CHAPTER V

SOLUTION OF $f(x+y) = f(x)f(y)$ AND $f(xy) = f(x)f(y)$

In determining solutions of the functional equation

$$(A) \qquad g(xoy^{-1}) = g(x)g(y) + f(x)f(y)$$

treated in the previous chapters, it turns out that certain solutions of (A) are expressible in terms of homomorphisms from an abelian group G into the multiplicative group M(F), where F is a field of characteristic different from 2. In this chapter, we shall characterize these homomorphisms for the case where $G = \mathbb{R}^n$, $F = \triangle$ and $G = \mathbb{R}^*$, $F = \mathbb{C}^*$.

5.1 Solution of $f(x+y) = f(x)f(y)$

Theorem 5.1.1 Let V be a vector space over a field F with $\mathcal{B} = \{ V_\alpha . \alpha \in I \}$ as a basis. Let f be a function on V into a commutative group G'. Then f satisfies

$$(5.1.1.1) \qquad f(x+y) = f(x)f(y),$$

iff there exists a family $\{ f_\alpha : \alpha \in I \}$ of homomorphisms from the additive group of F into G' such that for any $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$ in V, we have

$$f(x) = f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} f_{\alpha_i}(a_i) .$$

<u>Proof</u> Assume that $f : V \longrightarrow G'$ satisfies (5.1.1.1), for each $V_\alpha \in \mathcal{B}$, define $f_\alpha(a) = f(aV_\alpha)$. Observe that for each $\alpha \in I$, $f_\alpha : F \longrightarrow G'$, and

$$\begin{aligned} f_\alpha(a+b) &= f((a+b)V_\alpha), \\ &= f(aV_\alpha + bV_\alpha), \\ &= f(aV_\alpha)\, f(bV_\alpha) \\ &= f_\alpha(a) f_\alpha(b). \end{aligned}$$

For any $x \in V$, we have $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$, where $a_i \in F$, $V_{\alpha_i} \in \mathcal{B}$.

Hence $$f(x) = f\Big(\sum_{i=1}^{n} a_i V_{\alpha_i}\Big).$$

By (5.1.1.1), we have

$$f(x) = \prod_{i=1}^{n} f(a_i V_{\alpha_i}).$$

Hence $$f(x) = \prod_{i=1}^{n} f_{\alpha_i}(a_i).$$

Conversely, assume that $\{ f_\alpha : \alpha \in I \}$ is a family of homomorphisms on the additive group of F into G' and f is given by $f\big(\sum_{i=1}^{n} a_i V_{\alpha_i}\big) = \prod_{i=1}^{n} f_{\alpha_i}(a_i)$. Then for any $x, y \in V$, we may write

$$x = \sum_{i=1}^{n} a_i V_{\alpha_i}, \quad y = \sum_{i=1}^{n} b_i V_{\alpha_i},$$

where $a_i, b_i \in F$ and $V_{\alpha_i} \in \mathcal{B}$.

Hence,

$$
\begin{aligned}
f(x+y) &= f\left(\sum_{i=1}^{n} a_i V_{\alpha_i} + \sum_{i=1}^{n} b_i V_{\alpha_i}\right), \\
&= f\left(\sum_{i=1}^{n} (a_i+b_i) V_{\alpha_i}\right), \\
&= \prod_{i=1}^{n} f_{\alpha_i}(a_i+b_i), \\
&= \prod_{i=1}^{n} \left(f_{\alpha_i}(a_i)\, f_{\alpha_i}(b_i)\right), \\
&= \prod_{i=1}^{n} f_{\alpha_i}(a_i) \prod_{i=1}^{n} f_{\alpha_i}(b_i), \\
&= f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) f\left(\sum_{i=1}^{n} b_i V_{\alpha_i}\right), \\
&= f(x)f(y).
\end{aligned}
$$

<u>Lemma 5.1.2</u> Let h be a homomorphism from the additive group Q of rational numbers into a commutative group G'. Then $h(na) = (h(a))^n$, for all $a \in Q$ and all $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers.

<u>Proof</u> Let $a \in Q$. Since h is a homomorphism, hence $h(0) = 1$

Therefore $h(0.a) = h(0) = 1 = (h(a))^0$.

Assume that k is a non-negative integer such that

$$h(k.a) = (h(a))^k .$$

Then

$$h((k+1)a) = h(ka + a),$$

$$= h(ka)h(a),$$

$$= (h(a))^k \, h(a),$$

$$= (h(a))^{k+1} .$$

Hence $h(na) = (h(a))^n$ for all non-negative integers n.

For any negative integer m, $-m$ is a positive integer.

Hence

$$1 = h(0) = h(ma + (-m)a),$$

$$= h(ma)(h(a))^{-m} .$$

Therefore $h(ma) = (h(a))^m$.

Thus $h(na) = (h(a))^n$ for all $n \in \mathbb{Z}$.

Theorem 5.1.3 h is a homomorphism from Q into G' , where G' is R^+ or $\triangle$, iff there exists $r \in G'$ such that $h(a) = r^a$, for $a \in Q$.

Proof Assume that h is a homomorphism from Q into G' . Let $a \in Q$. Then $a = \frac{p}{q}$, where p,q are integers, $q \neq 0$.

We have

$$(h(\tfrac{p}{q}))^q = h(q.\tfrac{p}{q}),$$

$$= h(p),$$

$$= h(p.1),$$

$$= (h(1))^p.$$

Hence $$h(\tfrac{p}{q}) = (h(1))^{\frac{p}{q}},$$

i.e. we have $h(a) = r^a$ where $r = h(1) \in G'$.

Conversely, assume that there exists $r \in G'$ such that

$$h(a) = r^a, \text{ for } r \in G'.$$

Then

$$h(a+b) = r^{a+b} = r^a . r^b,$$

$$= h(a)h(b).$$

Hence h is a homomorphism.

<u>Theorem 5.1.4</u> Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. A function $f : \mathbb{R} \longrightarrow G'$, where G' is $\mathbb{R}^+$ or $\triangle$, satisfies

(5.1.4.1) $$f(x+y) = f(x)f(y)$$

iff there exists a function b on H into G' such that for each $x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$, we have

$$f(\sum_{i=1}^{n} a_i V_{\alpha_i}) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}.$$

Proof Assume that $f : \mathbb{R} \longrightarrow G'$ satisfies (5.1.4.1) By Theorem 5.1.1, we see that f must be of the form

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} f_{\alpha_i}(a_i),$$

where f_{α_i} is a homomorphism from $\mathbb{Q}$ into G'.

By Theorem 5.1.3, each f_{α_i} must be of the form

$$f_{\alpha_i}(a) = b_{\alpha_i}^{a} \quad , \text{ for some } b_{\alpha_i} \in G'.$$

Let $b : H \longrightarrow G'$ be defined by $b(V_{\alpha_i}) = b_{\alpha_i}$.

Then we have,

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} f_{\alpha_i}(a_i)$$

$$= \prod_{i=1}^{n} b_{\alpha_i}^{a_i}$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} .$$

Conversely, assume that there exists a function b on H to G' such that f is defined by

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} ,$$

then, for any $x = \sum_{i=1}^{n} a_i V_{\alpha_i}$, $y = \sum_{i=1}^{n} a_i' V_{\alpha_i}$ in $\mathbb{R}$, we have

$$\begin{aligned} f(x+y) &= f\left(\sum_{i=1}^{n} a_i V_{\alpha_i} + \sum_{i=1}^{n} a_i' V_{\alpha_i}\right), \\ &= f\left(\sum_{i=1}^{n} (a_i + a_i') V_{\alpha_i}\right), \\ &= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'} , \\ &= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'} , \\ &= f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) f\left(\sum_{i=1}^{n} a_i' V_{\alpha_i}\right), \\ &= f(x)f(y). \end{aligned}$$

<u>Corollary 5.15</u> Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. A function $f : \mathbb{R} \longrightarrow C^*$ satisfies

$$(5.1.5.1) \qquad f(x+y) = f(x)f(y)$$

iff there exists a function c on H into C^* such that for each $x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, we have

$$f\left(\sum_{i=1}^{n} a_i V_{\alpha_i}\right) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} .$$

<u>Proof</u> Assume that $f : \mathbb{R} \rightarrow C^*$ satisfies (5.1.5.1).

Let $g(x) = |f(x)|$ and $h(x) = \frac{f(x)}{g(x)}$

Observe that $g : \mathbb{R} \rightarrow \mathbb{R}^+$, and

$h : \mathbb{R} \rightarrow \triangle$.

Hence,

$$
\begin{aligned}
g(x+y) &= |f(x+y)|, \\
&= |f(x)\ f(y)|, \\
&= |f(x)|\ |f(y)|, \\
&= g(x)g(y).
\end{aligned}
$$

Also,

$$
\begin{aligned}
h(x+y) &= \frac{f(x+y)}{g(x+y)}, \\
&= \frac{f(x)f(y)}{g(x)g(y)}, \\
&= \frac{f(x)}{g(x)} \cdot \frac{f(y)}{g(y)}, \\
&= h(x)\ h(y) .
\end{aligned}
$$

Therefore, by using Theorem 5.1.4 there exists a function b_1, on H into $\mathbb{R}^+$ and a function b_2 on H into $\triangle$ such that for each $x = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, we have

$$g(x) = \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i} ,$$

and

$$h(x) = \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} .$$

Let $c : H \longrightarrow C^*$ be defined by

$$c(V_{\alpha_i}) = b_1(V_{\alpha_i}) \; b_2(V_{\alpha_i}).$$

So we have,

$$\begin{aligned} f(x) &= g(x).\ h(x) \\ &= \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i} , \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} , \\ &= \prod_{i=1}^{n} (b_1(V_{\alpha_i}) \; b_2(V_{\alpha_i}))^{a_i} , \\ &= \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} . \end{aligned}$$

Conversely, if c is a function on H into C^*, and f is defined by

$$f(\sum_{i=1}^{n} a_i V_{\alpha_i}) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} ,$$

then it can be verified in the same way as in Theorem 5.1.4, that $f(x+y) = f(x)f(y)$.

<u>Theorem 5.1.6</u> Let $f : \mathbb{R}^n \longrightarrow \triangle$ be function. f satisfies

(5.1.6.1) $\qquad f(x+y) = f(x)f(y),$

iff for each $i = 1, \ldots, n$, there exists a function f_i on $\mathbb{R}$ to Δ satisfying

$$f_i(x+y) = f_i(x)f_i(y)$$

such that for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$f(x) = \prod_{i=1}^{n} (f_i o p_i)(x),$$

where the p_i's are given by $p_i(x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$.

<u>Proof</u> Assume that f satisfies (5.1.6.1)

For each $i = 1, \ldots, n$, let $\pi_i : \mathbb{R} \longrightarrow \mathbb{R}^n$ be defined by

$$\pi_i(x) = xe_i ,$$

where $e_i = (\delta_{i1}, \ldots, \delta_{in})$, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Set $f_i = fo\pi_i$,

hence $f_i : \mathbb{R} \longrightarrow \Delta$ and

$$\begin{aligned} f_i(x+y) &= (fo\pi_i)(x+y) \\ &= f(\pi_i(x+y)), \\ &= f((x+y)e_i) \\ &= f(xe_i + ye_i), \end{aligned}$$

$$= f(xe_i)f(ye_i),$$

$$= f(\pi_i(x))f(\pi_i(y)),$$

$$= f_i(x)f_i(y).$$

Also, from $f_i = fo\pi_i$, we have

$$f_i o p_i = (fo\pi_i)\, op_i ,$$

where p_i is defined by $p_i(x_1, \ldots, x_n) = x_i$.

Hence , for any $x = (x_1, \ldots, x_n)$, we have

$$f_i o\, p_i(x) = f(\pi_i(p_i(x_1, \ldots, x_n))),$$

$$= f(\pi_i(x_i)),$$

$$= f(x_i e_i).$$

Therefore

$$\prod_{i=1}^{n} f_i o p_i(x) = \prod_{i=1}^{n} f(x_i e_i),$$

$$= f(x_1 e_1)\ldots f(x_n e_n),$$

$$= f(x_1 e_1 + \ldots + x_n e_n),$$

$$= f(x_1, \ldots, x_n)$$

$$= f(x).$$

Conversely, assume that $f(x) = \prod_{i=1}^{n} f_i o p_i(x)$, where each f_i, $i=1, \ldots, n$, satisfies $f_i(x+y) = f_i(x)f_i(y)$ for all $x,y \in \mathbb{R}$.

We have

$$
\begin{aligned}
f(x+y) &= \prod_{i=1}^{n} (f_i(p_i(x+y)), \\
&= \prod_{i=1}^{n} f_i(x_i+y_i), \\
&= \prod_{i=1}^{n} (f_i(x_i)f_i(y_i)), \\
&= \prod_{i=1}^{n} f_i(x_i) \prod_{i=1}^{n} f_i(y_i), \\
&= \prod_{i=1}^{n} f_i(p_i(x)), \prod_{i=1}^{n} f_i(p_i(y)), \\
&= f(x)f(y).
\end{aligned}
$$

Corollary 5.1.7 By using Theorem 5.1.4, we see that $f : \mathbb{R}^n \longrightarrow \Delta$ satisfies $f(x+y) = f(x)f(y)$ if, and only if for $j = 1, \ldots, n$, there exist functions b_j on H, where H is a Hamel basis of $\mathbb{R}$ over Q, into Δ such that for each $x = (\sum_{i=1}^{m} a_{1i}V_{\alpha_i}, \ldots, \sum_{i=1}^{m} a_{ni}V_{\alpha_i})$

we have

$$f(x) = \prod_{j=1}^{n}\prod_{i=1}^{m} b_j(V_{\alpha_i})^{a_{ji}} .$$

5.2 Solution of $f(xy) = f(x)f(y)$

Lemma 5.2.1 Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. A function $f : (\mathbb{R}^+, .) \longrightarrow G'$, where G' is $\mathbb{R}^+$ or Δ, satisfies

$$(5.2.1.1) \qquad f(xy) = f(x)f(y)$$

iff there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function b on H into G' such that for each x in $\mathbb{R}^+$, if $g^{-1}(x) = \sum_{i=1}^{n} a_iV_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$; we have $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

Proof Assume that $f : \mathbb{R}^+ \longrightarrow G'$ satisfies (5.2.1.1). Since $\mathbb{R}$ is isomorphic to $\mathbb{R}^+$, hence there exist an isomorphism g from $\mathbb{R}$ onto $\mathbb{R}^+$ such that for each x in $\mathbb{R}^+$, $g^{-1}(x) = \sum_{i=1}^{n} a_iV_{\alpha_i} \in \mathbb{R}$.

Put $h = f \circ g$. Since f and g are homomorphisms, so is h. By Theorem 5.1.4, there exists a function b on H into G' such that for each $x = \sum_{i=1}^{n} a_iV_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$, we have

$$h(\sum_{i=1}^{n} a_iV_{\alpha_i}) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$$

Hence for each x in $\mathbb{R}^+$,

$$f(x) = fg(g^{-1}(x))$$

$$= h(g^{-1}(x))$$

$$= h(\sum_{i=1}^{n} a_i V_{\alpha_i})$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} .$$

Conversely, assume that there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b : H \longrightarrow G'$ such that for each x in $\mathbb{R}^+$,

$g^{-1}(x) = \sum_{i=1}^{n} a_i V_{\alpha_i}$; and f is defined by $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

Since g is an isomorphism so is g^{-1}. Then for any x, y in $\mathbb{R}^+$,

$g^{-1}(x) = \sum_{i=1}^{n} a_i V_{\alpha_i}$, $g^{-1}(y) = \sum_{i=1}^{n} a_i' V_{\alpha_i}$, we have

$$g^{-1}(xy) = g^{-1}(x) + g^{-1}(y) = \sum_{i=1}^{n} a_i V_{\alpha_i} + \sum_{i=1}^{n} a_i' V_{\alpha_i}$$

$$= \sum_{i=1}^{n} (a_i + a_i') V_{\alpha_i} .$$

Hence, $$f(x)f(y) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'}$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'}$$

$$= f(xy).$$

<u>Theorem 5.2.2</u> Let $H = \{ V_\alpha : \alpha \in I \}$ be a Hamel basis of $\mathbb{R}$ over Q.

A function $f : (\mathbb{R}^*, \cdot) \longrightarrow (\mathbb{R}^+, \cdot)$, satisfies

(5.2.2.1) $$f(xy) = f(x)f(y),$$

iff there exist an isomorphism $g : \mathbb{R} \to \mathbb{R}^+$ and a function $b : H \to \mathbb{R}^+$ such that for each x in $\mathbb{R}^*$, if $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$; we have $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

<u>Proof</u> Assume that $f : (\mathbb{R}^*, \cdot) \longrightarrow (\mathbb{R}^+, \cdot)$ satisfies (5.2.2.1).

Then
$$\begin{aligned} (f(-1))^2 &= f(-1)f(-1) \\ &= f((-1)(-1)) \\ &= f(1) \\ &= 1. \end{aligned}$$

Hence $f(-1) = 1$.

Let $f_1 = f|_{\mathbb{R}^+}$. Observe that f_1 is a homomorphism from $\mathbb{R}^+$ to $\mathbb{R}^+$. By Lemma 5.2.1, there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b : H \longrightarrow \mathbb{R}^+$ such that for each x in $\mathbb{R}^+$, $g^{-1}(x) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$ we have

$$f(x) = f_1(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} .$$

Let x be any element of $\mathbb{R}^- = \mathbb{R}^* - \mathbb{R}^+$. Therefore $-x \in \mathbb{R}^+$. It follows that $g^{-1}(|x|) = g^{-1}(-x) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$.

Thus,

$$\begin{aligned} f(x) &= f((-1)(-x)) \\ &= f(-1)f(-x) \\ &= f(-x) \\ &= f_1(-x) \\ &= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} . \end{aligned}$$

Hence for each x in $\mathbb{R}^*$, $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i}$, we have

$$f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} .$$

Conversely, assume that there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b : H \longrightarrow \mathbb{R}^+$ such that for each x in $\mathbb{R}^*$, $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i}$, we have $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

For any x, y in $\mathbb{R}^*$, $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i}$, $g^{-1}(|y|) = \sum_{i=1}^{n} a_i' V_{\alpha_i}$, it follows that $g^{-1}(|xy|) = g^{-1}(|x||y|) = g^{-1}(|x|) + g^{-1}(|y|)$ $= \sum_{i=1}^{n} (a_i + a_i') V_{\alpha_i}$. Hence

$$f(x)f(y) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'}$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'}$$

$$= f(xy).$$

<u>Theorem 5.2.3</u> Let $H = \{ V_\alpha : \alpha \in I \}$ be a Hamel basis of $\mathbb{R}$ over Q.

A function $f : (\mathbb{R}^*, .) \longrightarrow (\Delta, .)$, satisfies

(5.2.3.1) $\quad f(xy) = f(x)f(y)$

iff there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b : H \longrightarrow \Delta$ such that for each x in $\mathbb{R}^*$, if $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$,

where $V_{\alpha_i} \in H$; we have

(5.2.3.2) $$f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \qquad \text{for all } x \text{ in } \mathbb{R}^*; \text{ or}$$

(5.2.3.3) $$f(x) = \begin{cases} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} & \text{if } x > 0 \\ -\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} & \text{if } x < 0 . \end{cases}$$

<u>Proof</u> Assume that $f : \mathbb{R}^* \longrightarrow \Delta$ satisfies (5.2.3.1).

By using the same argument as in the proof of Theorem 5.2.2, It can be shown that $(f(-1))^2 = 1$. Hence $f(-1) = 1$ or -1.

Let $f_1 = f|_{\mathbb{R}^+}$. Then f_1 is a homomorphism from $\mathbb{R}^+$ into Δ.

By Lemma 5.2.1, there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b : H \longrightarrow \Delta$ such that for each x in $\mathbb{R}^+$, $g^{-1}(|x|) = g^{-1}(x) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$, where $V_{\alpha_i} \in H$, we have $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

Let x be any element of $\mathbb{R}^- = \mathbb{R}^* - \mathbb{R}^+$. Again, by using the same argument as in the proof of Theorem 5.2.2, it can be shown that $f(x) = f(-1)f(-x) = f(-1)\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$ where $g^{-1}(|x|) = g^{-1}(-x) = \sum_{i=1}^{n} a_i V_{\alpha_i}$. If $f(-1) = 1$, then $f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$. If $f(-1) = -1$, then $f(x) = -\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}$.

Hence f is of the forms (5.2.3.2) or (5.2.3.3).

Conversely, assume that $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ is an isomorphism, $b : H \longrightarrow \Delta$, and $f : \mathbb{R}^* \longrightarrow \Delta$ is given by

(5.2.3.2) $$f(x) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i} \quad \text{for all } x \text{ in } \mathbb{R}^* \text{; or}$$

(5.2.3.3) $$f(x) = \begin{cases} \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}, & x > 0 \\ -\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}, & x < 0, \end{cases}$$

where a_i's are such that $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i}$.

By using the same argument as in the proof of Theorem 5.2.2 it can be shown that f given by (5.2.3.2) satisfies (5.2.3.1). Suppose that f is given by (5.2.3.3). Let x,y be any elements of $\mathbb{R}^*$. Therefore $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i}$, $g^{-1}(|y|) = \sum_{i=1}^{n} a_i' V_{\alpha_i}$ and hence $g^{-1}(xy) = \sum_{i=1}^{n} (a_i + a_i') V_{\alpha_i}$. If both x and y belong to $\mathbb{R}^+$ we are done. First, let us assume that x, y $\in \mathbb{R}^-$. Therefore xy $\in \mathbb{R}^+$. Hence

$$f(x)f(y) = \left(-\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}\right)\left(-\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'}\right)$$

$$= \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'}$$

$$= f(xy).$$

Next, we assume that x $\in \mathbb{R}^+$, y $\in \mathbb{R}^-$. Then xy $\in \mathbb{R}^-$. It follows that

$$f(x)f(y) = \prod_{i=1}^{n} b(V_{\alpha_i})^{a_i}\left(-\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i'}\right)$$

$$= -\prod_{i=1}^{n} b(V_{\alpha_i})^{a_i + a_i'}$$

$$= f(xy).$$

Note that if x $\in \mathbb{R}^-$, y $\in \mathbb{R}^+$, then a similar argument shows that $f(x)f(y) = f(xy)$.

In any case we have $f(x)f(y) = f(xy)$ for all x,y in $\mathbb{R}^*$.

<u>Theorem 5.2.4</u> Let $H = \{ V_\alpha : \alpha \in I \}$ be a Hamel basis of $\mathbb{R}$ over Q. A function $f : (R^*, \cdot) \longrightarrow (\mathbb{C}^*, \cdot)$ satisfies

(5.2.4.1) $$f(xy) = f(x)f(y)$$

iff there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $c : H \longrightarrow \mathbb{C}^*$ such that for each x in R^*, $g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R}$

where $V_{\alpha_i} \in H$, we have

(5.2.4.2) $$f(x) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} \quad \text{for all } x \text{ in } \mathbb{R}^*; \quad \text{or}$$

(5.2.4.3) $$f(x) = \begin{cases} \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i}, & x > 0 \\ -\prod_{i=1}^{n} c(V_{\alpha_i})^{a_i}, & x < 0 \end{cases}$$

<u>Proof</u> Assume that $f : \mathbb{R}^* \longrightarrow \mathbb{C}^*$ satisfies (5.2.4.1)

Let $\phi(x) = |f(x)|$ and $h(x) = \dfrac{f(x)}{\phi(x)}$

Observe that $\phi : \mathbb{R}^* \longrightarrow \mathbb{R}^+$,

and $h : \mathbb{R}^* \longrightarrow \Delta$

Hence
$$\begin{aligned} \phi(xy) &= |f(xy)|, \\ &= |f(x)f(y)|, \\ &= |f(x)||f(y)|, \\ &= \phi(x)\,\phi(y). \end{aligned}$$

By Theorem 5.2.2, there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b_1 : H \longrightarrow \mathbb{R}^+$ such that for each x in $\mathbb{R}^*$,

$$g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R} ,$$

we have
$$\phi(x) = \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i} .$$

Observe that
$$\begin{aligned} h(xy) &= \frac{f(xy)}{\phi(xy)} \\ &= \frac{f(x)f(y)}{\phi(x)\phi(y)} \\ &= \frac{f(x)}{\phi(x)} \cdot \frac{f(y)}{\phi(y)} \\ &= h(x)h(y) . \end{aligned}$$

By Theorem 5.2.3, there exist an isomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ and a function $b_2 : H \longrightarrow \triangle$ such that for each x in $\mathbb{R}^*$,

$$g^{-1}(|x|) = \sum_{i=1}^{n} a_i V_{\alpha_i} \in \mathbb{R} ,$$

where $V_{\alpha_i} \in H$, we have

(5.2.4.4)
$$h(x) = \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} \quad \text{for all } x \text{ in } \mathbb{R}^*; \text{ or}$$

(5.2.4.5)
$$h(x) = \begin{cases} \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} , & x > 0 \\ -\prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} , & x < 0 . \end{cases}$$

Let $c : H \longrightarrow \mathbb{C}^*$ be defined by

$$c(V_{\alpha_i}) = b_1(V_{\alpha_i})\, b_2(V_{\alpha_i}) .$$

So we have

$$h(x) = \emptyset(x)h(x)$$

$$= \left(\prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i}\right) h(x) .$$

If h(x) is of the form (5.2.4.4) , then

$$f(x) = \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i} \prod_{i=1}^{n} b_2(V_{\alpha_i})^{a_i} ,$$

$$= \prod_{i=1}^{n} b_1(V_{\alpha_i})^{a_i}\ b_2(V_{\alpha_i})^{a_i} ,$$

(5.2.4.2) $$= \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} .$$

If h(x) is of the form (5.2.4.5), then we have

(5.2.4.3) $$f(x) = \begin{cases} \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} & , x > 0 \\ -\prod_{i=1}^{n} c(V_{\alpha_i})^{a_i} & , x < 0 . \end{cases}$$

Conversely, assume that $f : \mathbb{R}^* \longrightarrow \mathbb{C}^*$ is given by (5.2.4.2) or (5.2.4.3)

It can be verified in the same way as in the proof of Theorem 5.2.3 that f satisfies (5.2.4.1).

5.3 Continuous Solution of $f(x + y) = f(x)f(y)$

In this section, we shall determine all the continuous solutions of $f(x + y) = f(x)f(y)$, where f is a function from $\mathbb{R}^n$ into $\triangle$.

Lemma 5.3.1 Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying

(5.3.1.1) $$g(x + y) = g(x) + g(y).$$

Then $g(x) = bx$ for some b in $\mathbb{R}$.

Proof We first claim that $g(na) = ng(a)$ for all integers n and $a \in \mathbb{R}$.

Since g is a homomorphism, hence $g(0) = 0$.

Therefore $$g(0.a) = g(0) = 0 = 0.\ g(a) .$$

Assume that k is a non-negative integer such that

$$g(ka) = kg(a) .$$

Then,
$$\begin{aligned} g((k+1)a) &= g(ka + a) , \\ &= g(ka) + g(a) , \\ &= kg(a) + g(a) , \\ &= (k + 1)g(a). \end{aligned}$$

For any negative integer m, -m is a positive integer. Hence

$$\begin{aligned} 0 = g(0) &= g(ma + (-m)a), \\ &= g(ma) + g((-m)a), \\ &= g(ma) + (-m)\, g(a), \end{aligned}$$

Thus $g(ma) = mg(a)$.

Therefore $g(na) = ng(a)$ for all integer n.

For $r = \frac{p}{q}$, where p,q are integers and $q \neq 0$. we have

$$\begin{aligned} qg(r) &= qg(\tfrac{p}{q}), \\ &= g(q.\ \tfrac{p}{q}), \\ &= g(p), \\ &= g(p.1), \\ &= pg(1). \end{aligned}$$

Thus $g(r) = \frac{p}{q}\, g(1) = rg(1)$.

Let $x \in \mathbb{R}$. Since the set of rational numbers is dense in $\mathbb{R}$, we can find a sequence $\{r_n\}$ of rational numbers converging to x. Since g is continuous, hence

$$\lim_{n \to \infty} g(r_n) = g(x) .$$

But $\lim_{n \to \infty} g(r_n) = \lim_{n \to \infty} r_n\, g(1) = xg(1)$.

Therefore $g(x) = xg(1)$, $x \in \mathbb{R}$.

Thus $g(x) = bx$, where $b = g(1) \in \mathbb{R}$.

<u>Theorem 5.3.2</u> Let $g : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, .)$ be a continuous function. g satisfies

(5.3.2.1) $$g(x + y) = g(x)\, g(y)$$

iff f is of the form

(5.3.2.2) $$g(x) = e^{ax}$$ for some a in $\mathbb{R}$.

<u>Proof</u> Assume that g satisfies (5.3.2.1).

Let $h(x) = \ln x$, $x > 0$,

Put $f = hog$.

Since both h and g are continuous, hence f is also continuous. We also have

$$\begin{aligned} f(x + y) &= h(g(x + y) , \\ &= \ln (g(x + y)), \\ &= \ln(g(x)g(y)) \\ &= \ln g(x) + \ln g(y) \\ &= h(g(x)) + h(g(y)), \\ &= f(x) + f(y). \end{aligned}$$

Therefore, by Lemma 5.3.1, there exists $a \in \mathbb{R}$ such that for all

$x \in \mathbb{R}$, $$f(x) = ax.$$

Then,

$$\ln(g(x)) = h(g(x)),$$
$$= f(x),$$
$$= ax .$$

Therefore $g(x) = e^{ax}$, where $a \in \mathbb{R}$.

Conversely, let $g(x) = e^{ax}$ for some a in $\mathbb{R}$.

Thus $$g(x+y) = e^{a(x+y)},$$
$$= e^{ax + ay},$$
$$= e^{ax} . e^{ay},$$
$$= g(x)g(y) .$$

<u>Remark 5.3.3</u> Observe that the function g given in (5.3.2.2) is an isomorphism iff the element a is different from zero.

<u>Theorem 5.3.4</u> Let $I : (\mathbb{R}, +) \longrightarrow \Delta$ be a continuous function. I satisfies

(5.3.4.1) $$I(x + y) = I(x)I(y)$$

iff there exists $k \in \mathbb{R}$ such that $I(x) = e^{ikx}$.

<u>Proof</u> Assume that $I : \mathbb{R} \longrightarrow \Delta$ is given by $I(x) = e^{ikx}$ for some k in $\mathbb{R}$. Then $I(x + y) = e^{ik(x+y)} = e^{ikx} \cdot e^{iky} = I(x)I(y)$.

Conversely, assume that I satisfies (5.3.4.1).

Let $f : \mathbb{R}/\mathbb{Z} \longrightarrow \Delta$ be given by $f(\bar{x}) = e^{2\pi ix}$ where $\bar{x}$ denotes the equivalence class containing x. Observe that f defines an open isomorphism on $\mathbb{R}/\mathbb{Z}$ to Δ.

Put $\beta = f^{-1} o I$.

Since both I and f^{-1} are continuous, hence β is also continuous. We also have

$$\begin{aligned}
\beta(x + y) &= f^{-1} o I(x + y) \\
&= f^{-1}(I(x)I(y)) \\
&= f^{-1}(I(x)) + f^{-1}(I(y)). \\
&= \beta(x) + \beta(y).
\end{aligned}$$

Therefore by Theorem 2.3.1, there exists $a \in \mathbb{R}$ such that $\beta(x) = \varphi(ax)$ where φ is the canonical mapping from $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$

Then

$$\begin{aligned}
f^{-1} o I(x) &= \beta(x) \\
&= \varphi(ax) \\
&= \overline{ax}.
\end{aligned}$$

Hence,

$$I(x) = f(\overline{ax})$$

$$= e^{2\pi iax}$$

$$= e^{ikx} \quad \text{where } k = 2\pi a \in \mathbb{R}.$$

Therefore $I(x) = e^{ikx}$ where $k \in \mathbb{R}$.

<u>Theorem 5.3.5</u> Let $h : (\mathbb{R}, +) \longrightarrow (\mathbb{C}^*, .)$ be a continuous function. h satisfies

(5.3.5.1) $$h(x+y) = h(x)h(y)$$

iff there exists $c \in \mathbb{C}$ such that $h(x) = e^{cx}$.

<u>Proof</u> Assume that $h : \mathbb{R} \longrightarrow \mathbb{C}^*$ is given by $h(x) = e^{cx}$ for some $c \in \mathbb{C}$. Then

$$h(x + y) = e^{c(x + y)} = e^{cx + cy} = e^{cx} e^{cy} = h(x)\, h(y).$$

Conversely, assume that h satisfies (5.3.5.1).

Let $g(x) = |h(x)|$ and $I(x) = \dfrac{h(x)}{g(x)}$.

Observe that $g : \mathbb{R} \longrightarrow \mathbb{R}^+$,

and $I : \mathbb{R} \longrightarrow \Delta$.

Since h is continuous, so are g and I.

Also, $g(x + y) = |h(x + y)|$

$= |h(x)\ h(y)|$

$= |h(x)|\,|h(y)|$

$= g(x)\ g(y).$

By using Theorem 5.3.2, we get $g(x) = e^{ax}$ for some $a \in \mathbb{R}$.

Observe that $I(x + y) = \dfrac{h(x + y)}{g(x + y)}$,

$= \dfrac{h(x)\ h(y)}{g(x)\ g(y)}$,

$= \dfrac{h(x)}{g(x)} \cdot \dfrac{h(y)}{g(y)}$,

$= I(x)\ I(y).$

By using Theorem 5.3.4, we get $I(x) = e^{ikx}$ for some $k \in \mathbb{R}$.

Thus $h(x) = I(x)g(x)$,

$= e^{ikx} \cdot e^{ax}$,

$= e^{(a + ik)x}$

$= e^{cx}$, where $c = (a + ik) \in \mathbb{C}$.

<u>Theorem 5.3.6</u> Let $f : \mathbb{R}^n \longrightarrow \Delta$ be a continuous function. f satisfies

(5.3.6.1) $f(x + y) = f(x)f(y)$

iff there exist $k_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that for each $x = (x_1, \ldots, x_n)$ we have $f(x) = e^{i(k_1x_1+\ldots+k_nx_n)}$.

<u>Proof</u> Assume that f satisfies (5.3.6.1).

Using Theorem 5.1.5, there exist $f_i : \mathbb{R} \longrightarrow \Delta$ satisfying

$$f_i(x + y) = f_i(x)f_i(y) \quad , \quad i = 1, \ldots, n ,$$

such that for each $x \in \mathbb{R}^n$, we have

$$f(x) = \prod_{i=1}^{n}(f_i o p_i)(x),$$

where p_i, $i = 1, \ldots, n$, is given by $p_i(x_1, \ldots, x_n) = x_i$.

Such an f_i is given by $f_i = fo\pi_i$, where π_i is defined as in the proof of Theorem 5.1.6 .

Since f and π_i are continuous, hence each f_i is continuous.

By using Theorem 5.3.4, we have

$$f_j(x_j) = e^{ik_jx_j} \quad \text{for each } j = 1,\ldots, n \text{ and } k_j \in \mathbb{R}.$$

Hence,

$$f(x) = \prod_{i=1}^{n}(f_i o p_i)(x),$$

$$= f_1(x_1)\ldots f_n(x_n),$$

$$= e^{ik_1x_1} \ldots e^{ik_nx_n} ,$$

$$= e^{i(k_1x_1 + \ldots+k_nx_n)} .$$

Conversely, assume that there exist $k_j \in \mathbb{R}$, $j = 1, \ldots, n$ such that $f(x) = e^{i(k_1x_1 + \ldots + k_nx_n)}$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then we have

$$
\begin{aligned}
f(x+y) &= e^{i(k_1(x_1+y_1) + \ldots + k_n(x_n+y_n))}, \\
&= e^{i((k_1x_1 + \ldots + k_nx_n) + (k_1y_1 + \ldots + k_ny_n))}, \\
&= e^{i(k_1x_1 + \ldots + k_nx_n)} e^{i(k_1x_1 + \ldots + k_nx_n)}, \\
&= f(x)f(y).
\end{aligned}
$$

5.4 Continuous Solution of $f(xy) = f(x)f(y)$

In this section, we shall determine all the continuous solutions of $f(xy) = f(x)f(y)$, where f is a function from R^* into C^*.

Lemma 5.4.1 Let $f : (\mathbb{R}^+, \cdot) \longrightarrow \mathbb{C}^*$ be a continuous function. f satisfies

$$(5.4.1.1) \qquad f(xy) = f(x)f(y)$$

iff there exists $c \in \mathbb{C}$ such that $f(x) = x^c$, where $x^c = e^{c \ln x}$.

Proof Assume that $f : (\mathbb{R}^+, \cdot) \longrightarrow C^*$ is given by $f(x) = x^c$ for some c in $\mathbb{C}$. It can be verified that f satisfies (5.4.1.1) Conversely, assume that f satisfies (5.4.1.1).

Let $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ be given by $g(x) = e^x$. Hence g is a continuous isomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$

Put $$h = f \circ g \ .$$

Since f and g are continuous, hence h is also continuous. We also have

$$\begin{aligned} h(x + y) &= f \circ g(x + y), \\ &= f(g(x + y)), \\ &= f(g(x)g(y)), \\ &= f(g(x))\, f(g(y)), \\ &= h(x)h(y). \end{aligned}$$

Therefore by Theorem 5.3.5, there exists $c \in \mathbb{C}$ such that for all $x \in \mathbb{R}$ $h(x) = e^{cx}$. Hence for all x in $\mathbb{R}^+$,

$$\begin{aligned} f(x) &= f(g(g^{-1}(x))) \\ &= h(g^{-1}(x)) \\ &= e^{cg^{-1}(x)} \\ &= e^{c \ln x} \\ &= x^c . \end{aligned}$$

<u>Theorem 5.4.2</u> Let $f : (\mathbb{R}^*, .) \longrightarrow \mathbb{C}^*$ be a continuous function . f satisfies

(5.4.2.1) $$f(xy) = f(x)f(y)$$

iff there exists $c \in \mathbb{C}$ such that

(5.4.2.2) $$f(x) = |x|^c \quad \text{for all } x \text{ in } \mathbb{R}^* \text{ ; or}$$

(5.4.2.3) $$f(x) = \begin{cases} |x|^c & , x > 0 \\ -|x|^c & , x < 0 . \end{cases}$$

<u>Proof</u> Assume that $f : (\mathbb{R}^*, .) \longrightarrow \mathbb{C}^*$ is given by (5.4.2.2) or (5.4.2.3). Then it can be verified that f satisfies (5.4.2.1) . Conversely, assume that f satisfies (5.4.2.1). Then it can be verified in the same way as in the proof of Theorem 5.2.2 that $(f(-1))^2 = 1$. Hence $f(-1) = 1$ or -1. Let $f_1 = f|_{\mathbb{R}^+}$. Then f_1 is a continuous homomorphism from $\mathbb{R}^+$ to $\mathbb{C}^*$. By Lemma 5.4.1, $f_1(y) = y^c = |y|^c$ for some $c \in \mathbb{C}$.

Let x be any element of $\mathbb{R}^- = \mathbb{R}^* - \mathbb{R}^+$. Therefore $-x \in \mathbb{R}^+$ Thus,

$$\begin{aligned} f(x) &= f((-1)(-x)), \\ &= f(-1)f(-x), \\ &= f(-1)(-x)^c, \\ &= f(-1)\ |x|^c . \end{aligned}$$

If $f(-1) = 1$, then $f(x) = |x|^c$, hence f is of the form (5.4.2.2).

If $f(-1) = -1$, then $f(x) = -|x|^c$, hence f is of the form (5.4.2.3).

Hence f is of the forms (5.4.2.2) or (5.4.2.3).

5.5 Existence of Discontinuous Solution of $f(x + y) = f(x)f(y)$

The purpose of this section is to provide some examples of a discontinuous solution of $f(x + y) = f(x)f(y)$, where f is a function from $(\mathbb{R}^n, +)$ into $(\triangle, \cdot)$. For simplicity, we give examples of discontinuous solutions from $\mathbb{R}^3$ to $\triangle$.

Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. By using Corollary 5.1.7, any function $f : \mathbb{R}^3 \longrightarrow \triangle$ satisfying $f(x + y) = f(x)f(y)$ must be of the form

$$f\left(\sum_{i=1}^{m} a_{1i} V_{\alpha_i}, \sum_{i=1}^{m} a_{2i} V_{\alpha_i}, \sum_{i=1}^{m} a_{3i} V_{\alpha_i}\right) = \prod_{j=1}^{3} \prod_{i=1}^{m} b_j(V_{\alpha_i})^{a_{ji}},$$

where b_1, b_2, b_3 are functions on H into $\triangle$.

Let us denote such function f by f_{b_1, b_2, b_3}. Hence each triple $b = (b_1, b_2, b_3)$, where $b_j : H \longrightarrow \triangle$, $j = 1, 2, 3$, defines a function f_b satisfying $f_b(x + y) = f_b(x)f_b(y)$. A discontinuous function f_b satisfying this equation can be obtained by choosing suitable functions b_1, b_2 and b_3. We shall first constructed $b_j : H \longrightarrow \triangle$, $j = 1,2,3$, which will make f_b a discontinuous solution of $f(x + y) = f(x)f(y)$.

Choose three distinct elements $V_{\alpha_1}, V_{\alpha_2}, V_{\alpha_3}$ of H and three nonzero complex number z_1, z_2, z_3, such that $|z_i| = 1, i = 1, 2, 3,$ and not all z_i's are 1.

Define $b_j : H \to \Delta$, $j = 1, 2, 3,$ by putting

$$b_1(V_{\alpha_1}) = z_1, \quad b_1(V_\alpha) = 1 \quad \text{for all } \alpha \neq \alpha_1$$
$$b_2(V_{\alpha_2}) = z_2, \quad b_2(V_\alpha) = 1 \quad \text{for all } \alpha \neq \alpha_2$$
$$b_3(V_{\alpha_3}) = z_3, \quad b_3(V_\alpha) = 1 \quad \text{for all } \alpha \neq \alpha_3.$$

By Corollary 5.1.7 f_b satisfies $f_b(x + y) = f_b(x)f_b(y)$. Next, we show that f_b is not continuous.

Suppose that f_b is continuous. By Theorem 5.3.6, there exist $k_i \in \mathbb{R}$, $i = 1, 2, 3,$ such that for any $x = (x_1, x_2, x_3) \in \mathbb{R}$ we have $f_b(x_1, x_2, x_3) = e^{i(k_1x_1 + k_2x_2 + k_3x_3)}$.

Observe that $f_b(V_{\alpha_1}, 0, 0) = b_1(V_{\alpha_1})^1 = z_1$,

and $f_b(V_{\alpha_1} + V_{\alpha_2}, 0, 0) = b_1(V_{\alpha_1})^1 \, b_1(V_{\alpha_2})^1 = z_1 \cdot 1 = z_1$

Therefore $f_b(V_{\alpha_1}, 0, 0) = f_b(V_{\alpha_1} + V_{\alpha_2}, 0, 0)$.

Since $f_b(x_1, x_2, x_3) = e^{i(k_1x_1 + k_2x_2 + k_3x_3)}$, hence

$$e^{ik_1V\alpha_1} = f_b(V\alpha_1, 0, 0)$$

$$= f_b(V\alpha_1 + V\alpha_2, 0, 0)$$

$$= e^{ik_1(V\alpha_1 + V\alpha_2)}.$$

Therefore $e^{ik_1V\alpha_2} = 1$. Thus $ik_1V\alpha_2 = 2k\pi i$ for some integer k. Since $V\alpha_2 \in H$, we have $V\alpha_2 \neq 0$. Therefore $k_1 = \frac{2k\pi}{V\alpha_2}$, Similarly we can show that

$k_1 = \frac{2k'\pi}{V\alpha_3}$ where k' is an integer. Hence

$2k\pi V\alpha_3 = 2k'\pi V\alpha_2$. But $V\alpha_2$, $V\alpha_3$ are linearly independent. Hence $k = k' = 0$. Therefore $k_1 = 0$. By a similar argument we can show that $k_2 = k_3 = 0$. Therefore $f_b(x) = 1$ for all $x = (x_1, x_2, x_3)$.

By the choice of z_i's, we may assume that $z_1 \neq 1$. Hence $f_b(V\alpha_1, 0, 0) = z_1 \neq 1$, which is a contradiction. Therefore $f_b(x)$ cannot be of the form $e^{i(k_1x_1 + k_2x_2 + k_3x_3)}$, i.e. f_b is not continuous. Hence there exists a discontinuous solution of $f(x + y) = f(x)f(y)$.

It can be seen that if we choose n distinct elements $V_{\alpha_1}, \ldots, V_{\alpha_n}$ in H and any n non-zero complex numbers $z_1, \ldots, z_n$ such that $|z_i| = 1$, $i=1, \ldots, n$, and not all z_i's are 1 and define $b_j : H \longrightarrow \Delta$ by

$$b_j(V_{\alpha_i}) = \begin{cases} z_j & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases}$$

then $f_b : \mathbb{R}^n \longrightarrow \Delta$, defined by

$$f_b\Big(\sum_i a_{1i}V_{\alpha_i}, \ldots, \sum_i a_{ni}V_{\alpha_i}\Big) = \prod_j \prod_i b_j(V_{\alpha_i})^{a_{ji}},$$

is a discontinuous solution of $f(x + y) = f(x)f(y)$.

5.6 Existence of Discontinuous Solution of $f(xy) = f(x)f(y)$

Theorem 5.6.1 There exist discontinuous solutions of

$$f(x + y) = f(x)f(y)$$

on $\mathbb{R}$ to $\mathbb{C}^*$.

Proof Let $H = \{V_\alpha : \alpha \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q. By using Corollary 5.1.5, any function $f : \mathbb{R} \longrightarrow \mathbb{C}^*$ satisfying $f(x + y) = f(x)f(y)$ must be of the form

$$f\Big(\sum_{i=1}^{n} a_i V_{\alpha_i}\Big) = \prod_{i=1}^{n} c(V_{\alpha_i})^{a_i}$$

where c is a function on H into $\mathbb{C}^*$.

Choose two distinct elements V_{α_1}, V_{α_2} of H and nonzero complex number z_1 such that $z_1 \neq 1$.

Define $c : H \longrightarrow \mathbb{C}^*$ by putting

$$c(V_{\alpha_1}) = z_1, \quad c(V_\alpha) = 1, \quad \text{for all } \alpha \neq \alpha_1 .$$

By Corollary 5.1.5, c defines a function f_c satisfying $f_c(x + y) = f_c(x)f_c(y)$. Similar arguments as given in the proof in Section 5.5 can be used to show that f_c is not continuous. Hence there exist discontinuous solutions of $f(x + y) = f(x)f(y)$.

<u>Theorem 5.6.2</u> Let $g : \mathbb{R}^+ \longrightarrow \mathbb{C}^*$ be a function such that $g = h \circ \ln$ where h is a function on $\mathbb{R}$ into $\mathbb{C}^*$. Then g is continuous if and only if h is continuous.

<u>Proof</u> Assume that h is continuous. Then $g = h \circ \ln$, being the composition of two continuous function, is also continuous.

Coversely, Assume that g is continuous. Let O be any open set in $\mathbb{C}^*$. Since g is continuous and ln is open, hence $\ln(g^{-1}(O))$ is an open set in $\mathbb{R}$. However $h^{-1}(O) = \ln(g^{-1}(O))$, which implies that $h^{-1}(O)$ is open. Hence h is continuous.

<u>Theorem 5.6.3</u> Let $f : \mathbb{R}^* \longrightarrow \mathbb{C}^*$ be a function such that $f = g \circ h$ where g is a function on $\mathbb{R}^+$ into $\mathbb{C}^*$ and $h : \mathbb{R}^* \longrightarrow \mathbb{R}^+$ is defined by $h(x) = |x|$. Then f is continuous if and only if g is continuous.

<u>Proof</u> Since $h : \mathbb{R}^* \longrightarrow \mathbb{R}^+$, defined by $h(x) = |x|$, is continuous and open, hence we can verify in the same way as in the proof of Theorem 5.6.2 that f is continuous if and only if g is continuous

<u>Theorem 5.6.4</u> There exist discontinuous solutions of

$$f(xy) = f(x)f(y)$$

on $\mathbb{R}^*$ to $\mathbb{C}^*$.

<u>Proof</u> Let $h : \mathbb{R} \longrightarrow \mathbb{C}^*$ be a discontinuous solution of

$$h(x + y) = h(x)h(y).$$

The existence of such h is guaranteed by Theorem 5.6.1
Let $g = h \circ \ln$, $f = g \circ k$ where $k : \mathbb{R}^* \longrightarrow \mathbb{R}^+$ is defined by $k(x) = |x|$. By Theorem 5.6.3, f is continuous if and only if g is continuous. By Theorem 5.6.2, g is continuous if and only if h is continuous. Hence f is discontinuous. Therefore discontinuous solutions of $f(xy) = f(x)f(y)$ on $\mathbb{R}^*$ to $\mathbb{C}^*$ exist.