# CHAPTER II

# TOPOLOGICAL CONCEPTS

## 2.1 Basic Concepts

A <u>topological space</u> is an ordered pair $(X, \tau)$ where $X$ is a set and $\tau$ is a family of subsets of $X$ satisfying the following conditions :

a) $\phi$ and $X$ are elements of $\tau$.

b) The intersection of any two members of $\tau$ is in $\tau$.

c) The arbitrary union of members of $\tau$ is in $\tau$.

The family $\tau$ is called the <u>topology</u> of the space $(X, \tau)$. Occasionally, we shall denote any topological space $(X, \tau)$ simply by $X$. The members of $\tau$ are called <u>$\tau$-open</u> sets of $X$ (or simply <u>open sets</u> of $X$). A subset $A$ of $X$ is said to be <u>closed</u> if and only if its relative complement $X \setminus A$ is open. If a topological space $X$ has the property that for any $x$, $y$ in $X$ there exist open sets $O_1$, $O_2$ such that $x \in O_1$, $y \in O_2$ and $O_1 \cap O_2 = \phi$, we say that $X$ is a <u>Hausdorff space</u>. For any topological space $(X, \tau)$ it can be shown that if $Y$ is any subset of $X$, then the family

$$\delta = \{T \cap Y : T \in \tau\}$$

is a topology on $Y$; it is called the <u>relative topology</u> of $Y$ and the topological space $(Y, \delta)$ is called a subspace of $(X, \tau)$.

By a <u>neighborhood</u> of a point x in a topological space X, we mean a set N for which there exists an open set O such that $x \varepsilon O \subseteq N$. The <u>boundary</u> of a subset A, denoted by $\partial A$, is defined to be the set of all $x \varepsilon X$ such that each neighborhood of x intersects both A and $X \setminus A$. The <u>interior</u> of A is defined to be the set of all $x \varepsilon X$ such that A is a neighborhood of x in X. An x in interior of A is called an <u>interior point</u> of A. By the <u>closure</u> of a subset A, denoted by $\bar{A}$, we mean a set of all points x in X such that each neighborhood of x intersects A.

A subcollection $\mathcal{B}$ of a topology $\tau$ is said to be a <u>base</u> of $\tau$ provided the following condition holds : for each $T \varepsilon \tau$ and $x \varepsilon T$, there is a $Wx \varepsilon \mathcal{B}$ such that $x \varepsilon Wx \subseteq T$, or equivalently, each T in $\tau$ is a union of members of $\mathcal{B}$. It can be shown that if a family $\mathcal{B}$ of subsets of a set X has the properties:

i) the union of sets in $\mathcal{B}$ is X,

ii) for each $B_1, B_2 \varepsilon \mathcal{B}$, $B_1 \cap B_2$ is the union of members of $\mathcal{B}$,

then $\mathcal{B}$ is a base for some topology for X. This tolopogy consists of all sets that can be written as union of sets in $\mathcal{B}$.

A function f from a topological space $(X, \tau)$ into a topological space $(Y, \delta)$ is <u>continuous at a point x</u> if and only if given any neighborhood Vy of the point $y = f(x)$, there is a neighborhood Ux of the point x such that $f(Ux) \subseteq Vy$. The mapping f is said to be <u>continuous on X</u> if it is continuous at every point of X. If f is a bijection such that f and $f^{-1}$ are continuous, then f is called a homeomorphism. Any two topological spaces are homeomorphic if there exists a homeomorphism between them. If f is a homeomorphism from X to the subspace f(X) of Y, then f is called an embedding of X into Y.

A topological space $(X, \tau)$ is said to be connected if and only if X is not the union of two nonempty, disjoint open sets. It can be shown that if f is a continuous function of $(X, \tau)$ into $(Y, \delta)$ and $(X, \tau)$ is connected, then f(X) is connected. A connected subspace of $(X, \tau)$ is not properly contained in any larger connected subspace is called a component of $(X, \tau)$.

Let $\sim$ be an equivalence relation on a set X. For each $x \in X$, the equivalence class of x under $\sim$ is the subset $x/\sim = \{y \in X : x \sim y\}$ of X, and each $y \in x/\sim$ is a representative of this equivalence class. The quotient of X under $\sim$ is the set

$$X/\sim \quad = \quad \{x/\sim : x \in X\}$$

of all the equivalence classes, and the quotient map induced by $\sim$ is the surjection

$$p \ : \ X \to X/\sim$$

which sends each element of X to its equivalence class under $\sim$. We shall call an equivalence class under $\sim$ as an identified point. For any topological space X and any equivalence relation $\sim$ on X, it can be shown that the collection

$$\tau = \{V \subseteq X/\sim \; : \; p^{-1}(V) \text{ is open in } X\}$$

is a topology on the quotient set $X/\sim$. This topology is called *the quotient topology induced by* $\sim$.

## 2.2 The Space $\mathbb{R}^n$

The space $X = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, \text{ for all } i = 1, \ldots, n\}$ of all n-tuples of real numbers can be made into a vector space over $\mathbb{R}$ by defining

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and

$$\theta(x_1, \ldots, x_n) = (\theta x_1, \ldots, \theta x_n) \;,$$

where $x_i$, $y_i \in \mathbb{R}$, for all $i = 1, \ldots, n$ and $\theta \in \mathbb{R}$. It can be shown that

$$\mathcal{B} = \{\prod_{i=1}^{n} O_i \; : \; O_i \text{ is an open interval in } \mathbb{R}, \text{ for all } i = 1, \ldots, n\}$$

form a base for some topology $\tau$ of X. We shall denote this vector space X together with this topology by $\mathbb{R}^n$ and refer to it as the *n-dimensional Euclidean space*.

If x and y are two distinct points in $\mathbb{R}^n$, then by *the line segment joining x and y*, we mean the set

$$q(x, y) = \{\theta_1 x + \theta_2 y : \theta_1, \theta_2 \in \mathbb{R}, 0 < \theta_1 < 1, 0 < \theta_2 < 1, \text{ and } \theta_1 + \theta_2 = 1\}.$$

The point x, y will be called *the endpoints of q(x, y)*.

By <u>the line passing through x and y</u>, we mean the set

$$l(x, y) = \{\theta_1 x + \theta_2 y : \theta_1, \theta_2 \in \mathbb{R} \text{ and } \theta_1 + \theta_2 = 1\}.$$

If x, y and z are three distinct points in $\mathbb{R}^3$ such that they are not in the same line, then by <u>the plane passing through x, y and z</u>, we mean the set

$$P(x, y, z) = \{\theta_1 x + \theta_2 y + \theta_3 z : \theta_1, \theta_2, \theta_3 \in \mathbb{R} \text{ and } \theta_1 + \theta_2 + \theta_3 = 1\}.$$

## 2.3 Topological Sums and Connected Sums

Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be any two topological spaces such that $X_1 \cap X_2 = \phi$. Let $X = X_1 \cup X_2$, it can be shown that

$$\tau = \{A : A \subseteq X, A \cap X_1 \in \tau_1 \text{ and } A \cap X_2 \in \tau_2\}$$

is a topology on X. The topological space $(X, \tau)$ will be called the <u>topological sum</u> of $X_1$ and $X_2$ and will be denoted by $(X_1, \tau_1) + (X_2, \tau_2)$, or simply by $X_1 + X_2$. If $X_1$, $X_2$ are Hausdorff, then it can be shown that $X_1 + X_2$ is also Hausdorff.

Let $E_1$ and $E_2$ be closed subsets of two disjoint topological spaces $X_1$ and $X_2$, respectively. Assume that $\partial E_1$ and $\partial E_2$ are not empty and homeomorphic. Let h be a homeomorphism from $\partial E_1$ to $\partial E_2$. Then the <u>h-connected sum</u> of $\overline{X_1 - E_1}$ and $\overline{X_2 - E_2}$, denoted by $\overline{X_1 - E_1} \textcircled{h} \overline{X_2 - E_2}$, is the quotient space of the topological sum $\overline{X_1 - E_1} + \overline{X_2 - E_2}$ obtained by identifying the points x and h(x) for all points x in $\partial E_1$.

## 2.4 Surfaces

By a surface we mean a Hausdorff space such that each point has a neighborhood homeomorphic to $\mathbb{R}^2$.

If S is any surface, by a disc on S, we mean a closed subset D of S such that D is homeomorphic to the unit disc, $\{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 \leq 1\}$.

If $D_1$ and $D_2$ are any two disjoint discs on S, then the topological space $\overline{S-(D_1 \cup D_2)}$ will be denoted by $S(D_1, D_2)$. We observe that $\partial(D_1 \cup D_2)$ consists of two components, each being homeomorphic to the unit circle, $\{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}$.

In what follows we shall define what we mean by a surface "S attached by a handle".

Let a surface S be given. Let

$$C = B \cup L \cup U ,$$

where

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z < 1\} ,$$

$$L = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\} ,$$

$$U = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 1\} .$$

Let $D_1$, $D_2$ be two disjoint discs on S. Then there exist a homeomorphism $h_1$ from $\partial D_1$ to $\partial L$ and a homeomorphism $h_2$ from $\partial D_2$ to $\partial U$. Let $h = h_1 \cup h_2$. We see that h is a homeomorphism from $\partial D_1 \cup \partial D_2 = \partial(D_1 \cup D_2)$ to $\partial L \cup \partial U = \partial(L \cup U)$. Hence we may construct the h-connected sum $\overline{S-(D_1 \cup D_2)}$ Ⓗ $\overline{C-(L \cup U)}$. It can be shown that this h-connected sum is a surface. We shall refer to this resulting surface as _the surface S attached by a handle_.

If S is homeomorphic to the unit sphere, $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, we say that _S is an $S_0$_. If there exists a surface S' which is an $S_t$ such that S is homeomorphic to S' attached by a handle, we say that _S is an $S_{t+1}$_.