CHAPTER V

HARMONIC ANALYSIS ON $\mathbb{Z}$ AND $\mathbb{R}$

In this chapter, we shall consider $\mathbb{Z}$ to be a measure space, where the measure is just the counting measure. As usual we shall use

$$1 \qquad \int_{\mathbb{Z}} f(k)dk = \sum_{k=-\infty}^{\infty} f(k)$$

to denote the integral of $f : \mathbb{Z} \longrightarrow \mathbb{C}$ with respect to the counting measure on $\mathbb{Z}$. Thus a function $f : \mathbb{Z} \longrightarrow \mathbb{C}$ is integrable (with respect to the counting measure) if and only if

$$2 \qquad \int_{\mathbb{Z}} |f(k)|dk = \sum_{k=-\infty}^{\infty} |f(k)| < +\infty \quad .$$

The space of all such functions will be denoted by l^1.

Recall now if $f \in L^1(\mathbb{T})$, then its Fourier transform is defined to be the function $\hat{f}$

$$3 \qquad \hat{f}(k) = \int_{\mathbb{T}} f(\dot{t}) \overline{E_k(\dot{t})} d\dot{t} = \int_0^1 f(t) e^{-2\pi ikt} dt$$

where we have identified f with its associated 1-periodic $F : \mathbb{R} \rightarrow \mathbb{C}$. The Fourier series of f can then be written as

$$4 \qquad \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi ikt} = \int_{\mathbb{Z}} \hat{f}(k) e^{2\pi ikt} dk.$$

The central problem in harmonic analysis on $\mathbb{T}$ is to determine whether, and in what sense, the Fouries series (4) of f represents f.

Of course, we can do harmonic analysis over domains other than $\mathbb{T}$. For instance, over $\mathbb{Z}$ or $\mathbb{R}$. (In the latter case, we obtain the theory of Fouries integrals) and the corresponding central problems become the inversion problems and these have very nice complete solutions.

1. Harmonic Analysis on $\mathbb{Z}$

Let $f \in l^1$. In analogy with harmonic analysis on $\mathbb{T}$, we define the Fourier transform of f to be the function

$$\hat{f}(\dot{x}) = \int_{\mathbb{Z}} f(j)\, \overline{E_j(\dot{x})}\, dj \qquad (\dot{x} \in \mathbb{T})$$

$$= \sum_{j=-\infty}^{\infty} f(j)\, \overline{E_j(\dot{x})} .$$

The answer to the inversion is given by

1.1 Theorem. Let $f \in l^1(\mathbb{Z})$. Associate to f the function $\hat{f}$ defined on $\mathbb{T}$ whose value at $\dot{x}$ is

$$\text{(1-1)} \qquad \hat{f}(\dot{x}) = \sum_{j=-\infty}^{\infty} f(j)\, \overline{E_j(\dot{x})} .$$

Then

$$\text{(1-2)} \qquad \int_{\mathbb{T}} \hat{f}(\dot{x})\, E_k(\dot{x})\, d\dot{x} = f(k).$$

<u>Proof</u>. Since $f \in l^1(\mathbb{Z})$, $\sum_{k=-\infty}^{\infty} |f(k)| < \infty$.

Thus the series in (1-1) converges uniformly, and so

$$\int_{\mathbb{T}} \hat{f}(\dot{x})\, E_k(\dot{x})d\dot{x} = \int_{\mathbb{T}} \Big(\sum_{j=-\infty}^{\infty} f(j)\, \overline{E_j(\dot{x})}\Big)\, E_k(\dot{x})d\dot{x}$$

$$= \sum_{j=-\infty}^{\infty} \int_{\mathbb{T}} f(j)\, \overline{E_j(\dot{x})}\, E_k(\dot{x})d\dot{x}$$

$$= \lim_{n\to\infty} \sum_{j=-n}^{n} \int_{\mathbb{T}} f(j)\, \overline{E_j(\dot{x})}\, E_k(\dot{x})d\dot{x} \quad .$$

By the property of orthogonality of $\{E_n\}$, one gets

$$= \int_{\mathbb{T}} f(k)d\dot{x}$$

$$= f(k) \quad .$$

Hence the Theorem is now proved .

This proposition illustrates the simplicity of the elementary aspects of the harmonic analysis on $\mathbb{Z}$.

2. <u>Harmonic Analysis on $\mathbb{R}$</u> (Theory of Fourier Integrals)

2.1 <u>Definition</u>. Let m be Lebesgue measure on $\mathbb{R}$ divided by $\sqrt{2\pi}$. Define

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\, dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)dx,$$

where dx refers to ordinary Lebesgue measure, the p-norm by

$$(2) \qquad \|f\|_p = \Big\{ \int_{-\infty}^{\infty} |f(x)|^p\, dm(x)\Big\}^{\frac{1}{p}} \quad (1 \le p < \infty),$$

the convolution by

$$(3)\quad (f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dm(y) \quad (x \in \mathbb{R}),$$

and the Fourier transform of f by

$$(4)\quad \hat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dm(x) \quad (t \in \mathbb{R}).$$

Throughout this chapter, we shall write L^p for $L^p(\mathbb{R})$, C_0 for the space of all continuous functions on $\mathbb{R}$ which vanish at infinity, and $C_c(\mathbb{R})$ the space of all continuous complex functions on $\mathbb{R}$ whose support is compact.

2.2 Theorem. Suppose $f \in L^1$, and α and λ are real numbers.

(a) If $g(x) = f(x)\, e^{i\alpha x}$, then $\hat{g}(t) = \hat{f}(t-\alpha)$.

(b) If $g(x) = f(x-\alpha)$, then $\hat{g}(t) = \hat{f}(t)\, e^{-i\alpha t}$.

(c) If $g \in L^1$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\, \hat{g}(t)$.

Thus the Fourier transform converts multiplication by a character into translation, and vice versa, and it converts convolutions to pointwise products.

(d) If $g(x) = \overline{f(-x)}$, then $\hat{g}(t) = \overline{\hat{f}(t)}$.

(e) If $g(x) = f(x/\lambda)$ and $\lambda > 0$, then $\hat{g}(t) = \lambda \hat{f}(\lambda t)$

(f) If $g(x) = -ix\, f(x)$ and $g \in L^1$, then $\hat{f}$ is differentiable and $\hat{f}'(t) = \hat{g}(t)$.

Proof. To prove (a), let $g(x) = f(x)\, e^{i\alpha x}$. Then it follows by the formula 2.1 (4) that

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)\, e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-ix(t-\alpha)}\, dm(x)$$

$$= \hat{f}(t-\alpha) \quad .$$

To prove (b), let $g(x) = f(x-\alpha)$. By substituting $g(x) = f(x-\alpha)$ in the formula 2.1 (4), we then obtain,

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)\, e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} f(x-\alpha)\, e^{-ixt}\, dm(x)$$

$$= e^{-i\alpha t} \int_{-\infty}^{\infty} f(x-\alpha)\, e^{-i(x-\alpha)t}\, dm(x)$$

$$= e^{-i\alpha t}\, \hat{f}(t) \, .$$

To prove (c), let $g \in L^1$ and $h = f * g$. Then it follows by formula 2.1 (3), 2.1 (4) and Fubini's theorem that

$$\hat{h}(t) = \int_{-\infty}^{\infty} (f*g)(x) e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dm(y)\right) e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} e^{-itx}\, dm(x) \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dm(y).$$

$$= \int_{-\infty}^{\infty} g(y)\, e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x-y)\, e^{-it(x-y)}\, dm(x)$$

$$= \int_{-\infty}^{\infty} g(y) e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x)\, e^{-itx}\, dm(x)$$

$$= \hat{g}(t)\, \hat{f}(t).$$

To prove (d), let $g(x) = \overline{f(-x)}$. Then

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)\, e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} \overline{f(-x)\, e^{-i(-x)t}}\, dm(x)$$

$$= \overline{\int_{-\infty}^{\infty} f(-x)\, e^{-i(-x)t}\, dm(x)}$$

$$= \overline{\hat{f}(t)} \,.$$

To prove (e), let $g(x) = f(x/\lambda)$ and $\lambda > 0$.

Then we obtain

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x)\, e^{-ixt}\, dm(x)$$

$$= \int_{-\infty}^{\infty} f(x/\lambda)\, e^{-ixt}\, dm(x)$$

$$= \lambda \int_{-\infty}^{\infty} f(x/\lambda)\, e^{-i\frac{x}{\lambda}(\lambda t)}\, dm(x/\lambda)$$

$$= \lambda\, \hat{f}(\lambda t) \,.$$

To prove (f), consider

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dm(x)$$

and

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dm(x).$$

Then

$$(1)\quad \frac{\hat{f}(s)-\hat{f}(t)}{s-t} = \frac{\int_{-\infty}^{\infty} f(x)\,(e^{-ixs}-e^{-ixt})\,dm(x)}{s-t}$$

$$= \frac{\int_{-\infty}^{\infty} f(x)\,e^{-ixt}\,(e^{-ix(s-t)}-1)\,dm(x)}{s-t}$$

$$= \int_{-\infty}^{\infty} f(x)\,e^{-ixt}\,\varphi(x,\,s-t)\,dm(x) \quad (s \neq t),$$

where $\varphi(x, u) = (e^{-ixu} - 1)/u$. Since $|\varphi(x, u)| \leq |x|$ for all real $u \neq 0$ and since $\varphi(x, u) \longrightarrow -ix$ as $u \to 0$, the dominated convergence theorem applies to (1), if s tends to t, and we conclude that

$$(2)\quad \hat{f}'(t) = -i\int_{-\infty}^{\infty} x\,f(x)\,e^{-ixt}\,dm(x) .$$

2.3 Remark.

(a) In the preceding proof, the appeal to the dominated convergence theorem may seem to be illegitimate since the dominated convergence theorem deals only with countable sequences of functions. However, it does enable us to conclude that

$$\lim_{n\to\infty} \frac{\hat{f}(s_n)-\hat{f}(t)}{s_n - t} = -i\int_{-\infty}^{\infty} x\,f(x)\,e^{-ixt}\,dm(t)$$

for every sequence $\{s_n\}$ which converges to t; and this says exactly that

$$\lim_{s\to t} \frac{\hat{f}(s)-\hat{f}(t)}{s-t} = -i\int_{-\infty}^{\infty} x\,f(x)\,e^{-ixt}\,dm(t).$$

We shall encounter similar situation again, and shall apply the dominated convergence theorem to them without further comment.

2.4 Theorem. For any function f on $\mathbb{R}$ and every $y \in \mathbb{R}$, let f_y be the translate of f defined by

(1) $$f_y(x) = f(x-y) \qquad (x \in \mathbb{R}) .$$

If $1 \leq p < \infty$ and if $f \in L^p$, the mapping

(2) $$y \longmapsto f_y$$

is a uniformly continuous mapping of $\mathbb{R}$ into $L^p(\mathbb{R})$.

Proof. Given any $\varepsilon > 0$, Since $f \in L^p$ and since $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, there exists a continuous function g whose support lies in a bounded interval $[-A, A]$ such that $\|f - g\|_p < \varepsilon$. The uniform continuity of g shows that there exists a $\delta \in (0, A)$ such that $|s-t| < \delta$ implies $|g(s) - g(t)| < (3A)^{-1/p}\varepsilon$. If $|s-t| < \delta$, it follows that

$$\int_{-\infty}^{\infty} |g(x-s) - g(x-t)|^p dx < (3A)^{-1}\varepsilon^p(2A + \delta) < \varepsilon^p$$

so that $\|g_s - g_t\|_p < \varepsilon$.

Note that L^p - norms (relative to Lebesgue measure) are translation invariant so that $\|f\|_p = \|f_s\|_p$.

Thus

$$\|f_s - f_t\|_p \leq \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p$$
$$= \|(f-g)_s\|_p + \|g_s - g_t\|_p + \|(g-f)_t\|_p < 3\varepsilon$$

whenever $|s-t| < \delta$. This completes the proof.

2.5 Theorem. If $f \in L^1$, then $\hat{f} \in C_0$ and

$$(1) \qquad \|\hat{f}\|_\infty \leq \|f\|_1 .$$

Proof. The inequality (1) follows from 2.1 (4), since for any $t \in R$,

$$|\hat{f}(t)| = \left|\int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dm(x)\right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)\, e^{-ixt}|\, dm(x)$$

$$= \int_{-\infty}^{\infty} |f(x)|\, dm(x) = \|f\|_1$$

So that $\|\hat{f}\| \leq \|f\|_1$. If $t_n \to t$, then

$$(2) \quad |\hat{f}(t_n) - \hat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x)|\, |e^{-it_n x} - e^{-itx}|\, dm(x).$$

Since $|e^{-it_n x} - e^{-itx}| \leq |e^{-it_n x}| + |e^{-itx}| = 2$, the integrand is bounded by $2|f(x)|$ and tends to 0 for every x, as $n \to \infty$. Hence $\hat{f}(t_n) \to \hat{f}(t)$, by the dominated convergence theorem. Thus $\hat{f}$ is continuous.

Since $e^{\pi i} = -1$, 2.1 (4) gives

$$(3) \quad \hat{f}(t) = - \int_{-\infty}^{\infty} f(x)\, e^{-it(x+\pi/t)}\, dm(x)$$

$$= - \int_{-\infty}^{\infty} f(x - \pi/t)\, e^{itx}\, dm(x).$$

Hence

$$(4) \quad 2\hat{f}(t) = \int_{-\infty}^{\infty} \{f(x) - f(x - \pi/t)\}\, e^{-itx}\, dm(x),$$

so that

$$(5) \quad 2\,|\hat{f}(t)| \leq \|f - f_{\pi/t}\|_1 ,$$

which tends to 0 as $t \to \pm\infty$, by theorem 2.4.

2.6 A pair of Auxiliary Functions. In this proof of the inversion theorem it will be convenient to know a positive function H which has a positive Fourier transform whose integral is easily calculated.

Put

$$(1) \qquad H(t) = e^{-|t|}$$

and define

$$(2) \quad h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t)\, e^{itx}\, dm(t) \qquad (\lambda > 0).$$

Putting $H(\lambda t) = e^{-|\lambda t|}$, we get

$$h_\lambda(x) = \int_{-\infty}^{\infty} e^{-|\lambda t|} e^{itx}\, dm(t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\lambda t|} e^{itx}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\lambda t|} (\cos tx + i \sin tx)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\lambda t|} \cos tx\, dt \quad \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{|\lambda t|} \sin tx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} e^{-|\lambda t|} \cos tx\, dt + \frac{i}{\sqrt{2\pi}} \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} e^{|\lambda t|} \sin tx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to -\infty} \int_{\alpha}^{0} e^{\lambda t} \cos tx\, dt + \lim_{\beta \to \infty} \int_{0}^{\beta} e^{\lambda t} \cos tx\, dt$$

$$+ \frac{i}{\sqrt{2\pi}} \left\{ \lim_{\alpha \to -\infty} \int_{\alpha}^{0} e^{-\lambda t} \sin tx\, dt + \lim_{\beta \to \infty} e^{-\lambda t} \sin tx\, dt \right\}$$

Integrate by part, one gives

$$\frac{1}{\sqrt{2\pi}}\left\{\frac{\lambda}{\lambda^2+x^2}\lim_{\alpha\to-\infty}\left(1-\cos x\alpha\, e^{\lambda\alpha}-\frac{x\sin x\alpha\, e^{\lambda\alpha}}{\lambda}\right)\right.$$

$$\left.+\frac{\lambda}{\lambda^2+x^2}\lim_{\beta\to\infty}\left(1-\cos x\beta\, e^{-\lambda\beta}\ \frac{x}{\lambda}\sin x\beta\, e^{-\lambda\beta}\right)\right\}$$

$$+\frac{i}{\sqrt{2\pi}}\left\{\frac{\lambda}{\lambda^2+x^2}\lim_{\alpha\to-\infty}\left(\sin x\alpha\, e^{\lambda\alpha}-\frac{x}{\lambda}(i-\cos x\alpha\, e^{\lambda\alpha})\right)\right.$$

$$\left.+\frac{\lambda}{\lambda^2+x^2}\lim_{\beta\to\infty}\left(-\sin x\beta\, e^{-\lambda\beta}\cdot\frac{x}{\lambda}(\cos x\beta\, e^{-\lambda\beta}-1)\right)\right\}.$$

We claim that $\lim_{\alpha\to-\infty}\cos x\alpha\, e^{\lambda\alpha}=0$; that is $\lim_{\alpha\to\infty}\cos x\alpha\, e^{-\lambda\alpha}=0.$

Given any $1>\varepsilon>0$, choose $N_0\in(\mathbb{Z}>0)$ such that $N_0>\frac{1}{\lambda}\ln\frac{1}{\varepsilon}$. Then for all $\alpha>N_0$

$$\left|\cos x\alpha\, e^{-\lambda\alpha}\right|\leq\left|e^{-\lambda\alpha}\right|\leq\left|e^{-\lambda N_0}\right|<e^{\ln\varepsilon}=\varepsilon\ .$$

Similary, we have $\lim_{\alpha\to-\infty}\sin x\alpha\, e^{\lambda\alpha}=-\lim_{\alpha\to\infty}\sin x\alpha\, e^{-\lambda\alpha}$ $=0.$

Hence

$$(3)\quad h_\lambda(x)=\frac{1}{\sqrt{2\pi}}\cdot\frac{2\lambda}{\lambda^2+x^2}+\frac{i}{\sqrt{2\pi}}\left(-\frac{x}{\lambda}+\frac{x}{\lambda}\right)$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\lambda^2+x^2}$$

and hence

$$\int_{-\infty}^{\infty} h_{\lambda}(x)\, dm(x) = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\ \frac{\lambda}{\lambda^2 + x^2}\, dm(x)$$

$$= \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + x^2}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + (x/\lambda)^2}\, d(x/\lambda)$$

$$= \frac{1}{\pi} \lim_{\substack{\beta \to \infty \\ \alpha \to -\infty}} \int_{\alpha}^{\beta} \frac{1}{1 + (x/\lambda)^2}\, d(x/\lambda)$$

By putting $\mathcal{U}(t) = \dfrac{\sin t}{\cos t}$, we obtain

$$= \frac{1}{\pi} \lim_{\substack{\beta \to \frac{\pi}{2} \\ \alpha \to -\frac{\pi}{2}}} \int_{\alpha}^{\beta} \frac{\mathcal{U}'(t)}{1 + \mathcal{U}^2(t)}\, dt$$

$$= \frac{1}{\pi} \lim_{\substack{\beta \to \frac{\pi}{2} \\ \alpha \to -\frac{\pi}{2}}} \int_{\alpha}^{\beta} \frac{(1/\cos^2 t)}{(1 + \sin^2 t/\cos^2 t)}\, dt$$

$$= \frac{1}{\pi} \lim_{\substack{\beta \to \frac{\pi}{2} \\ \alpha \to -\frac{\pi}{2}}} (\beta - \alpha)$$

$$= 1$$

Hence

$$(4) \qquad \int_{-\infty}^{\infty} h_{\lambda}(x)\, dm(x) = 1 ,$$

Note also that $0 < H(t) \leq 1$ and that $H(\lambda t) \to 1$ as $\lambda \to 0$.

2.7 Proposition. If $f \in L^1$, then

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t)\, \hat{f}(t)\, e^{ixt} dm(t).$$

Proof. Apply Fubini's theorem, we get

$$\begin{aligned}
(f * h_\lambda)(x) &= \int_{-\infty}^{\infty} f(x-y)\, h_\lambda(y)\, dm(y) \\
&= \int_{-\infty}^{\infty} f(x-y)\Big(\int_{-\infty}^{\infty} H(\lambda t) e^{ity} dm(t)\Big)\, dm(y) \\
&= \int_{-\infty}^{\infty} f(x-y)\, dm(y) \int_{-\infty}^{\infty} H(\lambda t)\, e^{ity} dm(t) \\
&= \int_{-\infty}^{\infty} H(\lambda t)\, dm(t) \int_{-\infty}^{\infty} f(x-y)\, e^{ity} dm(y) \\
&= \int_{-\infty}^{\infty} H(\lambda t)\, dm(t) \int_{-\infty}^{\infty} f(y)\, e^{it(x-y)} dm(y) \\
&= \int_{-\infty}^{\infty} H(\lambda t)\, \hat{f}(t)\, e^{itx} dm(t)\ .
\end{aligned}$$

2.8 Theorem. If $g \in L^\infty$ and g is continuous at a point x, then

(1) $$\lim_{\lambda \to 0} (g * h_\lambda)(x) = g(x).$$

Proof. On account of 2.6 (4), we have

$$\begin{aligned}
(g * h_\lambda)(x) - g(x) &= \int_{-\infty}^{\infty} [g(x-y) - g(x)]\, h_\lambda(y)\, dm(y) \\
&= \int_{-\infty}^{\infty} [g(x-y) - g(x)]\, \lambda^{-1} h_1\Big(\frac{y}{\lambda}\Big)\, dm(y) \\
&= \int_{-\infty}^{\infty} g(x-\lambda s) - g(x)]\, h_1(s)\, dm(s).
\end{aligned}$$

The last integrand is dominated by $2\|g\|_\infty \, h_1(s)$ and converges to 0 pointwise for every s, as $\lambda \to 0$. Hence (1) follows from the dominated convergence theorem .

2.9 <u>Theorem</u>. If $1 \leq p < +\infty$ and $f \in L^p$, then

(1) $$\lim_{\lambda \to 0} \| f * h_\lambda - f \|_p = 0 \quad .$$

<u>Proof</u>. Since $h_\lambda \in L^q$, where q is the exponent conjugate to p, $(f * h_\lambda)(x)$ is defined for every x. Because of 2.6 (4) we have

(2) $$(f * h_\lambda)(x) - f(x) = \int_{-\infty}^{\infty} [f(x-y) - f(x)] \, h_\lambda(y) \, dm(y)$$

By Jensen's Inequality, we obtain

(3) $$|(f * h_\lambda)(x) - f(x)|^p \leq \int_{-\infty}^{\infty} |f(x-y) - f(x)|^p . |h (y)|^p dm(y)$$

By using 2.6 (2), 2.6 (3) and applying Jensen's Inequality again, we have

$$\begin{aligned} |h_\lambda(y)|^p &= (h_\lambda(y))^p \\ &= \left(\int_{-\infty}^{\infty} H(\lambda t) \, e^{ity} dm(t) \right)^p \\ &= \left(\int_{-\infty}^{\infty} e^{-|\lambda t|} . e^{ity} dm(t) \right)^p \\ &\leq \int_{-\infty}^{\infty} e^{-|\lambda p t|} . e^{ipty} dm(t) \end{aligned}$$

By putting $t' = pt$, one gets

$$= \frac{1}{P} \int_{-\infty}^{\infty} e^{-|\lambda t'|} \, e^{it'y} dm(t')$$

$$\leq \int_{-\infty}^{\infty} e^{-|\lambda t'|} e^{it'y} \, dm(t')$$

$$= \int_{-\infty}^{\infty} H(\lambda t') \, e^{it'y} \, dm(t')$$

$$= h_{\lambda}(y) \ .$$

Hence

(4) $|(f * h_{\lambda})(x) - f(x)|^p \leq \int_{-\infty}^{\infty} |f(x-y) - f(x)|^p . h_{\lambda}(y) \, dm(y)$

Integrate (3) with respect to x and apply Fubini's theorem:

(5) $\| f * h_{\lambda} - f \|_p^{\,p} \leq \int_{-\infty}^{\infty} \| f_y - f \|_p^{\,p} \, h_{\lambda}(y) \, dm(y)$.

If $g(y) = \| f_y - f \|_p^{\,p}$ then g is bounded and continuous, by Theorem 2.4, and $g(0) = 0$. Then

(6) $\| f * h_{\lambda} - f \|_p^{\,p} \leq \int_{-\infty}^{\infty} g(y) \, h_{\lambda}(y) \, dm(y)$

$$= \int_{-\infty}^{\infty} g(0-y) \, h_{\lambda}(y) \, dm(y)$$

$$= \quad g * h_{\lambda}(0) \quad .$$

By 2.8 , we obtain $\lim_{\lambda \to 0} (g * h_{\lambda}(0)) = g(0) = 0$.

Thus $\lim_{\lambda \to 0} \| f * h_{\lambda} - f \|_p = 0$.

This completes the proof.

2.10 <u>The Inversion Theorem</u>. If $f \in L^1$ and $\hat{f} \in L^1$, and if

(1) $g(x) = \int_{-\infty}^{\infty} \hat{f}(t) \, e^{ixt} \, dm(t) \qquad (x \in \mathbb{R})$,

then $g \in C_0$ and $f(x) = g(x)$ a.e .

Proof. By Proposition 2.7,

(2) $$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t)\, \hat{f}(t)\, e^{ixt}\, dm(t).$$

The integrands on the right side of (2) are bounded by $|\hat{f}(t)|$, and since $H(\lambda t) \to 1$ as $\lambda \to 0$, the right side of (2) converges to $g(x)$, for every $x \in \mathbb{R}$, by the dominated convergence theorem.

By Theorem 2.9 we see that $f * h_\lambda$ converges to f in L^p, and so by the proof of Theorem 1.26 there exists a subsequence $f * \lambda_n$ converging pointwise almost everywhere to $f(x)$, that is

(3) $$\lim_{n \to \infty} (f * h_{\lambda_n})(x) = f(x) \quad \text{a.e}$$

Hence $f(x) = g(x)$ a.e. That $g \in C_0$ follows from Theorem 2.5.

2.11 The Uniqueness Theorem. If $f \in L^1$ and $\hat{f}(t) = 0$ for all $t \in \mathbb{R}$, then $f(x) = 0$ a.e .

Proof. Since $\hat{f} = 0$ we have $\hat{f} \in L^1$, and the result follows from the inversion theorem.