

CHAPTER III

THEORY



3.1 Cyclic Extraction

3.1.1 Unsteady State Process and Cyclic Steady State

Cyclic extraction is an unsteady-state process in which the concentrations of both phases (extract & raffinate) in all parts of the column change with time. Since the concentration of raffinate and extract phases is a function of time. Therefore the process is a unsteady state process.

Despite the fact that cyclic extraction is an unsteady state process, the whole system is said to be in a cyclic steady-state condition when the concentration at any point within the column varies with time in the same manner from one cycle to another as shown in Figure 3.1

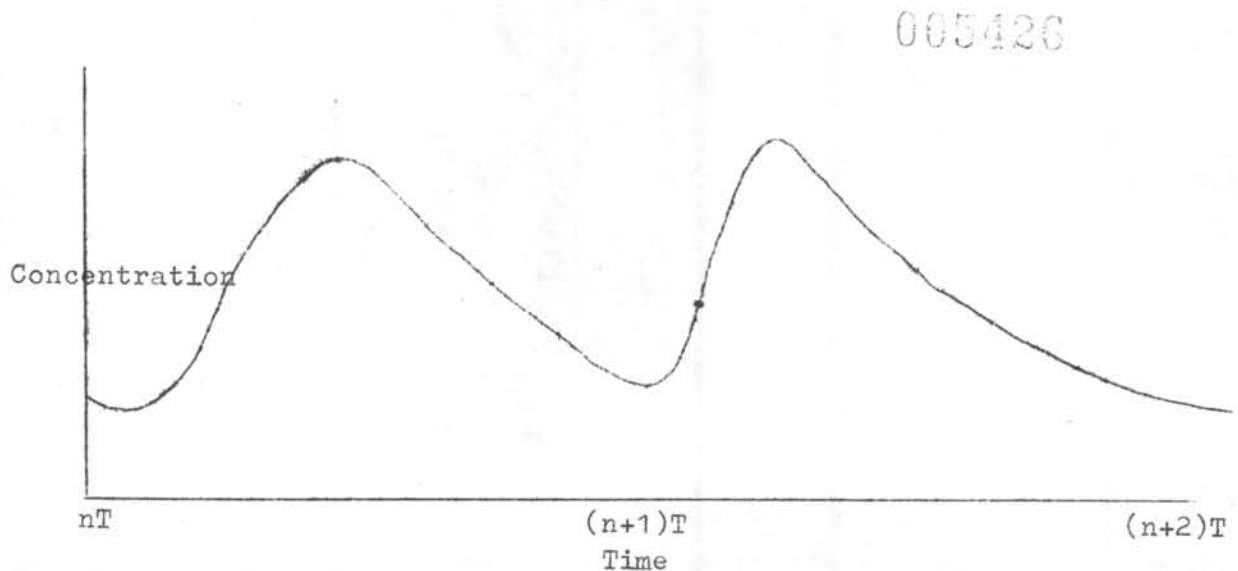


Figure 3.1 Concentration fluctuation at cyclic steady state

The concentration of raffinate and extract leaving the column is constant from one cycle to another when the whole system has reached the cyclic steady state.

3.2 Interphase Mass Transfer

In most of mass transfer operation, two insoluble phases are brought in to contact to permit transfer of constituent substances between them. Therefore we are now concerned with the simultaneous application of the diffusional mechanism for each phase. We have seen that the rate of diffusion within each phase is depend on concentration gradient. At the same time the concentration gradients of the two phase system are indicative of the departure from equilibrium which exists between phases. It is necessary, therefore to consider both the diffusional phenomena and equilibrium in order to describe the system fully. The common principles for all systems involving the distribution of a substance between two insoluble phases are:-

1. At a fixed set of conditions, referring to temperature and pressure, there exists a set of equilibrium relationships which may be shown graphically in the form of an equilibrium-distribution curve by plotting the equilibrium concentrations in the two phases one against the other.
2. For a system in equilibrium, there is no net diffusion of the components between the phases
3. For a system not in equilibrium, diffusion of the components between the phases will occur in such a maner as to being the system to a condition of equilibrium.

3.3 Effect of Mass Transfer on Drop Formation

For liquid-liquid mass transfer operations in which drops of one liquid are dispersed throughout another, the size distribution of the drops is of great importance because the rate of mass transfer depends directly on the interfacial area. The size distribution of drops resulting from the injection of one liquid of an immiscible pair into the another has been studied as a function of flow conditions and physical properties of each phase by Hayworth and Treybal (1950) and Keith and Hixson (1955).

3.4 Number of Ideal Stage

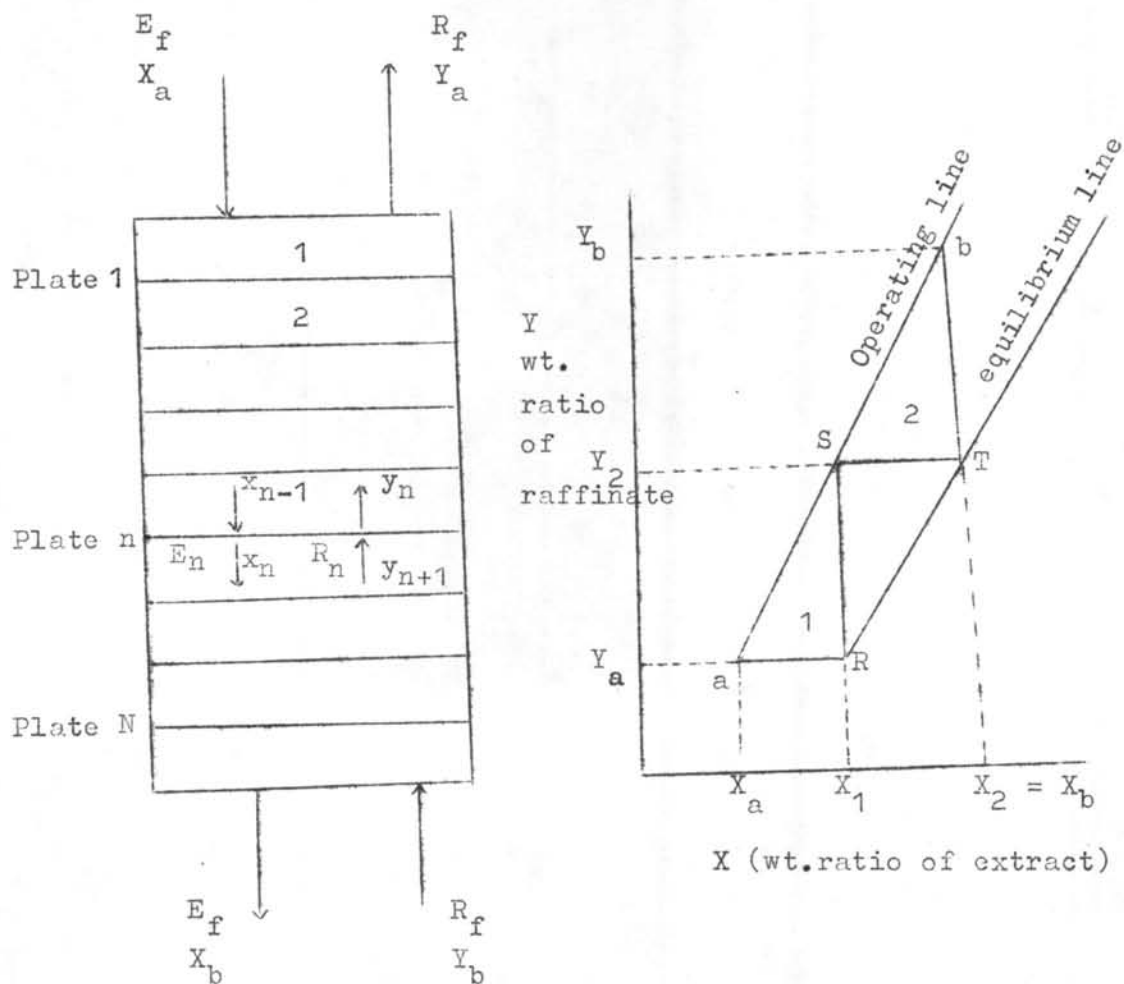


Figure 3.2 Graphical determination of number of ideal stages

The ideal stage is a standard to which an actual stage may be compared. In an ideal stage, the raffinate phase leaving the stage is equilibrium with extract phase leaving the same stage.

To use ideal stages in design, it is necessary to apply a correct factor, called the stage efficiency or plate efficiency, which relates ideal stage to an actual one.

3.4.1 Graphical Method for Evaluating Number of Ideal Stage.

The graphical determination of the number of ideal stages is generally satisfactory method. The simplest graphical method, is based on the use of the operating line and equilibrium line. An example of stage calculation for simple counter current is shown in Figure 3.2. The operating line and equilibrium line may be plotted by knowing all four of compositions at both ends of the cascade. The ends of operating line are point a, having coordinates (X_a, Y_a) and point b, having coordinates (X_b, Y_b) . It is desired to determine the number of ideal stages to accomplish the raffinate phase concentration change Y_b to Y_a and extract phase concentration change X_a to X_b . The problem is solved as follows.

The concentration of raffinate (light phase) leaving the top stage, stage 1, is Y_a or Y_1 . If the stage is ideal X_1 , the concentration of extract phase (heavy phase) leaving this stage is equilibrium. Then the point (X_1, Y_1) must line on the equilibrium line. This fact fixed point R found by moving horizontal line from point a to equilibrium line. The abscissa of point R is X_1 . The operating line is now used. It passes through all points having coordinates of the type (X_n, Y_{n+1}) , and since X_1 is known, Y_2 is found by moving vertical from point R

to the operating line at point S, the coordinates of which are (X_1, Y_2) . The step, or triangle, defined by points a, R and S represents one ideal stage, the first one in this column. The second stage is located graphically on the diagram by repeating the same construction, passing horizontally to the equilibrium curve at point T, having coordinates (X_2, Y_2) , and vertically to the operating line again at point b, having coordinates (X_2, Y_b) . For the situation shown in Figure 3.2, The second stage is the last, as the concentration of light phase leaving that stage is Y_b , and the heavy phase leaving is X_b , which are the desired terminal concentrations. Two ideal stages are required for the separation.

3.4.2 Method for Calculating the Number of Ideal Stages

When the operating and equilibrium lines are both straight over a given concentration range X_a to X_b , the number of ideal stage can be calculated by formula, and graphical construction is unnecessary. Formulas for this purpose are derived as follows.

Let the equation of the equilibrium line be

$$Y_e = mX_e + B \dots\dots\dots(3.1)$$

where by definition, m and B are constant. If stage n is ideal

$$Y_n = mX_n + B \dots\dots\dots(3.2)$$

substitution for X_n into equation of overall material balance around level a & n+1 we get,

$$E_f \cdot X_a + R_f \cdot Y_{n+1} = E_f \cdot X_n + R_f \cdot Y_a \dots\dots\dots(3.3)$$

for ideal stage and constant E_f/R_f

$$Y_{n+1} = \frac{E_f (Y_n - B)}{mR_f} + Y_a - \frac{E_f \cdot Y_a}{R_f} \dots\dots\dots(3.4)$$

Equation (3.8) can be used to calculate, step by step, the value of Y_{n+1} for each stage starting with stage 1. The method may be followed with the aid of Figure 3.2

For stage 1. using $n=1$ in equation (3.8) and noting that $Y_1 = Y_a$

$$Y_2 = AY_a - AY_a^* + Y_a = Y_a(1+A) - AY_a^*$$

For stage 2. using $n = 2$ in equation (3.8) and eliminating Y_2

$$\begin{aligned} Y_3 &= AY_2 - AY_a^* + Y_a = A Y_a(1+A) - AY_a^* - AY_a^* + Y_a \\ &= Y_a(1+A+A^2) - Y_a^*(A+A^2) \end{aligned}$$

These equation may be generalized for the n^{th} stage, giving

$$Y_{n+1} = Y_a(1+A+A^2 + \dots + A^n) - Y_a^*(A+A^2 + \dots + A^n)$$

for the entire cascade, $n=N$, the total number of stages,

and
$$Y_{n+1} = Y_{N+1} = Y_b$$

Then

$$Y_b = Y_a(1+A+A^2 + \dots + A^N) - Y_a^*(A+A^2 + \dots + A^N) \quad \text{---(3.9)}$$

Equation (3.9) can be written

$$Y_b = Y_a \left(\frac{1-A^{N+1}}{1-A} \right) - Y_a^* A \left(\frac{1-A^N}{1-A} \right) \quad \text{---(3.10)}$$

Equation (3.10) is a form of the Kremer equation. It can be used as such as in the form of chart relating N, A , and the terminal concentrations. It can also be put it to a simpler form by the following method.

Equation (3.8) is, for stage N

$$Y_b = AY_N - AY_a^* + Y_a \quad \text{---(3.11)}$$

It can be seen from Figure 3.2 that $Y_N = Y_b^*$ and Eq.(3.11) can be written

$$Y_a = Y_b - A(Y_b^* - Y_a^*) \quad \text{-----}(3.12)$$

Collecting terms in Equation (3.10) containing A^{N+1} gives

$$A^{N+1}(Y_a - Y_a^*) = A(Y_b - Y_a^*) + (Y_a - Y_b) \quad \text{-----}(3.13)$$

Substituting $Y_a - Y_b$ from (3.12) into equation (3.13) give

$$A^N(Y_a - Y_a^*) = Y_b - Y_a^* - Y_b^* + Y_a = Y_b - Y_b^* \quad \text{-----}(3.14)$$

Taking logarithms of equation (3.14) and solving for N gives

$$N = \frac{\log \frac{Y_b - Y_b^*}{Y_a - Y_a^*}}{\log A} \quad \text{-----}(3.15)$$

and from equation (3.12)

$$\frac{Y_b - Y_a}{Y_b^* - Y_a^*} = A \quad \text{-----}(3.16)$$

Equation (3.15) can be written

$$N = \frac{\log \frac{Y_b - Y_b^*}{Y_a - Y_a^*}}{\frac{Y_b - Y_a}{Y_b^* - Y_a^*}} \quad \text{-----}(3.17)$$

The various concentration differences in equation (3.17) are shown in Figure (3.4)

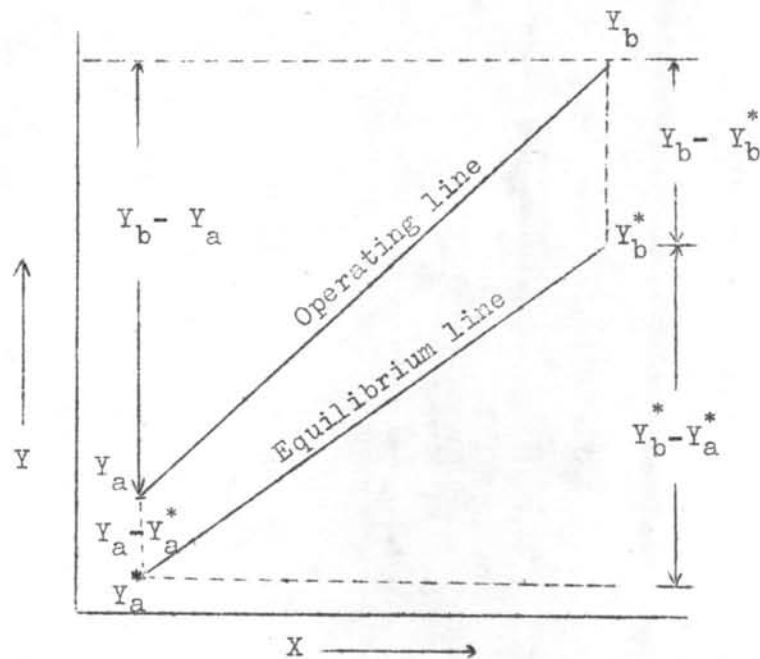


Figure 3.4 Concentration differences in equation (3.17)

Heavy phase Form of Equation (3.17). The choice of Y as the concentration coordinate in the above equations rather than X is arbitrary and equations analogous to equation (3.15) and (3.17) in X can be derived

They are

$$N = \frac{\log \frac{X_b^* - X_b}{X_a^* - X_a}}{\log A} = \frac{\log \frac{X_b^* - X_b}{X_a^* - X_a}}{\log \frac{X_b^* - X_a^*}{X_b - X_a}} \quad \text{-----(3.18)}$$

where X^* is the equilibrium concentration corresponding to Y. The concentration differences in equation (3.18) are shown in Figure 3.5

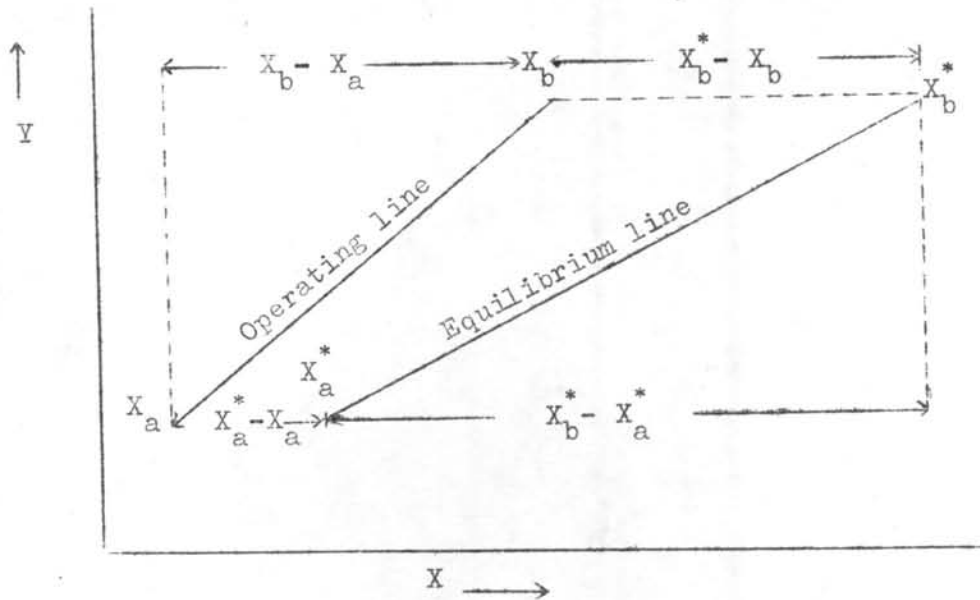


Figure 3.5 Concentration differences in equation (3.18)

3.5 The Transfer Unit

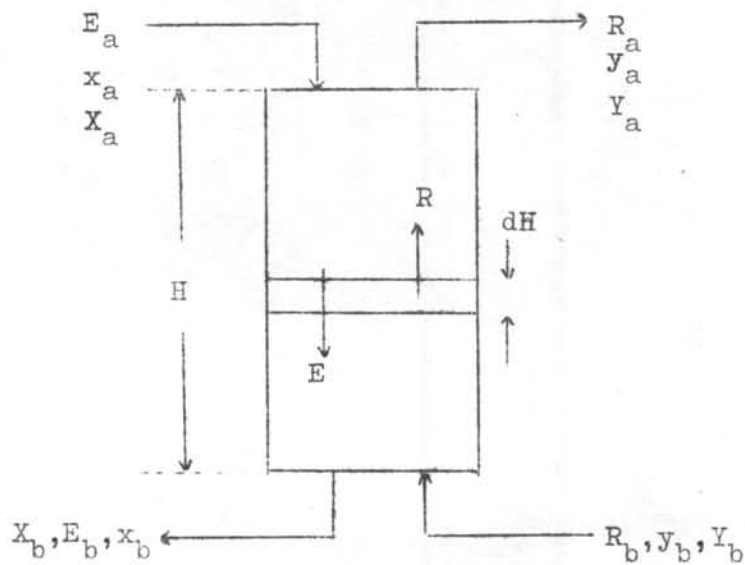


Figure 3.6 Extraction with counter current contact

3.5.1 Individual-film Transfer Unit

Consider the extraction column of Fig. 3.6, where raffinate and extract phase flow countercurrently. The raffinate phase, enter at a rate R_b moles/(hr)(sq.ft.of tower), with a concentration of distributed solute y_b mole fraction, and leaves at a rate R_a with concentration y_a . Similarly, the extract phase enters at a rate E_a moles/(hr)(sq.ft.), undergoes a concentration change x_a to x_b , and leaves at a rate E_b . For a crosssectional area of the tower is s sq.ft. and the total interfacial surface between phase A ft^2 , the exposed interfacial surface is a sq.ft/cu.ft. of tower. At some position in the tower where raffinate and extract rates are R and E , a differential change in concentration of these streams occurs over a differential height dH . This change in concentration resulted from the diffusion of distributed solute from phase R to phase E because of the concentration gradients.

If N_s is the total rate of transfer of solute moles/hr.sq.ft., then the rate of transfer for the differential section can be described by

$$dN_s = d(Ry) = k_R dA C_{RM} (y - y_i) \quad \text{-----}(3.19)$$

where C_{RM} is the average of values C_R and C_{Ri}

This equation limits where solvents are completely immiscible and relatively dilute solutions. Since the total raffinate rate R varies from one end of the tower to the other, but the solute-free raffinate, $R(1 - y_R)$, remains constant. Consequently,

$$d(Ry) = R(1 - y) \frac{d(y)}{(1 - y)} = \frac{R dy}{(1 - y)} \quad \text{-----}(3.20)$$

The mass transfer coefficient K_R includes a term $(1-y)$ in which varies throughout the tower. The quantity $k_R(1-y)_{iM}$ is more likely to be constant. In addition, $dA = aSdH$. Equation 3.19 may be modified,

$$\frac{(1-y)_{iM} R dy}{(1-y)} = k_R a S (1-y)_{iM} C_{RM} (y-y_i) dH \quad \text{----- (3.21)}$$

$$\frac{(1-y)_{iM} dy}{(1-y)(y-y_i)} = \frac{k_R a S (1-y)_{iM} C_{RM} dH}{R} \quad \text{----- (3.22)}$$

Since the terms $(1-y)_{iM}$ and $(1-y)$ are usually nearly unity, the left hand portion of eq. 3.22 is the concentration change per unit of concentration difference causing the change, and represents a measure of the difficulty of extraction. This in turn is designated as NTU, the number of transfer units, which when multiplied by experimentally determined factor HTU, the height per transfer unit, gives the height of the tower. Thus,

$$(NTU) (HTU) = H \quad \text{----- (3.23)}$$

where

$$\begin{aligned} NTU_R &= \int_{y_a}^{y_b} \frac{(1-y)_{iM} dy}{(1-y)(y-y_i)} \\ &= \int_0^H \frac{dH}{HTU_R} \\ &= \frac{H}{HTU_R} \quad \text{----- (3.24)} \end{aligned}$$

Consider equation (3.22) and (3.24) shows that HTU_R and the mass transfer coefficient are related

$$HTU_R = \frac{R}{k_R a (1-y_{iM}) C_{Rav} S} \quad \text{----- (3.25)}$$

In similar fashion, the concentration differences in terms of the extract phase might have been used, which

$$NTU_E = \int_{x_a}^{x_b} \frac{(1-x)_i M dx}{(1-x)(x-x_i)} = \int_0^H \frac{dH}{HTU_E} = \frac{H}{HTU_E} \quad \text{---(3.26)}$$

$$HTU_E = \frac{E}{k_E a (1-x)_i M C_{Eav}} \quad \text{-----(3.27)}$$

3.5.2 Over-all Transfer Units

The practical difficulties entering into the use of true equilibrium interfacial concentrations x_i and y_i have led to the introduction of overall mass transfer coefficient K_E and K_R , which express in terms of overall concentration gradients $(y-y^*)$ and $(x-x^*)$

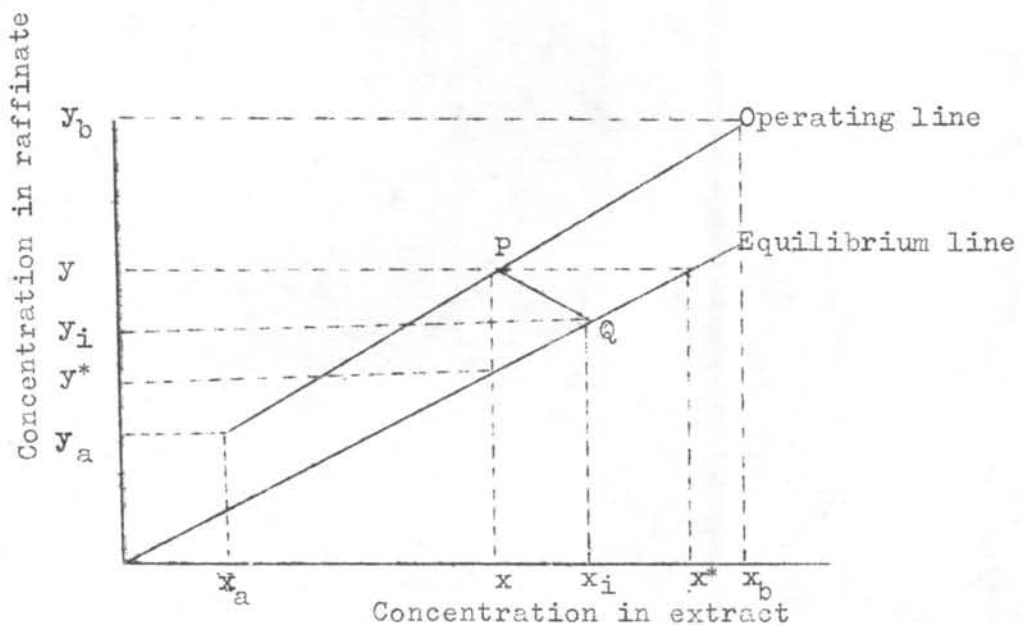


Figure 3.7 Operation diagram for continuous counter-current extraction

In the absence of mass transfer coefficient,

$$m = \frac{x_i}{y_i} = \frac{x^*}{y} = \frac{x}{y^*} \quad \text{-----}(3.28)$$

be truly constant over the range of concentrations encountered in the design. This is a serious restriction, and in practice the equations are usually used regardless of constancy of m

From

$$dN_S = K_E ds (C_E^* - C_E) = K_E dS C_{oE}$$

$$dN_S = K_R ds (C_R - C_R^*) = K_R dS C_{oR}$$

We obtain

$$\begin{aligned} NTUR_o &= \int_{y_a}^{y_b} \frac{(1-y)_{oM} dy}{(1-y)(y-y^*)} \\ &= \int_{y_a}^{y_b} \frac{dy}{(1-y) \ln \left(\frac{1-y^*}{1-y} \right)} = \frac{H}{HTUR_o} \quad \text{---}(3.29) \end{aligned}$$

$$HTUR_o = \frac{R}{K_R a (1-y)_{oM} C_{Rav} S} \quad \text{-----}(3.30)$$

and

$$\begin{aligned} NTUE_o &= \int_{x_a}^{x_b} \frac{(1-x)_{oM} dx}{(1-x)(x-x^*)} \\ &= \int_{x_a}^{x_b} \frac{dx}{(1-x) \ln \left(\frac{1-x^*}{1-x} \right)} = \frac{H}{HTUE_o} \quad \text{----}(3.31) \end{aligned}$$

$$HTUE_o = \frac{E}{K_E a (1-x)_{oM} C_{Eav} S} \quad \text{-----}(3.32)$$

These equations are used for design, equation (3.29) in cases where the principle diffusional resistance lies in the R phase (m large), Equation (3.31) in cases where the principle diffusional resistance lies in the E phase (m small).

To determine the value of $NTUR_O$ or $NTUR_E$, graphical integration of the respective expressions is ordinarily required, for which in turn, operating diagrams, refer to Figure 3.7, where in operating diagram is shown, including equilibrium curve and operating line, directions for obtaining which have previously been described. For any point P on the operating line, the vertical distance to the equilibrium curve gives the overall concentration difference $(y - y^*)$, while the horizontal distance gives $(x^* - x)$. To evaluate $NTUR_O$, either the quantity $\frac{(1-y)_{OM}}{(1-y)(y-y^*)}$ or the quantity $\frac{1}{(1-y)\ln\frac{(1-y^*)}{(1-y)}}$ is calculated for as many points on the operating line as required to give a smooth curve when plotted against y . The area under the resulting curve between the limits x_b and x_a is the required value. $NTUE_O$ may be obtained in a similar manner. Equation (3.24) and equation (3.25) could be evaluated in the same fashion also, if the position of points corresponding to Q could be located.

Considerable effort has been put into the problems of reducing the tedium of the graphic integration for $NTUR_O$ and $NTUE_O$. If $(1-y^*)$ and $(1-y)$ differ by no more than a factor of 2, an arithmetic average rather than a logarithmic average for $(1-y)_{OM}$ incurs an error of 1.5 percent, thus

$$(1-y)_{OM} = \frac{(1-y^*) + (1-y)}{2} \quad \text{-----}(3.33)$$

and substitution in equation (3.29) leads to

$$NTUR_o = \int_{y_a}^{y_b} \frac{dy}{(y-y^*)} + \frac{1}{2} \ln \frac{(1-y_a)}{(1-y_b)}$$

Similarly, $NTUE_o = \int_{x_a}^{x_b} \frac{dx}{(x^*-x)} + \frac{1}{2} \ln \frac{(1-x_b)}{(1-x_a)}$ ----(3.35)

and graphic integration is made of a curve of $\frac{1}{(y-y^*)}$ vs y for $NTUR_o$. Weight fraction are usually more convenient than mole fraction, since the triangular equilibrium diagrams from which the operating diagrams are plotted use this unit most frequently. Thus,

$$x = \frac{r X_W}{r X_W + 1 - X_W} \text{ -----(3.36)}$$

where r is the ratio of molecular weight of non solute substitution in equation (3.29) and (3.31) leads to results that are too awkward to use directly, but if the arithmetic rather than logarithmic average is used for $(1-X_W)_{oM}$, there is obtained

$$NTUR_o = \int_{Y_{Wa}}^{Y_{Wb}} \frac{dY_W}{(Y_W - Y_W^*)} + \frac{1}{2} \ln \frac{(1-Y_{Wa})}{1-Y_{Wb}} + \frac{1}{2} \ln \frac{Y_{Wa}(r-1) + 1}{Y_{Wb}(r-1) + 1} \text{ -----(3.37)}$$

and $NTUE_o = \int_{X_{Wa}}^{X_{Wb}} \frac{dX_W}{(X_W^* - X_W)} + \frac{1}{2} \ln \frac{(1-X_{Wb})}{1-X_{Wa}} + \frac{1}{2} \ln \frac{X_{Wb}(r-1) + 1}{X_{Wa}(r-1) + 1} \text{ -----(3.38)}$

weight ratio, $X = X_W/(1-X_W)$ and $Y = Y_W/(1-Y_W)$

are sometimes convenient since the operating line on the operating diagram is then frequently a straight line. Substitution in equation (3.29) and (3.31) leads to

$$\begin{aligned} \text{NTUE}_o &= \int_{Y_a}^{Y_b} \frac{(1+rY)_{oM} dY}{(1+rY)(Y-Y^*)} \\ &= \int_{Y_a}^{Y_b} \frac{dY}{(Y-Y^*)} + \frac{1}{2} \ln \frac{(1-rY_a)}{(1-rY_b)} \quad \text{-----}(3.39) \end{aligned}$$

and

$$\begin{aligned} \text{NTUE}_o &= \int_{X_a}^{X_b} \frac{(1+rX)_{oM} dX}{(1+rX)(X-X^*)} \\ &= \int_{X_a}^{X_b} \frac{dX}{(X-X^*)} + \frac{1}{2} \ln \frac{1+rX_b}{1+rX_a} \quad \text{-----}(3.40) \end{aligned}$$

where the right hand part of preceding equations include an approximation equation equivalent to that of equation (3.33). Mole ratio concentration units leads to equations identical with equation (3.39) and (3.40) with the exception that r is omitted.

Further simplification can be introduced by evaluation of the integrals of equation (3.34) to (3.40) formally rather than graphically. This can be accomplished with more or less precision depending on the validity of the assumption necessary to permit formal integration. For dilute solutions $(1-Y)$ and $(1-X)$ for system using mass ratio are nearly unity, and R_f and E_f are substantially constant, where R_f is the weight of solute free raffinate phase (gm/hr), and E_f is the weight of solute-free extract phase (gm/hr). A material balance over the

lower portion of the tower of figure 3.6 then becomes

$$R_f(Y - Y_a) = E_f(X - X_a) \quad \text{-----}(3.41)$$

$$\text{For } m = X/Y^*$$

Substitution $X = mY^*$ and rearrangement leads to

$$Y^* = \frac{R_f}{mE_f} (Y - Y_a) + \frac{X_a}{m} \quad \text{-----}(3.42)$$

Define $\xi = \text{diffusion coefficient} = mE_f/R_f$

and inserting in the integral of equation (3.39)

$$NTUR_o = \int_{Y_a}^{Y_b} \frac{dY}{Y - \frac{1}{\xi} (Y - Y_a) + \frac{X_a}{m}} + \frac{1}{2} \ln \frac{1+rY_a}{1+rY_b} \quad \text{---}(3.43)$$

For dilute solution $(1-rY_a) = 1$ and $(1-rY_b) = 1$ and also the term

$$\ln \frac{(1-rY_a)}{(1-rY_b)} = 0$$

So equation 3.43 can be reduced to

$$\begin{aligned} NTUR_o &= \int_{Y_a}^{Y_b} \frac{dr}{Y - \left[\frac{1}{\xi} (Y - Y_a) + \frac{X_a}{m} \right]} \\ &= \int_{Y_a}^{Y_b} \frac{dY}{(1-\frac{1}{\xi}) Y + \left(\frac{1}{\xi} Y_a - \frac{X_a}{m} \right)} \\ &= \frac{1}{1-\frac{1}{\xi}} \int_{Y_a}^{Y_b} \frac{d \left[\left(1-\frac{1}{\xi}\right) Y + \left(\frac{1}{\xi} Y_a - \frac{X_a}{m} \right) \right]}{\left[\left(1-\frac{1}{\xi}\right) Y + \left(\frac{1}{\xi} Y_a - \frac{X_a}{m} \right) \right]} \\ &= \frac{1}{1-\frac{1}{\xi}} \ln \left[\left(1-\frac{1}{\xi}\right) Y + \frac{1}{\xi} Y_a - \frac{X_a}{m} \right] \Big|_{Y_a}^{Y_b} \quad \text{---}(3.44) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(1 - \frac{1}{\xi}\right)Y_b + \left(\frac{1}{\xi}Y_a - \frac{X_a}{m}\right)}{\left(1 - \frac{1}{\xi}\right)Y_a + \left(\frac{1}{\xi}Y_a - \frac{X_a}{m}\right)} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(1 - \frac{1}{\xi}\right)Y_b + \left(\frac{1}{\xi}Y_a - \frac{X_a}{m}\right)}{\left[Y_a - \frac{1}{\xi}Y_a + \frac{1}{\xi}Y_a - \frac{X_a}{m}\right]} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(1 - \frac{1}{\xi}\right)Y_b + \frac{1}{\xi}Y_a - \frac{X_a}{m}}{\left[Y_a - \frac{X_a}{m}\right]} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(1 - \frac{1}{\xi}\right)Y_b + \frac{1}{\xi}Y_a - \frac{X_a}{m}}{\left(Y_a - \frac{X_a}{m}\right)} - \frac{1}{\xi} + \frac{1}{\xi} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(\xi Y_b - Y_b + Y_a - \frac{\xi X_a}{m} - Y_a + \frac{X_a}{m}\right)}{\xi \left(Y_a - \frac{X_a}{m}\right)} + \frac{1}{\xi} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(\xi - 1\right)Y_b - \left(\xi - 1\right)\frac{X_a}{m}}{\xi \left(Y_a - \frac{X_a}{m}\right)} + \frac{1}{\xi} \right] \\
&= \frac{1}{1 - \frac{1}{\xi}} \ln \left[\frac{\left(Y_b - \frac{X_a}{m}\right)\left(\xi - 1\right)}{\xi \left(Y_a - \frac{X_a}{m}\right)} + \frac{1}{\xi} \right]
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1 - \frac{1}{\Sigma}} \ln \left[\frac{(Y_b - X_{a/m}) (1 - \frac{1}{\Sigma}) + \frac{1}{\Sigma}}{(Y_a - X_{a/m})} \right] \\
 \text{NTUR}_o &= \frac{\ln \left[\frac{(Y_b - X_{a/m}) (1 - \frac{1}{\Sigma}) + \frac{1}{\Sigma}}{(Y_a - X_{a/m})} \right]}{(1 - \frac{1}{\Sigma})} \quad \text{-----}(3.45)
 \end{aligned}$$

In the case of no solute contain in extract phase $X_a = 0$ leads to:-

$$\text{NTUR}_o = \frac{\ln \left[\left(\frac{Y_b}{Y_a} \right) (1 - \frac{1}{\Sigma}) + \frac{1}{\Sigma} \right]}{(1 - \frac{1}{\Sigma})} \quad \text{-----}(3.46)$$

In similar maner we can prove that

$$\text{NTUE}_o = \frac{\ln \left[\left(\frac{X_b - mY_a}{X_a - mY_a} \right) (1 - \Sigma) + \Sigma \right]}{1 - \Sigma} \quad \text{-----}(3.47)$$

$$X_a = 0; \quad = \frac{\ln \left[\left(1 - \frac{1}{\Sigma} \frac{X_b}{Y_a} \right) (1 - \Sigma) + \Sigma \right]}{1 - \Sigma} \quad \text{-----}(3.48)$$

So by using equation (3.29) and (3.31) we can evaluate HTUR_o and HTUE_o if we know the height of column.