# CHAPTER II

## PRELIMINARIES

In this thesis, we assume a basic knowledge of the Euclidean Plane [3]. In this chapter we will define the fundamental concepts which are the foundations of integral geometry.

2.1 <u>Definition</u> : A <u>coordinate system</u> on the Euclidean plane X is a homeomorphism $\Phi$ of an open subset on X onto an open set of $\mathbb{R}^2$

2.2 <u>Example</u> : Choose two orthogonal straight lines and choose two directions on each straight line. This allows us to define a homeomorphism $\Phi : X \longmapsto \mathbb{R}^2$ by using directed distances . We call this coordinate system a <u>Rectangular Cartesian Coordinate System</u>.

2.3 <u>Example</u> : Pick a closed infinite ray 1 . Set $U = X - 1$ and define a homeomorphism $\Psi : U \longmapsto \mathbb{R}^2$ by using angle and distance. We remove 1 because we want $\Psi$ to be continuous and call this coordinate system a <u>Polar Coordinate System</u>.

2.4 <u>Definition</u> : Let $U \subset X$ be an open set and $\Phi$ be a homeomorphism of U onto an open set of $\mathbb{R}^2$. We call the pair $(U, \Phi)$ a <u>coordinate neighborhood.</u>

2.5 <u>Definition</u> : Let $U \subset R^n$ be an open set and $f = (f_1, \ldots, f_m)$ be a map from U to $R^m$. We call f a <u>$c^1$ - map</u> if

1. f is continuous
2. $\frac{\partial f_i}{\partial x_j}$ exists, $\forall$ points $\in U$ and is continuous where $\begin{cases} i = 1, \ldots, m \\ j = 1, \ldots, n \end{cases}$

2.6 <u>Definition</u> : Two coordinate neighborhoods $(U_1, \Phi_1)$ and $(U_2, \Phi_2)$ are <u>$c^1$- related</u> if

1. $U_1 \cap U_2 \neq \emptyset$
2. $\Phi_2 \circ \Phi_1^{-1} : \Phi_1 (U_1 \cap U_2) \longrightarrow \Phi_2 (U_1 \cap U_2)$ and $\Phi_1 \circ \Phi_2^{-1} : \Phi_2 (U_1 \cap U_2) \longrightarrow \Phi_1 (U_1 \cap U_2)$ are $c^1$- maps

These two functions are shown in Figure 1 .

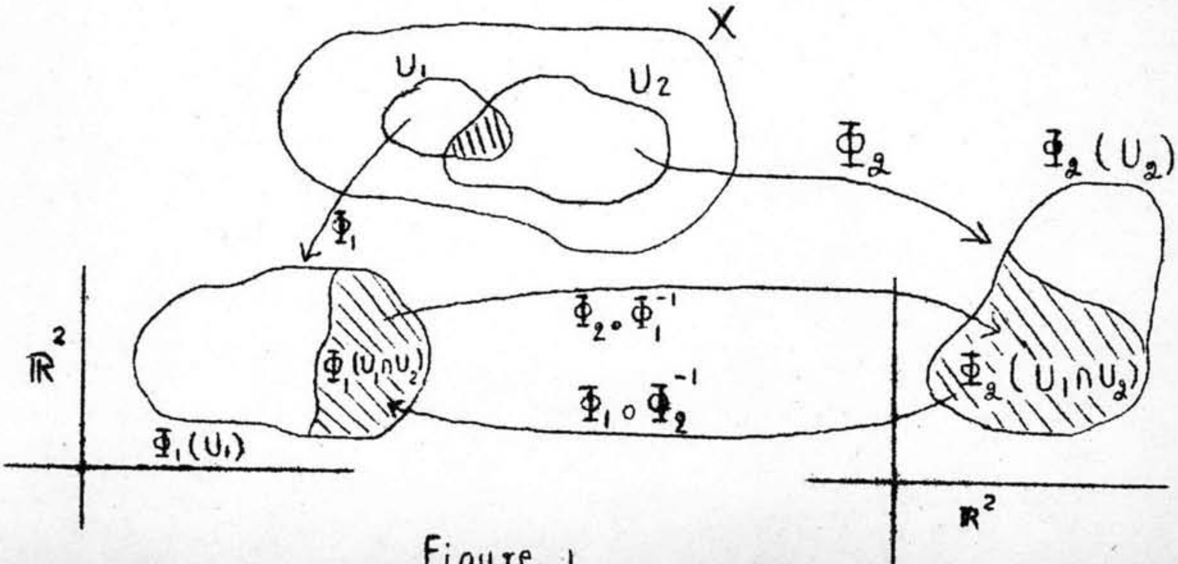


Figure 1.

2.7 Definition : An atlas is a set of coordinate neighborhoods $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ such that $\{U_\alpha\}_{\alpha \in I}$ cover X and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ are all $c^1$- related to each other for all pairs with nonempty intersection.

2.8 Notation : If $F : R^n \to R^n$ then Jac (F) will denote the Jacobian determinant of the function F .

2.9 Remark : If we choose a fixed rectangular cartesian coordinate neighborhood $(X, \Phi_0)$ then we see that $\{(X, \Phi_0)\}$ is an atlas . We can form a new atlas $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in I}$ in the following way :

The coordinate neighborhood $(U_1, \Psi_1)$ belongs to the new atlas $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in I}$ if

1. $(X, \Phi_0)$ and $(U_1, \Psi_1)$ are $c^1$- related
2. Jac $(\Phi_0 \circ \Psi_1^{-1}) > 0$ (note that if Jac $(\Phi_0 \circ \Psi_1^{-1}) > 0$

   then Jac $(\Psi_1 \circ \Phi_0^{-1}) > 0$)

Let $(U_2, \Psi_2)$ be another coordinate neighborhood such that $U_1 \cap U_2 \neq \emptyset$ and is $c^1$- related to $(X, \Phi_0)$ . Claim that $(U_1, \Psi_1)$ and $(U_2, \Psi_2)$ are $c^1$- related to each other. To see this, note that

$$\psi_1 \circ \psi_2^{-1} = \psi_1 \circ \Phi_0^{-1} \circ \Phi_0 \circ \psi_2^{-1} \quad \text{and}$$

$$\psi_2 \circ \psi_1^{-1} = \psi_2 \circ \Phi_0^{-1} \circ \Phi_0 \circ \psi_1^{-1} \quad \text{are } c^1\text{- maps since}$$

$\psi_1 \circ \Phi_0^{-1}$ , $\Phi_0 \circ \psi_2^{-1}$ , $\psi_2 \circ \Phi_0^{-1}$ and $\Phi_0 \circ \psi_1^{-1}$

are all $c'$ - maps. and $\text{Jac}(\psi_1 \circ \psi_2^{-1}) = \text{Jac}(\psi_1 \circ \Phi_0^{-1})\text{Jac}(\Phi_0 \circ \psi_2^{-1}) > 0$

$$\text{Jac}(\psi_2 \circ \psi_1^{-1}) = \text{Jac}(\psi_2 \circ \Phi_0^{-1})\, \text{Jac}(\Phi_0 \circ \psi_1^{-1}) > 0$$

2.10 Example : Let $(X, \Phi_0)$ be a rectangular cartesian coordinate neighborhood and $(U, \psi)$ be the polar coordinate neighborhood which has the same origin and the positive part of x - axis as its removed closed infinite ray. We see that $(X, \Phi_0)$ and $(U, \psi)$ are $c'$ - related. Because $X \cap U \neq \emptyset$ and

$$\Phi_0 \circ \psi^{-1}(r, \theta) = (r \cos \theta , r \sin \theta)$$

$$\psi \circ \Phi_0^{-1}(x,y) = \left(\sqrt{x^2 + y^2} , \tan^{-1} \frac{y}{x}\right) \text{ are } c' \text{- maps}$$

Consider $\text{Jac}(\Phi_0 \circ \psi^{-1}) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} > 0$$

and $$\text{Jac}\,(\psi_1 \circ \Phi_0^{-1}) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r > 0$$

Thus $(U, \psi)$ belong to the new atlas.

2.11 Note : From now on, when we study the Euclidean Plane we shall only use coordinate neighborhoods obtained from the above constructed atlas.

2.12 Definition : Given a function $f : X \longrightarrow R$ and a coordinate neighborhood $(U, \Phi)$ on X . The function $f \circ \Phi^{-1} : \Phi(U) \longrightarrow R$ is called the local representation of the function f in terms of the coordinate neighborhood $(U, \Phi)$ . This function is shown in Figure 2.

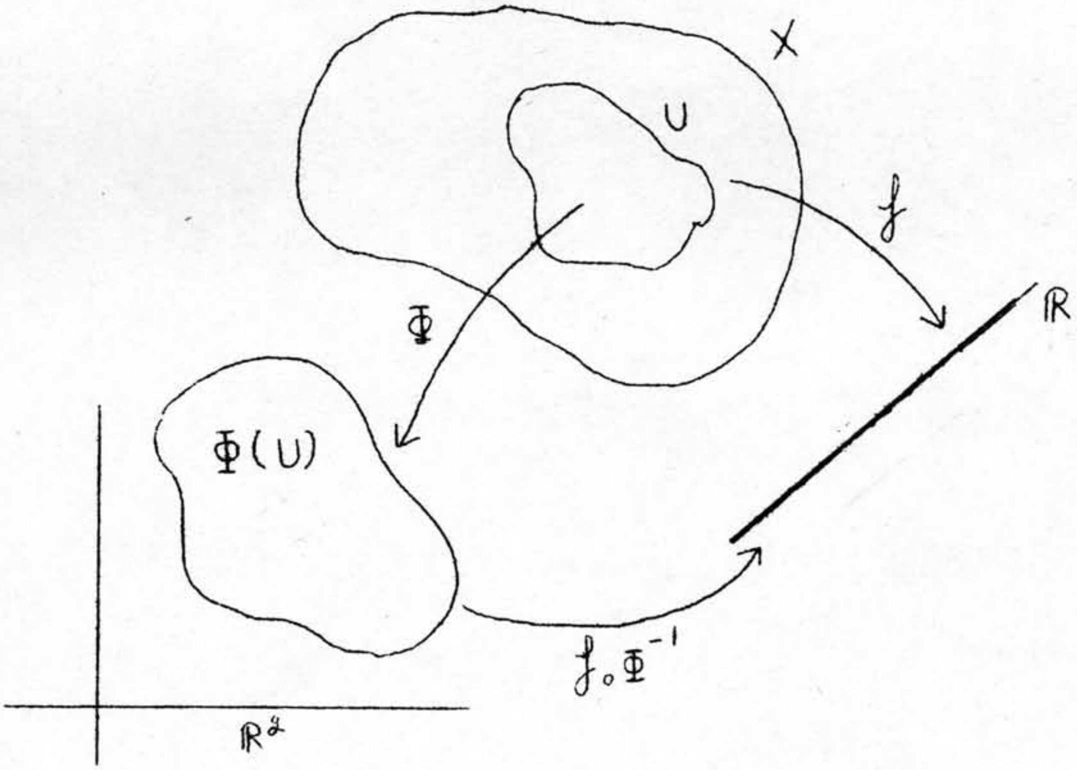


Figure 2.

2.13 <u>Example</u> : Fix $P_0 \in X$ and define $f : X \longrightarrow \mathbb{R}$ by $f(P) = d(P, P_0)$ the distance from P to $P_0$.

In terms of a rectangular cartesian coordinate neighborhood $(X, \Phi_0)$ with $P_0$ as origin, the local representation is $(f_\circ \Phi_0^{-1})(x, y) = \sqrt{x^2 + y^2}$

In terms of a polar coordinate neighborhood $(U, \Psi)$ with $P_0$ as origin, the local representation is $(f_\circ \Psi^{-1})(r, \theta) = r$

2.14 <u>Definition</u> : On $R^2$ we have two projections $\pi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$.

If $(X, \Phi)$ is a coordinate neighborhood then $\pi_1 \circ \Phi : X \longrightarrow \mathbb{R}$ and $\pi_2 \circ \Phi : X \longrightarrow \mathbb{R}$ are called the <u>coordinate functions</u> of the coordinate neighborhood $(X, \Phi)$.

2.15 <u>Example</u> : If $(X, \Phi_0)$ is a rectangular cartesian coordinate neighborhood we usually write $x = \pi_1 \circ \Phi_0$ and $y = \pi_2 \circ \Phi_0$. The functions x and y are called the coordinate functions of $(X, \Phi_0)$. We usually denote $\Phi_0$ by $(x, y)$.

If $(U, \Psi)$ is a polar coordinate neighborhood we usually write $r = \pi_1 \circ \Psi$ and $\theta = \pi_2 \circ \Psi$. The functions r and $\theta$ are called the coordinate functions of $(U, \Psi)$. We usually denote $\Psi$ by $(r, \theta)$.

2.16 <u>Definition</u> : Let $f : X \longrightarrow R$, f is called <u>differentiable at $P \in X$</u> if $\exists$ a coordinate neighborhood $(U, \Phi)$ such that the function $f \circ \Phi^{-1} : \Phi(U) \longrightarrow \mathbb{R}$ is a differentiable function.

Claim that if f is differentiable at P with respect to one coordinate neighborhood then f is differentiable at P with respect to all coordinate neighborhoods. To prove this, let $(V, \Psi)$ be another coordinate neighborhood which belongs to the same atlas. Then $f \circ \Psi^{-1} = f \circ \Phi^{-1} \circ \Phi \circ \Psi^{-1}$ is a differentiable function since $f \circ \Phi^{-1}$ and $\Phi \circ \Psi^{-1}$ are differentiable functions. We see that it is true for all coordinate neighborhoods, so this definition is well - defined.

2.17 <u>Definition</u> : A <u>differential 1 - form</u> $\omega$ on X is a correspondence which assigns to each coordinate neighborhood $(U, \Phi)$ an ordered pair of continuous functions $(f_1, g_1)$ where $f_1 : \Phi(U) \to \mathbb{R}$ and $g_1 : \Phi(U) \to \mathbb{R}$ such that if $(f_1, g_1)$ corresponds to $(U, \Phi)$ and $(f_2, g_2)$ corresponds to $(V, \psi)$ and $U \cap V \neq \emptyset$ then

$$(f_1, g_1) = (f_2 \circ (\psi \circ \Phi^{-1}), g_2 \circ (\psi \circ \Phi^{-1}))\, \mathrm{Mat}\, (\psi \circ \Phi^{-1}) \text{ on } \Phi(U \cap V)$$

where $\mathrm{Mat}\,(\psi \circ \Phi^{-1})$ is the $2 \times 2$ matrix whose determinant is the Jacobian.

To prove this is well - defined. We have

$$\omega : (U, \Phi) \longmapsto (f_1, g_1) \quad \text{and}$$

$$\omega : (V, \psi) \longmapsto (f_2, g_2) \quad \text{such that}$$

$$(f_1, g_1) = (f_2 \circ (\psi \circ \Phi^{-1}), g_2 \circ (\psi \circ \Phi^{-1}))\, \mathrm{Mat}\, (\psi \circ \Phi^{-1}) \quad \ldots.(1)$$

Let $(W, \theta)$ be another coordinate neighborhood such that

$$\omega : (W, \theta) \longmapsto (f_3, g_3) \quad \text{and}$$

$$(f_2, g_2) = (f_3 \circ (\theta \circ \psi^{-1}), g_3 \circ (\theta \circ \psi^{-1}))\, \mathrm{Mat}\, (\theta \circ \psi^{-1}) \ldots..(2)$$

We want to show that

$$(f_1, g_1) = (f_3 \circ (\theta \circ \Phi^{-1}), g_3 \circ (\theta \circ \Phi^{-1}))\, \mathrm{Mat}\, (\theta \circ \Phi^{-1})$$

To prove this, represent (2) in (1) we get

$$(f_1, g_1) = \left(f_3 \circ (\theta \circ \psi^{-1}), g_3 \circ (\theta \circ \psi^{-1})\right) \circ (\psi \circ \Phi^{-1})\, \mathrm{Mat}\, (\theta \circ \psi^{-1})$$
$$\mathrm{Mat}\, (\psi \circ \Phi^{-1})$$

$$= \left(f_3 \circ (\theta \circ \Phi^{-1}),\ g_3 \circ (\theta \circ \Phi^{-1})\right) \text{ Mat } \left(\theta \circ \psi^{-1}\right) \left(\psi \circ \Phi^{-1}\right)$$

$$= \left(f_3 \circ (\theta \circ \Phi^{-1}),\ g_3 \circ (\theta \circ \Phi^{-1})\right) \text{ Mat } \left(\theta \circ \Phi^{-1}\right)$$

where the last equality follows from the chain rule of advanced calculus[1].

Therefore, this is well - defined.

2.18 Definition : The differential 1 - form $\omega$ is called differentiable at $P \in X$ if $\exists$ a coordinate neighborhood $(U, \Phi)$ and $U \ni P$ such that the functions $(f, g)$ corresponding to $(U, \Phi)$ are differentiable at $\Phi(P)$.

Claim that if $\omega$ is differentiable at $P \in X$ with respect to one coordinate neighborhood then $\omega$ is differentiable at $P \in X$ with respect to all coordinate neighborhoods. Proof is the same as the function case.

2.19 Definition : The differential 1 - form is differentiable in a neighborhood $U \subset X$ if this differential 1 - form is differentiable at $P$, $\forall P \in U$.

2.20 Definition : A differential 2 - form on X is a correspondence which assigns to each coordinate neighborhood $(U, \Phi)$ a continuous function $f_1$ where $f_1 : \Phi(U) \to \mathbb{R}$ such that if $f_1$ corresponds to $(U, \Phi)$ and $f_2$ corresponds to $(V, \psi)$ and $U \cap V \neq \emptyset$ then

$$f_1 = f_2 \circ (\psi \circ \Phi^{-1}) \text{ Jac } (\psi \circ \Phi^{-1}) \text{ on } \Phi(U \cap V)$$

We prove that this is well - defined in the same way that we proved differential 1 - forms were well - defined .

2.21 <u>Definition</u> : The differential 2 - form is <u>differentiable at P $\varepsilon$ X</u> if $\exists$ a coordinate neighborhood $(U, \Phi)$ and $U \ni P$ such that the function f cooresponding to $(U, \Phi)$ is differentiable at $\Phi(P)$ .

2.22 <u>Definition</u> : The differential 2 - form is <u>differentiable in a neighborhood $U \subset X$</u> if this differential 2 - form is differentiable at P , $\forall P \varepsilon U$ .

2.23 <u>Definition</u> : A <u>density</u> on X is a correspondence which assigns to each coordinate neighborhood $(U, \Phi)$ a positive continuous function $f_1$ where $f_1 : \Phi(U) \rightarrow \mathbb{R}$ such that if $f_1$ corresponds to $(U, \Phi)$ and $f_2$ corresponds to $(V, \Psi)$ and $U \cap V \neq \emptyset$ then

$$f_1 = f_2 \circ (\Psi \circ \Phi^{-1}) \left| \text{Jac} ( \Psi \circ \Phi^{-1} ) \right| \quad \text{on} \quad \Phi(U \cap V)$$

2.24 <u>Remark</u> : Since Jac $(\Psi \circ \Phi^{-1}) > 0$ on X, there is no need to distinguish differential 2 - forms from densities. However, for sets of straight lines we can not **always** guarantee that Jac $(\Psi \circ \Phi^{-1}) > 0$ so we must integrate densities and not differential 2 - forms in this case. We shall say more about this in Chapter III .

2.25 <u>Definition</u> : If $f : X \to \mathbb{R}$ is a function and $\omega$ is a differential 1 - form on X such that

$$\omega : (U, \Phi) \longmapsto (f_1, g_1)$$

we can define $f\omega$ as a differential 1 - form such that

$$f\omega : (U, \Phi) \longmapsto ((f_\circ \Phi^{-1}) f_1, (f_\circ \Phi^{-1}) g_1)$$

Claim that this is well - defined. To see this let $(V, \Psi)$ be another coordinate neighborhood such that

$$\omega : (V, \Psi) \longmapsto (f_2, g_2) \text{ and}$$

$$f\omega : (V, \Psi) \longmapsto ((f_\circ \Psi^{-1}) f_2, (f_\circ \Psi^{-1}) g_2)$$

We must prove that

$$((f_\circ \Phi^{-1}) f_1, (f_\circ \Phi^{-1}) g_1) = \Big[((f_\circ \Psi^{-1}) f_2)_\circ (\Psi_\circ \Phi^{-1}), ((f_\circ \Psi^{-1}) g_2)_\circ (\Psi_\circ \Phi^{-1})\Big] \text{Mat} (\Psi_\circ \Phi^{-1})$$

$$= \Big[(f_\circ \Phi^{-1})(f_2 \circ (\Psi_\circ \Phi^{-1})), (f_\circ \Phi^{-1})(g_2 \circ (\Psi_\circ \Phi^{-1}))\Big] \text{Mat} (\Psi_\circ \Phi^{-1})$$

$$= (f_\circ \Phi^{-1}) \Big[f_2 \circ (\Psi_\circ \Phi^{-1}), g_2 \circ (\Psi_\circ \Phi^{-1})\Big] \text{Mat} (\Psi_\circ \Phi^{-1})$$

To see this, since $\omega$ is a differential 1 - form, we have

$$(f_1, g_1) = (f_2 \circ (\Psi_\circ \Phi^{-1}), g_2(\Psi_\circ \Phi^{-1})) \text{Mat} (\Psi_\circ \Phi^{-1})$$

Hence ,

$$\left((f\circ\Phi^{-1})\, f_1,\ (f\circ\Phi^{-1})\, g_1\right) = (f\circ\Phi^{-1})(f_1,\ g_1)$$

$$= (f\circ\Phi^{-1})(f_2\circ(\psi\circ\Phi^{-1}),\ g_2\circ(\psi\circ\Phi^{-1}))$$

$$\text{Mat}\ (\psi\circ\Phi^{-1})$$

Therefore, this is well - defined .

2.26 Definition : If $f : X \to \mathbb{R}$ is a function and $\omega$ is a differential 2 - form on X such that

$$\omega :\ (U, \Phi) \longmapsto f_1$$

we can define $f\omega$ as a differential 2 - form such that

$$f\omega : (U, \Phi) \longmapsto ((f\circ\Phi^{-1})\ f_1)$$

We prove that this is well - defined in the same way that we proved $f\omega$ was a differential 1 - form.

2.27 Definition : If $\omega_1$ and $\omega_2$ are two differential 1 - forms on X such that

$$\omega_1 \ :\ (U, \Phi) \longmapsto (f_1,\ g_1) \quad \text{and}$$

$$\omega_2 \ :\ (U, \Phi) \longmapsto (h_1,\ k_1)$$

we can define $\omega_1 + \omega_2$ as a differential 1 - form on X such that

$$\omega_1 + \omega_2 : (U, \Phi) \longmapsto (f_1 + h_1,\ g_1 + k_1)$$

Claim that this is well - defined . Let $(V, \psi)$ be another coordinate neighborhood such that

$$\omega_1 : (V, \psi) \longmapsto (f_2, g_2)$$

$$\omega_2 : (V, \psi) \longmapsto (h_2, k_2) \quad \text{and}$$

$$\omega_1 + \omega_2 : (V, \psi) \longmapsto (f_2 + h_2, g_2 + k_2)$$

We must prove that

$$(f_1 + h_1, g_1 + k_1) = \left[(f_2 + h_2)_\circ(\psi_\circ \Phi^{-1}), (g_2 + k_2)_\circ(\psi_\circ \Phi^{-1})\right] \text{Mat}(\psi_\circ \Phi^{-1}).$$

To see this, since $\omega_1, \omega_2$ are differential 1 - forms we have

$$(f_1, g_1) = (f_{2\circ}(\psi_\circ \Phi^{-1}), g_{2\circ}(\psi_\circ \Phi^{-1}))\text{Mat}(\psi_\circ \Phi^{-1}) \quad \text{and}$$

$$(h_1, k_1) = (h_{2\circ}(\psi_\circ \Phi^{-1}), k_{2\circ}(\psi_\circ \Phi^{-1}))\text{Mat}(\psi_\circ \Phi^{-1})$$

Hence,

$$(f_1 + h_1, g_1 + k_1) = (f_1, g_1) + (h_1, k_1)$$

$$= (f_{2\circ}(\psi_\circ \Phi^{-1}), g_{2\circ}(\psi_\circ \Phi^{-1}))\text{Mat}(\psi_\circ \Phi^{-1}) + (h_{2\circ}(\psi_\circ \Phi^{-1}), k_{2\circ}(\psi_\circ \Phi^{-1}))\text{Mat}(\psi_\circ \Phi^{-1})$$

$$= \left[(f_2, g_2)_\circ(\psi_\circ \Phi^{-1}) + (h_2, k_2)_\circ(\psi_\circ \Phi^{-1})\right]\text{Mat}(\psi_\circ \Phi^{-1}).$$

$$= \left[(f_2 + h_2)_\circ(\psi_\circ \Phi^{-1}), (g_2 + k_2)_\circ(\psi_\circ \Phi^{-1})\right] \text{Mat}(\psi_\circ \Phi^{-1}).$$

Therefore, this is well - defined.

2.28 Definition : If $\omega_1$ and $\omega_2$ are two differential 2 - forms on X such that

$$\omega_1 : (U, \Phi) \longmapsto f_1 \quad \text{and}$$

$$\omega_2 : (U, \Phi) \longmapsto g_1$$

We can define $\omega_1 + \omega_2$ as a differential 2 - forms on X such that

$$\omega_1 + \omega_2 : (U, \Phi) \longmapsto f_1 + g_1$$

We prove that this is well - defined is the same way that we proved $\omega_1 + \omega_2$ was a differential 1 - form .

2.29 Definition : If $f : X \to \mathbb{R}$ is a differentiable function then we can define a differential 1 - form df as follows :

If $(U, \Phi)$ is a coordinate neighborhood then

df : $(U, \Phi) \longmapsto (\partial_1(f \circ \Phi^{-1}), \partial_2(f \circ \Phi^{-1}))$ where $\partial_1(f \circ \Phi^{-1})$, $\partial_2(f \circ \Phi^{-1})$ are the first partial derivatives of each variable



Claim that this is well - defined. To see this, let $(V, \psi)$ be another coordinate neighborhood such that

df : $(V, \psi) \longmapsto (\partial_1(f \circ \psi^{-1}), \partial_2(f \circ \psi^{-1}))$

We must prove that

$$\left(\partial_1(f_\circ \Phi^{-1}), \partial_2(f_\circ \Phi^{-1})\right) = \Big[(\partial_1(f_\circ \psi^{-1})_\circ(\psi_\circ \Phi^{-1}), \partial_2(f_\circ \psi^{-1})_\circ(\psi_\circ \Phi^{-1})\Big] \text{Mat } (\psi_\circ \Phi^{-1})$$

But this is true from the chain rule of Advanced Calculus [1].

Therefore df is a differential 1 - form.

2.30 <u>Remark</u> : If $f : X \longrightarrow R$ and $g : X \longrightarrow R$ are continuous functions then we can define

1. $d(f \pm g) = df \pm dg$
2. $d(fg) = fdg + gdf$
3. $d(f_\circ g) = (f'_\circ g)dg$

Let $\omega$ be a differential 1 - form on X and $(X, \Phi_1)$ be a coordinate neighborhood so $\omega$ assigns to $(X, \Phi_1)$ an ordered pair of functions $(f_1, g_1)$ such that

$f_1 : \Phi_1(X) \longrightarrow \mathbb{R}$ and $g_1 : \Phi_1(X) \longrightarrow \mathbb{R}$ i.e. $\omega : (X, \Phi_1) \longmapsto (f_1, g_1)$

Let $(x_1, y_1)$ be the coordinate functions of $(X, \Phi_1)$. We get two differential 1 - forms $dx_1$ and $dy_1$ with respect to $(X, \Phi_1)$. The differential 1 - form $dx_1$ has 2 functions

$$\begin{aligned}(\partial_1(x_{1\circ}\Phi_1^{-1}), \partial_2(x_{1\circ}\Phi_1^{-1})) &= (\partial_1(\pi_1 \circ \Phi_1 \circ \Phi_1^{-1}), \partial_2(\pi_1 \circ \Phi_1 \circ \Phi_1^{-1})) \\ &= (\partial_1 \pi_1, \partial_2 \pi_1) \\ &= (1, 0)\end{aligned}$$

Similarly with respect to $(X, \Phi_1)$, the differential 1 - form $dy_1$ has 2 functions (0, 1)

Let $f = f_1 \circ \Phi_1$ and $g = g_1 \circ \Phi_1$

then $f : X \longrightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$

From definition 2.25 and 2.27 we can define $fdx_1 + gdy_1$ as the differential 1 - form such that $fdx_1 + gdy_1$ assigns to $(X, \Phi_1)$ the ordered pair of functions

$$\left[(f_\circ \Phi_1^{-1}) 1 , (f_\circ \Phi_1^{-1}) 0\right] + \left[(g_\circ \Phi_1^{-1}) 0 , (g_\circ \Phi_1^{-1}) 1\right]$$

$$= (f_\circ \Phi_1^{-1} , g_\circ \Phi_1^{-1})$$

$$= (f_1, g_1)$$

i.e $fdx_1 + gdy_1 : (X, \Phi_1) \longmapsto (f_1, g_1)$

thus $fdx_1 + gdy_1$ and $\omega$ assigns to $(X, \Phi_1)$ the same ordered pair of functions $(f_1, g_1)$

i.e $\omega = fdx_1 + gdy_1$ $\forall$ coordinate neighborhoods

therefore $\omega = f_1(x_1, y_1)dx_1 + g_1(x_1, y_1)dy_1$

Similarly for the differential 2 - form $\omega$, in terms of the coordinate neighborhood $(X, \Phi_1)$ we can write

$$\omega = f_1(x_1, y_1)\, dx_1 dy_1$$

2.31 <u>Definition</u> : If $\omega_1$ and $\omega_2$ are two differential 1 - forms on X such that

$$\omega_1 : (U, \Phi) \longmapsto (f_1, g_1) \text{ and}$$

$$\omega_2 : (U, \Phi) \longmapsto (h_1, k_1)$$

we can define $\omega_1 \omega_2$ as a differential 2 - form such that $\omega_1\omega_2 : (U, \Phi) \longmapsto f_1k_1 - h_1g_1$. We call $\omega_1\omega_2$ is the exterior product of $\omega_1$ and $\omega_2$.

Claim that this is well - defined, let $(V, \psi)$ be another coordinate neighborhood. Let $(x_1, y_1)$ and $(x_2, y_2)$ be the coordinate functions of $(U, \Phi)$ and $(V, \psi)$ respectively such that $x_2 = \varphi_1(x_1, y_1)$ and $y_2 = \varphi_2(x_1, y_1)$

Then in the coor. neighborhood $(U, \Phi)$ we have

$$\omega_1 = f_1(x_1, y_1)\, dx_1 + g_1(x_1, y_1)\, dy_1$$

$$\omega_2 = h_1(x_1, y_1)\, dx_1 + k_1(x_1, y_1) dy_1$$

$$\omega_1\omega_2 = \left[f_1(x_1, y_1)k_1(x_1, y_1) - g_1(x_1, y_1)h_1(x_1, y_1)\right] dx_1 dy_1$$

and in the coor. neighborhood $(V, \psi)$ we have

$$\omega_1 = f_2(x_2, y_2)dx_2 + g_2(x_2, y_2)dy_2$$

$$\omega_2 = h_2(x_2, y_2)dx_2 + k_2(x_2, y_2)dy_2$$

$$\omega_1\omega_2 = \left[f_2(x_2, y_2)k_2(x_2, y_2) - g_2(x_2, y_2)h_2(x_2, y_2)\right] dx_2 dy_2$$

We want to show that

$$f_1(x_1,y_1)k_1(x_1,y_1) - g_1(x_1,y_1)h_1(x_1,y_1)$$
$$= \left[(f_2(x_2,y_2)k_2(x_2,y_2) - g_2(x_2,y_2)h_2(x_2,y_2)\right] \frac{\partial(\varphi_1,\varphi_2)}{\partial(x_1, y_1)}$$

since $\omega_1$ is a diff. 1 - form , we have

$$(f_1(x_1, y_1), g_1(x_1y_1)) = (f_2(x_2,y_2), g_2(x_2,y_2))\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial y_1} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial y_1}\end{pmatrix}$$

i.e $f_1(x_1,y_1) = f_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial x_1}$

$+ g_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial x_1}$

and $g_1(x_1,y_1) = f_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial y_1} + g_2(\varphi_1(x_1,y_1),$
$\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial y_1}$

and since $\omega_2$ is a diff. 1 - form, so we have

$$(h_1(x_1,y_1), k_1(x_1,y_1)) = (h_2(x_2,y_2), k_2(x_2,y_2))\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial y_1} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial y_1}\end{pmatrix}$$

i.e $h_1(x_1,y_1) = h_2(\varphi_1(x_1,y_1), \varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial x_1}$

$+ k_2(\varphi_1(x_1,y_1), \varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial x_1}$

and $k_1(x_1,y_1) = h_2(\varphi_1(x_1,y_1), \varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial y_1}$

$+ k_2(\varphi_1(x_1,y_1) , \varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial y_1}$

so $f_1(x_1,y_1)k_1(x_1,y_1) - g_1(x_1,y_1)h_1(x_1,y_1)$

$$= \left[f_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial x_1} + g_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial x_1}\right]$$

$$\left[h_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial y_1} + k_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial y_1}\right]$$

$$- \left[f_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_1}{\partial y_1} + g_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial y_1}\right]$$

$$\left[h_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1)\frac{\partial\varphi_1}{\partial x_1} + k_2(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\frac{\partial\varphi_2}{\partial x_1}\right]$$

$$= \Big[(f_2k_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1)) - (g_2h_2)(\varphi_1(x_1,y_1),$$

$$\varphi_2(x_1,y_1))\Big]\left(\frac{\partial\varphi_1}{\partial x_1}\cdot\frac{\partial\varphi_2}{\partial y_1}\right)$$

$$+\Big[(f_2h_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1)) - (f_2h_2)(\varphi_1(x_1,y_1),$$

$$\varphi_2(x_1,y_1))\Big]\left(\frac{\partial\varphi_1}{\partial x_1}\ \frac{\partial\varphi_1}{\partial y_1}\right)$$

$$+\Big[(g_2h_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1)) - (f_2k_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\Big]$$

$$\left(\frac{\partial\varphi_1}{\partial y_1}\ \frac{\partial\varphi_2}{\partial x_1}\right)$$

$$+\Big[(g_2k_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1)) - (g_2k_2)(\varphi_1(x_1,y_1),\varphi_2(x_1,y_1))\Big]$$

$$\left(\frac{\partial\varphi_2}{\partial x_1}\ \frac{\partial\varphi_2}{\partial y_1}\right)$$

$$= \left[(f_2k_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1)) - (g_2h_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1))\right] \left(\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\varphi_2}{\partial y_1}\right)$$

$$+ \left[(g_2h_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1)) - (f_2k_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1))\right] \left(\frac{\partial\varphi_1}{\partial y_1}\frac{\partial\varphi_2}{\partial x_1}\right)$$

$$= \left[(f_2k_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1)) - (g_2h_2)(\varphi_1(x_1,y_1),\ \varphi_2(x_1,y_1))\right] \left(\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\varphi_2}{\partial y_1} - \frac{\partial\varphi_1}{\partial y_1}\frac{\partial\varphi_2}{\partial x_1}\right)$$

$$= \left[f_2(x_2,y_2)k_2(x_2,y_2) - g_2(x_2,y_2)h_2(x_2,y_2)\right] \frac{\partial(\varphi_1,\varphi_2)}{\partial(x_1,y_1)}$$

so, this is well - defined.

2.32 Definition : On n - dimensional space with coordinates $x_1, \ldots, x_n$ , a differential $p$ - form is an expression of the form

$$\sum_{i_1,i_2,\ldots,i_p=1}^{n} f_{i_1 \ldots\ldots i_p}(x_1, \ldots, x_n)\, dx_{i_1}\, dx_{i_2} \ldots dx_{i_p}$$

Where the sum is taken over all possible combinations of the $p$ indices, and the coefficients $f_{i_1 \ldots i_p}(x_1, \ldots, x_n)$ are assumed to be infinitely differentiable functions of the coordinates.

2.23 Example :

For n=2 with coordinates x,y where $p = 0, 1, 2$, we have

a diff. 0 - form is just a differentiable function f(x, y)

a diff. 1 - form is an expression f dx + g dy

a diff 2 - form is an expression f dx dy

For n=3 with coordinates x, y, z where $p = 0, 1, 2, 3$ we have

a diff 0 - form is just a differentiable function f(x, y, z)

a diff 1 - form is an expression f dx + g dy + h dz a <u>diff 2 - form</u> is an expression $f\,dx\,dy + g\,dy\,dz + h\,dz\,dx$

a <u>diff 3 - form</u> is an expression $f\,dx\,dy\,dz$.

The coefficients f, g, h are assumed to be infinitely differentiable functions of the coordinates.

2.34 <u>Remark</u> : -

If $\omega$ is the k - form and $\lambda$ is the m - form, symbolically represented by the sum

$$\omega = \sum a_{i_1 \ldots i_k}(x)\, dx_{i_1 \ldots \ldots}\, dx_{i_k}$$

and

$$\lambda = \sum b_{j_1 \ldots j_m}(x)\, dx_{j_1 \ldots \ldots}\, dx_{j_m}$$

on $G \subseteq \mathbb{R}^n$ where $x = (x_1, \ldots, x_n)$ and the indices $i_1, \ldots i_k$ range independently from 1 to n and also the indices $j_1, \ldots, j_m$ rang independently from 1 to n, then their exterior product, denoted by the symbol $\omega\lambda$, is defined to be the (k+m) - form

$$\omega\lambda = \sum a_{i_1 \ldots i_k}(x)\, b_{j_1 \ldots j_m}(x)\, dx_{i_1 \ldots}\, dx_{i_k}\, dx_{j_1} \ldots dx_{j_m}$$

In this sum, the indices $i_1, \ldots, i_k, j_1, \ldots, j_m$ range independently from 1 to n.

2.35 Definition : $A \subset$ Euclidean Plane X is a domain if $\exists$ a coordinate neighborhood $\{U, \Phi\}$ such that $U \supset A$ and $\Phi(A)$ is a domain in $\mathbb{R}^2$.

Let $\omega$ be a continuous density on X and $A \subset X$ be a domain. We define $m(A) = \int_A \omega$ in the following way :

Choose a coordinate neighborhood $(U, \Phi)$ such that $U \supset A$. $\omega$ assigns to $(U, \Phi)$ a function $f: \Phi(U) \longrightarrow \mathbb{R}$

Define $\int_A \omega = \iint_{\Phi(A)} f$ where $\iint_{\Phi(A)} f$ is a Lebesgue Integral if exists

If $\iint_{\Phi(A)} f$ does not exist we will not define $\int_A \omega$

Claim that this is well - defined, let $(V, \psi)$ be another coordinate neighborhood such that $V \supset A$ and

$$\int_A \omega = \iint_{\psi(A)} f_1$$

where $\omega : (V, \psi) \longmapsto f_1$

We must prove that $\iint_{\Phi(A)} f = \iint_{\psi(A)} f_1$

Since $\omega$ is a density and change variable in double integral, we get

$$\begin{aligned} \iint_{\psi(A)} f_1 &= \iint_{\psi(A)} f_o(\Phi_o \psi^{-1}) \left| \text{Jac}(\Phi_o \psi^{-1}) \right| \\ &= \iint_{\Phi(A)} f_o(\Phi \circ \psi^{-1}{}_o \psi_o \Phi^{-1}) \ \text{Jac}(\psi_o \Phi^{-1}) \left| \text{Jac}(\Phi_o \psi) \right| \\ &= \iint_{\Phi(A)} f \end{aligned}$$

Therefore, we can define $m(A) = \int_A \omega = \iint_{\Phi(A)} f$