdot)$ will be called a monoid if there exists an element e of S such that

$$e \cdot x = x \cdot e = x$$

for each x in S. Then e is called the identity element of S.

For any x in a monoid S, if there exists an element $x^{-1}$ in S such that

$$x \cdot x^{-1} = e = x^{-1} \cdot x$$

then $x^{-1}$ is called an <u>inverse</u> element of x.

<u>DEFINITION 2.3</u> Let x be an element in a monoid $(S, \cdot)$. For any non-negative integer n, we define

$$x^0 = e ,$$

$$x^{n+1} = x^n \cdot x .$$

<u>DEFINITION 2.4</u> Let $(S, \cdot)$ be a monoid. $(A, \cdot)$ will be a <u>submonoid</u> if and only if A is a nonempty subset of S, binary operation $\cdot$ on A is the restriction of $\cdot$ on S, and $(A, \cdot)$ is also a monoid.

Let S be a monoid and T be a subset of S. Let $\{H_i / i \in I\}$ be the family of all submonoid of S which contain T. We can verify that $\bigcap_{i \in I} H_i$ is a monoid. It is the smallest submonoid of S which contains T.

<u>DEFINITION 2.5</u> Let S be a monoid and T be a subset of S. By the <u>submonoid of S generated by T</u> we mean the smallest submonoid of S which contains T. We shall denote the submonoid of S generated by T by $\langle T \rangle$.

In case $T = \{a\}$, we write $\langle a \rangle$ inplace of $\langle T \rangle$.

<u>DEFINITION 2.6</u> Let S be a monoid. If there exists a in S such that $< a > = S$, we say that S is <u>cyclic</u>, and a is called a <u>generator</u> of S.

We can show that $< a > = \{a^n / n \in \mathbb{N}\}$. If the binary operation of the monoid S is denoted by + we denoted the identity by 0, and denote $x^n$ by $nx$ for all n in $\mathbb{N}$, for all x in S.

<u>DEFINITION 2.7</u> By a <u>group</u> we mean a monoid $(S, \cdot)$ in which for each x in S there exists an inverse element $x^{-1}$ of x in S. Furthermore, if

$$x \cdot y = y \cdot x$$

for all elements x, y of S. We say that $(S, \cdot)$ is an <u>abelian</u> or <u>commutative group</u>.

<u>DEFINITION 2.8</u> A mapping h of a monoid $(S, \cdot)$ into a monoid $(T, *)$ is a <u>homomorphism</u> provided that

$$h(x \cdot y) = h(x) * h(y)$$

for all x, y of S.

Let S be a monoid, T be a group. If h is a homomorphism from S into T then

$$h(e_s) = e_T ,$$

where $e_S$, $e_T$ are the identity elements of S and of T respectively.

<u>REMARK 2.9</u> Let $(S, +)$ be a cyclic monid with generator a, $(T, \cdot)$ be a group, h be a homomorphism from S into T. Then

$$h(na) = h(a)^n .$$

<u>DEFINITION 2.10</u> By a <u>field</u> we mean a triple $(F, +, \cdot)$, where $+, \cdot$ are two binary operations on F known as addition and multiplication respectively, such that the following hold:

(i) F forms a commutative group under addition,

(ii) $F^* = F - \{0\}$ forms a commutative group under multiplication,

(iii) F satisfies right and left distributive laws, i.e

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

for all a, b, c in F.

The additive identity 0, called the zero of the field, is unique; so is the additive inverse of each a, denoted by $-a$.

<u>DEFINITION 2.11</u> Let F be a field. If there is a least positive integer n such that $na = 0$ for all a in F, then F is said to have <u>characteristic n</u>. If no such n exists, F is said to have <u>characteristic zero</u>.

<u>DEFINITION 2.12</u> A field F is said to be <u>algebraically closed</u> if for every positive integer n and every $a_0, a_1, a_2, \ldots, a_n$ in F, $a_n \neq 0$, there exists x in F such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

It is well-known that the field $\mathbb{C}$ of complex numbers is algebraically closed.

DEFINITION 2.13 Let A be a subset of $\mathbb{R}^+ \cup \{0\}$. A point x in $\mathbb{R}^+ \cup \{0\}$ is a limit point of A if and only if every open interval which contains x, contains points of A other than x.

DEFINITION 2.14 A subset A of $\mathbb{R}^+ \cup \{0\}$ is dense in $\mathbb{R}^+ \cup \{0\}$ if and only if A is a subset of $\mathbb{R}^+ \cup \{0\}$ and every point in $\mathbb{R}^+ \cup \{0\}$ is a limit point of A.

It can be shown that if A is dense in $\mathbb{R}^+ \cup \{0\}$. Then for any x in $\mathbb{R}^+ \cup \{0\}$ there exists a sequence in A which converges to x.

We shall also make use of the following well-known fact.

THEOREM 2.15 If f is a function on $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{C}$ then f is continuous on $\mathbb{R}^+ \cup \{0\}$ if and only if for each sequence $\{x_n\}$ in $\mathbb{R}^+ \cup \{0\}$ which converges to a point x then the sequence $\{f(x_n)\}$ converges to f(x).

THEOREM 2.16 Let $\tilde{g}$ be a continuous homomorphism from $\mathbb{R}^+ \cup \{0\}$ into multiplicative group $\mathbb{C}^*$, Then

$$\tilde{g}(x) = \tilde{g}(1)^x$$

for all x in $\mathbb{R}^+ \cup \{0\}$.

PROOF Let $\tilde{g}$ be a continuous homomorphism from $\mathbb{R}^+ \cup \{0\}$ into multiplicative group $\mathbb{C}^*$. Then

(2.16.1) $\tilde{g}(0) = 1$ .

It can be verified by mathematical induction that, for any p, q in $\mathbb{P}$, we have

(2.16.2) $\tilde{g}\left(\frac{p}{q}\right) = \tilde{g}\left(\frac{1}{q}\right)^p$ .

Hence

(2.16.3) $\tilde{g}(1) = \tilde{g}\left(\frac{1}{q}\right)^q$ .

From (2.16.2) and (2.16.3), we have

(2.16.4) $\tilde{g}\left(\frac{p}{q}\right) = \tilde{g}(1)^{p/q}$

for all p, q in $\mathbb{P}$. From (2.16.1) and (2.16.4), we have

(2.16.5) $\tilde{g}(r) = \tilde{g}(1)^r$

for all r in $\mathbb{Q}^+ \cup \{0\}$ .

Let x be any point in $\mathbb{R}^+ \cup \{0\}$, $(r_n)$ be any sequence in $\mathbb{Q}^+ \cup \{0\}$ which tends to x. Since $\tilde{g}$ is continuous on $\mathbb{R}^+ \cup \{0\}$, hence

$$\tilde{g}(x) = \lim_{n \to \infty} \tilde{g}(r_n).$$

From (2.16.5), it follows that

$$\tilde{g}(x) = \lim_{n \to \infty} \tilde{g}(1)^{r_n},$$

$$= \tilde{g}(1)^x,$$

Therefore, $\tilde{g}(x) = \tilde{g}(1)^x$ for all x in $\mathbb{R}^+ \cup \{0\}$ .