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ON FUNDAMENTAL INVERSE SEMIGROUPS

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บทคัดย่อ

มันน์ได้ให้ลักษณะของเซมิกรุปผกผันหลักในรูปของเซมิกรุปของแมปปิง เข็นและฮซิห์ได้พิสูจน์ว่าทุก ๆ เซมิกรุปผกผันสมมาตรบนเซทใด ๆ เป็นเซมิกรุปผกผันหลักเสมอ ในวิทยา-นิพนธ์นี้เราได้ศึกษาต่างออกไปเกี่ยวกับคุณสมบัติของการเป็นเซมิกรุปผกผันหลัก

เราได้แสดงว่าโดยทั่วไปแล้ว เซมิกรุปย่อยผกผันและโฮโมมอร์ฟิคอิมเมจของเซมิ-กรุปผกผันหลักนั้นไม่จำเป็นต้องเป็นเซมิกรุปผกผันหลัก และเราได้พิสูจน์ว่าโฮโมมอร์ฟิซึมบนเซมิกรุปผกผันหลักซึ่งเป็น 1-1 บนเซทของไอเดมโพเทนท์ของเซมิกรุปนั้นเป็นไอซอมอร์ฟิซึม

ในวิทยานิพนธ์นี้ เราได้พิสูจน์ว่าไอเดียลใด ๆ ของเซมิกรุปผกผันหลักเป็นเซมิกรุปผกผันหลักเสมอ ยิ่งกว่านั้นยังพิสูจน์ได้ว่าเซมิกรุปผกผัน S เป็นเซมิกรุปผกผันหลักเมื่อและต่อเมื่อทุก ๆ ไอเดียลพรินซิพาลของเซมิกรุป S นั้นเป็นเซมิกรุปผกผันหลัก เราได้ให้ตัวอย่างเพื่อแสดงว่าเซมิกรุปรีส์คอเชียนของเซมิกรุปผกผันหลักนั้นไม่จำเป็นต้องเป็นเซมิกรุปผกผันหลัก อย่างไรก็ตามเราได้พิสูจน์สิ่งต่อไปนี้ ถ้าไอเดียล A ของเซมิกรุปผกผัน S และเซมิกรุปรีคอ-เชียน S/A เป็นเซมิกรุปผกผันหลักแล้ว S เป็นเซมิกรุปผกผันหลักด้วย

เราได้ศึกษาเซมิแลตติสของเซมิกรุปผกผันด้วยดังนี้ ให้ $S = \bigcup_{\alpha \in Y} S_\alpha$ เป็นเซมิแลตติส Y ของเซมิกรุปผกผัน S_α แต่ละไอเดียล I ของ Y ให้ $A_I = \bigcup_{\alpha \in I} S_\alpha$ แล้ว

A_I ต้องเป็นไอเดียลของ S สำหรับทุก ๆ ไอเดียล I ของ Y และเราได้พิสูจน์สิ่งต่อไปนี้

(๑) ถ้า S_α เป็นเซมิกรุปผกผันหลักทุก ๆ α ใน Y แล้ว S ต้องเป็นเซมิกรุปผกผันหลักด้วย แต่บทกลับของทฤษฎี นี้ไม่เป็นจริงโดยทั่วไป

(๒) A_α เป็นเซมิกรุปผกผันหลักทุก ๆ α ใน Y เมื่อและต่อเมื่อ S เป็นเซมิกรุปผกผันหลัก โดยกำหนดว่า A_α คือไอเดียล $A_{\alpha Y}$ ทุก ๆ α ใน Y

(๓) ถ้า S_α เป็นเซมิกรุปผกผันหลักทุก ๆ α ใน Y แล้วไอเดียล A_I และเซมิกรุปรีส์คอเชียน S/A_I เป็นเซมิกรุปผกผันหลักทุก ๆ ไอเดียล I ของ Y

Thesis Title On Fundamental Inverse Semigroups

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ABSTRACT

Munn has characterized a fundamental inverse semigroup as a certain semigroup of mappings. It has been shown by Chen and Hsieh that every symmetric inverse semigroup on any set is always fundamental. Different studies relating to the property of being fundamental are obtained in this thesis.

It is shown that an inverse subsemigroup and a homomorphic image of a fundamental inverse semigroup are not necessarily fundamental. A homomorphism from a fundamental inverse semigroup which is one-to-one on the set of its idempotents is an isomorphism.

It is proved that any ideal of a fundamental inverse semigroup is always fundamental. An inverse semigroup is fundamental if and only if all of its principal ideals are fundamental. An example to show that a Rees quotient semigroup of a fundamental inverse semigroup need not be fundamental is given. However, we have the following : If an ideal A of an inverse semigroup S and its

Rees quotient semigroup S/A are fundamental, then S is fundamental.

Semilattices of inverse semigroups are studies. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α, and for each ideal I of Y, let $A_I = \bigcup_{\alpha \in I} S_\alpha$. Then A_I is an ideal of S for all ideals I of Y. The following are proved :

(1) If S_α is fundamental for all $\alpha \in Y$, then S is fundamental, but the converse is not true in general.

(2) A_α is fundamental for all $\alpha \in Y$ if and only if S is fundamental, where A_α denotes the ideal $A_{\alpha Y}$ for all $\alpha \in Y$.

(3) If S_α is fundamental for all $\alpha \in Y$, then the ideal A_I and the Rees quotient semigroup S/A_I are fundamental for all ideals I of Y.

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CONTENTS

INTRODUCTION

Let S be a semigroup. An element 0 of S is a <u>zero</u> of S if $x0 = 0x = 0$ for all $x \in S$. An element 1 of S is an <u>identity</u> of S if for all $x \in S$, $x1 = 1x = x$.

A zero and an identity of a semigroup are unique.

Let S be a semigroup (with or without identity) and 1 be a symbol not representing any element in S. Define the multiplication $\cdot$ on $S \cup 1$ as follows : For $a, b \in S \cup 1$,

$$a \cdot b = \begin{cases} ab & \text{if } a, b \in S, \\ 1 & \text{if } a = b = 1, \\ a & \text{if } a \in S \text{ and } b = 1, \\ b & \text{if } a = 1 \text{ and } b \in S. \end{cases}$$

Under this multiplication, $S \cup 1$ is a semigroup with the identity 1. For a semigroup S, the notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has its identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

Let S be a semigroup. An element a of S is an <u>idempotent</u> of S if $a^2 = a$. We denote by E(S) the set of all idempotents of S, that is,

$$E(S) = \{ e \in S \mid e^2 = e \} .$$

S is called a <u>semilattice</u> if for all $a, b \in S$, $a^2 = a$ and $ab = ba$.

An element a of a semigroup S is <u>regular</u> if $a = axa$ for some

$x \in S$. A semigroup S is <u>regular</u> if every element of S is regular.

In any semigroup S, if $a, x \in S$ such that $a = axa$, then ax and xa are idempotents of S. Hence if S is a regular semigroup, then $E(S) \neq \phi$.

Let a and x be elements of a semigroup S such that $a = axa$. Then

(i) $aS = aS^1$ and $S^1a = Sa$, and

(ii) $aS = axS$ and $Sxa = Sa$.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if $a = axa$, $x = xax$. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a of S is denoted by a^{-1}. Then for any element a of the inverse semigroup S, we have

$$a = aa^{-1}a \quad , \quad a^{-1} = a^{-1}aa^{-1}$$

and aa^{-1}, $a^{-1}a \in E(S)$. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [[2], Theorem 1.17] . Hence, if S is an inverse semigroup, then E(S) is a semilattice. Every semilattice is obviously an inverse semigroup. For any elements a,b of an inverse semigroup S and $e \in E(S)$, we have

$$e^{-1} = e \;,\; (a^{-1})^{-1} = a \text{ and } (ab)^{-1} = b^{-1}a^{-1}$$ [[2], Lemma 1.18] .

Let S be an inverse semigroup. For any $a \in S$, $e \in E(S)$, we have that aa^{-1}, $a^{-1}a$, $a^{-1}ea$, aea^{-1} are all idempotents of S.

Every group is an inverse semigroup and the identity of the group is its only idempotent.

Let P be a nonempty set and $\leq$ be a relation on P. If the relation $\leq$ is reflexive, antisymmetric and transitive, then $\leq$ is called a partial order on P, and (P, $\leq$), or P, is called a partially ordered set.

The relation $\leq$ defined on an inverse semigroup S by

$$a \leq b \iff aa^{-1} = ab^{-1}$$

is a partial order on S [[3], Lemma 7.2] , and this partial order is called the natural partial order on the inverse semigroup S. We note that the restriction of the natural partial order $\leq$ on an inverse semigroup S to E(S) is as follows :

$$e \leq f \iff e = ef \quad (= fe) .$$

It then follows that if S is a semilattice, $a \leq b$ in S if and only if $a = ab$ ($= ba$) .

If S is an inverse semigroup and $a,b \in S$, then the following hold :

(i) $a \leq b$ if and only if $a = be$ for some $e \in E(S)$.

(ii) $a \leq b$ if and only if $a = fb$ for some $f \in E(S)$.

Let E be a semilattice. Then for any $e,f \in E$, $e \leq f$ if and only if $e = xf$ ($= fx$) for some $x \in E$.

Let S and T be semigroups and $\psi : S \to T$ be a map. ψ is a homomorphism from S into T if and only if

$$(ab)\psi = (a\psi)(b\psi)$$

for all $a,b \in S$. ψ is called an isomorphism if ψ is a homomorphism and one-to-one.

A semigroup T is called a homomorphic image of a semigroup S if there exists a homomorphism from S onto T.

Let a semigroup T be a homomorphic image of a semigroup S by a homomorphism ψ. If S is an inverse semigroup, then $T = S\psi$ is an inverse semigroup, for any $a \in S$, $(a\psi)^{-1} = a^{-1}\psi$ [[3], Theorem 7.36] , and moreover, for each $f \in E(T)$, there exists $e \in E(S)$ such that $e\psi = f$ [[3] , Lemma 7.34] , and hence

$$E(T) = \{ e\psi \mid e \in E(S) \} .$$

A reflexive, symmetric and transitive relation on a nonempty set X is an equivalence relation on X.

Let S be a semigroup. A relation ρ on S is called left compatible if for $a,b,c \in S$, $a\rho b$ imply $ca\rho cb$. Right compatibility is defined dually. An equivalence relation ρ on S is called a congruence on S if it is both left compatible and right compatible.

Let ρ be a congruence on a semigroup S. Let S/ρ denote the set of all ρ-classes on S, that is,

$$S/\rho = \{ a\rho \mid a \in S \}.$$

Define a multiplication on S/ρ by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a,b \in S) .$$

Then, under this operation, S/ρ is a semigroup which is called the quotient semigroup relative to the congruence ρ. If $i = \{(a,a) \mid a \in S\}$, then i is a congruence on S and we call it the identity congruence on S.

Let ρ be a congruence on a semigroup S. Then the mapping $\psi : S \longrightarrow S/\rho$ defined by

$$a\psi \quad = \quad a\rho \qquad\qquad (a \in S)$$

is an onto homomorphism and ψ will be denoted by $\rho^\natural$, and call it the *natural homomorphism* of S onto S/ρ . Therefore S/ρ is a homomorphic image of S.

Conversely, if $\psi : S \rightarrow T$ is a homomorphism from a semigroup S into a semigroup T, then the relation ρ on S defined by

$$a\rho b \iff a\psi = b\psi \qquad\qquad (a,b \in S)$$

is a congruence on S and $S/\rho \cong S\psi$, and ρ is called the *congruence on S induced by* ψ.

Let ρ be a congruence on an inverse semigroup S. Then S/ρ is an inverse semigroup, and for every $a\rho \in S/\rho$, $(a\rho)^{-1} = a^{-1}\rho$. Hence for all $a,b \in S$.

$$a\rho b \iff a^{-1}\rho\, b^{-1} .$$

For any $a\rho \in E(S/\rho)$, there exists $e \in E(S)$ such that $a\rho = e\rho$. Hence

$$E(S/\rho) = \{ e\rho \mid e \in E(S) \}.$$

Let S be a semigroup. A nonempty subset A of S is a *left ideal* of S if $xa \in A$ for all $x \in S$, $a \in A$. A *right ideal* of S is defined dually. A nonempty subset of S is an *ideal* (or two-sided ideal) of S if it is both a left ideal and a right ideal of S. An arbitrary intersection of left ideals, of right ideals and of ideals of a semigroup S if nonempty is a left ideal, a right ideal and an ideal of S; respectively.

Let A be a nonempty subset of a semigroup S. The *left ideal* of S generated by A is the intersection of all left ideals of S containing A. The right ideal of S generated by A is defined dually. The ideal of S generated by A is the intersection of all ideals of S containing A. If A contains only one element, say a, the left ideal of S generated by A is called the principal left ideal of S generated by a, the principal right ideal of S generated by a and the principal ideal of S generated by a are defined similarly.

Let a be an element of a semigroup S. Then we have S^1a, aS^1 and S^1aS^1 are the principal left ideal of S generated by a, the principal right ideal of S generated by a and the principal ideal of S generated by a ; respectively.

If S is a regular semigroup, then $S^1a = Sa$, $aS^1 = aS$ and $S^1aS^1 = SaS$ for all $a \in S$. If E is a semilattice, then an ideal I of E is principal if and only if $I = eE = Ee = EeE$ for some $e \in E$.

Let S be a semigroup. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{J}$ on S are defined as follow :

$a \mathcal{L} b$ if and only if $S^1a = S^1b$.

$a \mathcal{R} b$ if and only if $aS^1 = bS^1$.

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

$a \mathcal{J} b$ if and only if $S^1aS^1 = S^1bS^1$.

Note that $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ are equivalence relations on S and $\mathcal{H} \subseteq \mathcal{L}$, $\mathcal{H} \subseteq \mathcal{R}$, $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Moreover, $\mathcal{L}$ is right compatible on S and $\mathcal{R}$ is left compatible on S. These relations are called Green's relations on S. Equivalent definitions of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S can be given as follow : For $a,b \in S$,

$$a \mathcal{L} b \iff a = xb, \ b = ya \quad \text{for some } x,y \in S^1$$

and

$$a \mathcal{R} b \iff a = bx, \ b = ay \quad \text{for some } x,y \in S^1.$$

Let S be a regular semigroup and $a,b \in S$. Then

$$a \mathcal{L} b \iff Sa = Sb$$

$$\iff a = xb, \ b = ya \quad \text{for some } x,y \in S,$$

and

$$a \mathcal{R} b \iff aS = bS$$

$$\iff a = bx, \ b = ay \quad \text{for some } x,y \in S.$$

Any $\mathcal{H}$-class of S containing an idempotent e, H_e, is a subgroup of S [[2], Theorem 2.16], and moreover, H_e is the greatest subgroup of S having e as its identity.

Let S be an inverse semigroup. For $a \in S$, $a = aa^{-1}a$ so that $Sa = Sa^{-1}a$ and $aS = aa^{-1}S$ and hence $a \mathcal{L} a^{-1}a$ and $a \mathcal{R} a^{-1}a$.

Every $\mathcal{L}$-class and every $\mathcal{R}$-class of an inverse semigroup S contains exactly one idempotent [[2] , Theorem 1.17] .

Let S be a semigroup and let A be an ideal of S. Then the relation ρ defined by

$$a \rho b \quad \text{if and only if either } a,b \in A \text{ or } a = b$$

is a congruence on S and it is called the <u>Rees congruence induced by</u> A and the semigroup S/ρ is called the <u>Rees quotient semigroup induced by</u> A and denoted by S/A. Hence

$$a\rho = \begin{cases} \{a\} & \text{if } a \notin A, \\ A & \text{if } a \in A. \end{cases}$$

and S/A is a semigroup with zero, which is a homomorphic image of S, and for $a \in S$, $a\rho$ is the zero of S/A if and only if $a \in A$.

Let A be an ideal of a semigroup S. Then S is an inverse semigroup if and only if A and the Rees quotient S/A are both inverse semigroups [[3] , Corollary 7.37] .

Let S be a semigroup. The center of S, denoted by C(S), is the set $\{ a \in S \mid ax = xa \text{ for all } x \in S \}$.

Let Y be a semilattice and a semigroup $S = \bigcup_{\alpha \in Y} S_\alpha$ be a disjoint union of the subsemigroups S_α of S. The semigroup S is called a semilattice Y of semigroups S_α if $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, or equivalently, if $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ imply $ab \in S_{\alpha\beta}$.

A semilattice of inverse semigroups is an inverse semigroup [[3] , Theorem 7.5] . Then a semilattice Y of groups is an inverse semigroup.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α . To each $\alpha \in Y$, let e_α denote the identity of the group G_α. Then

$$E(S) = \{ e_\alpha \mid \alpha \in Y \} ,$$

and E(S) is contained in the center of S [[2], Lemma 4.8] . Because S is an inverse semigroup, $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in Y$ and hence $E(S) \cong Y$ by the isomorphism $e_\alpha \longmapsto \alpha$ $(\alpha \in Y)$. Moreover, S has an identity if and only if Y has an identity.

A semigroup S is called fundamental if the identity congruence on S is the only congruence on S contained in the Green's relation $\mathcal{H}$ of S.

In the first chapter, we introduce a significant result relating to fundamental inverse semigroups which has been given by Munn. He has charactorized a fundamental inverse semigroup as a certain semigroup of mappings. Including in this chapter, an example to show that an inverse subsemigroup and a homomorphic image of a fundamental inverse semigroup are not necessarily fundamental is given. It is also shown that any homomorphism from a fundamental inverse semigroup which is one-to-one on the set of its idempotents is an isomorphism.

Ideals and Rees quotient semigroups relating to the property of being fundamental are studied in the second chapter. It is shown that any ideal of a fundamental inverse semigroup is always fundamental. Let A be an ideal of an inverse semigroup S. It is proved that if A and the Rees quotient semigroup S/A are fundamental, then S is fundamental. An example to show that a Rees quotient semigroup of a fundamental inverse semigroup need not be fundamental is given.

In the last chapter, semilattices of inverse semigroups are studied. It is proved that a semilattice of fundamental inverse semigroups is fundamental. A weaker condition for a semilattice of inverse semigroups to be fundamental in term of ideals is also given in this chapter.