CHAPTER II

HOMOGENEOUS SPACES

In this chapter, we introduce an important class of spaces - the homogeneous spaces - which includes the spaces $C(\mathbb{T})$, $L^p(\mathbb{T})$ for $1 \leq p < +\infty$.

1. Definition. A homogeneous normed space on $\mathbb{T}$ is a normed space E of complex valued functions defined on $\mathbb{T}$ such that

(1) If $f \in E$ and $\dot{h} \in \mathbb{T}$, then the map $U_{\dot{h}} f : \dot{x} \longmapsto f(\dot{x} + \dot{h})$ is in E.

(2) $\| U_{\dot{h}} f \|_E = \| f \|_E$ for any $\dot{h} \in \mathbb{T}$

(3) $\lim_{\dot{h} \to \dot{h}_o} \| U_{\dot{h}} f - U_{\dot{h}_o} f \|_E = 0$ for any $\dot{h}_o \in \mathbb{T}$.

2. Definition. Let $C(\mathbb{T})$ denote the space of all complex - valued continuous functions defined on $\mathbb{T}$.

3. Proposition. $C(\mathbb{T})$ with the norm defined by $\| f \| = \sup_{x \in \mathbb{T}} |f(\dot{x})|$ is a linear space.

Proof. First we proceed to show that $C(\mathbb{T})$ is a linear space. For any $f, g \in C(\mathbb{T})$ and $\alpha \in \mathbb{C}$, $f + g \in C(\mathbb{T})$ and $\alpha f \in C(\mathbb{T})$. Then it is easily seen that $C(\mathbb{T})$ is a linear space. Next we will show that $\| \cdot \|$ defined by $\| f \| = \sup_{\dot{x} \in \mathbb{T}} |f(\dot{x})|$ is a norm on $C(\mathbb{T})$. By the definition, it is clear that $\| f \| \geq o$ for any $f \in C(\mathbb{T})$ and $\| f \| \neq o$ if and only if $f = o$. For any $f \in C(\mathbb{T})$ and any scalar λ, we have

$$\|\lambda f\| = \sup_{\dot{x} \in \Gamma} |\lambda f(\dot{x})|$$
$$= |\lambda| \cdot \sup_{\dot{x} \in \Gamma} |f(\dot{x})|$$
$$= |\lambda| \cdot \|f\|$$

For any f, g $\in C(\Gamma)$, we get

$$\|f+g\| = \sup_{\dot{x} \in \Gamma} |(f+g)(\dot{x})|$$
$$= \sup_{\dot{x} \in \Gamma} |f(\dot{x}) + g(\dot{x})|$$
$$\leq \sup_{\dot{x} \in \Gamma} |f(\dot{x})| + \sup_{\dot{x} \in \Gamma} |g(\dot{x})|$$
$$= \|f\| + \|g\| .$$

The norm as defined in proposition 3. is naturally called the <u>uniform norm</u>.

4. <u>Theorem</u>. $C(\Gamma)$ for the uniform norm is a homogeneous Banach space.

<u>Proof</u>. We will show first that $C(\Gamma)$ is a Banach space. Let $\{f_n\}$ be a Cauchy sequence in $C(\Gamma)$. For any $\varepsilon > 0$, there exists N such that for all m, n $\geq$ N, $\|f_n - f_m\|_\infty = \sup_{\dot{x} \in \Gamma} |f_n(\dot{x}) - f_m(\dot{x})| < \varepsilon/3$. In particular, for any $\dot{x} \in \Gamma$,

$$(*) \quad |f_n(\dot{x}) - f_m(\dot{x})| < \varepsilon/3 \text{ for all } m, n \geq N.$$

So that $\{f_n(\dot{x})\}$ is a Cauchy sequence of complex number, and therefore there exists $f(\dot{x})$ in $\mathbb{C}$ such that $\lim_{n \to \infty} f_n(\dot{x}) = f(\dot{x})$. By letting m in (*) tend to infinity, we then get $|f_n(\dot{x}) - f(\dot{x})| \leq \varepsilon/3$. Which implies $\sup_{\dot{x} \in \mathbb{T}} |f_n(\dot{x}) - f(\dot{x})| \leq \varepsilon/3 < \varepsilon$ for all $n \geq N$. That is $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$. Now we proceed to show that $f \in C(\mathbb{T})$. Given any $\dot{x}_0 \in \mathbb{T}$, by triangle inequality, we then get

$$|f(\dot{x}) - f(x_0)| \leq |f(\dot{x}) - f_N(\dot{x})| + |f_N(\dot{x}) - f_N(\dot{x}_0)| + |f_N(\dot{x}_0) - f(\dot{x}_0)|.$$

Since f_N is continuous at $\dot{x}_0$, there exists $\delta > 0$ such that $|\dot{x} - \dot{x}_0| < \delta$ implies $|f_N(\dot{x}) - f_N(\dot{x}_0)| < \varepsilon/3$, so that

$$|f(\dot{x}) - f(\dot{x}_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus $f \in C(\mathbb{T})$. Therefore $C(\mathbb{T})$ is a Banach space.

Now we will show that $C(\mathbb{T})$ is a homogeneous space. Let $f \in C(\mathbb{T})$ and $\dot{h} \in \mathbb{T}$. Since $\mathbb{T}$ is compact, f is uniformly continuous. Therefore $U_{\dot{h}} f \in C(\mathbb{T})$. Moreover

$$\|U_{\dot{h}} f\|_\infty = \sup_{\dot{x} \in \mathbb{T}} |f(\dot{x} + \dot{h})| = \sup_{\dot{x} \in \mathbb{T}} |f(\dot{x})| = \|f\|_\infty.$$

Since f is continuous on compact set, $\|U_{\dot{h}} f - U_{\dot{h}_0} f\|_\infty = \sup_{\dot{x} \in \mathbb{T}} |f(\dot{x} + \dot{h}) - f(\dot{x} + \dot{h}_0)|$ tends to zero as $\dot{h} \to \dot{h}_0$.

Hence $C(\mathbb{T})$ is a homogeneous space .

5. <u>Theorem</u>. For $1 \leq p < +\infty$, $L^p(\mathbb{T})$ is a homogeneous space.

<u>Proof</u>. Let $f \in L^p(\mathbb{T})$ and $\dot{h} \in \mathbb{T}$. We will show that $U_{\dot{h}} f \in L^p(\mathbb{T})$. The measurable property of $U_{\dot{h}} f$ will be proved first. Let U be any open set in $\mathbb{C}$. Since a translation is a homeomorphism of $\mathbb{C}$, $h + U$ is open in $\mathbb{C}$. Since f is measurable, $U_{\dot{h}} f$ is measurable. Consider the following :

$$\begin{aligned} \| U_{\dot{h}} f \|_p^p &= \int_{\mathbb{T}} | U_{\dot{h}} f(\dot{x}) |^p d\dot{x} \\ & \quad \int_{\mathbb{T}} | f(\dot{x} + \dot{h}) |^p d\dot{x} \\ & \quad \int_{\mathbb{T}} | f(\dot{x}) |^p d\dot{x} = \| f \|_p^p < +\infty . \end{aligned}$$

So that $U_{\dot{h}} f \in L^p(\mathbb{T})$ and $\| U_{\dot{h}} f \|_p = \| f \|_p$.

To prove the remaining condition, let $\varepsilon > 0$ be given. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, there exists a sequence $\{ f_n \}$ of continuous functions converging to f in $L^p(\mathbb{T})$, and so there exists n_0 such that for all $n \geq n_0$, $\| f - f_n \|_p < \varepsilon/3$.

We have

$$\begin{aligned} U_{\dot{h}}(f - f_{n_0})(\dot{x}) &= (f - f_{n_0})(\dot{x} - \dot{h}) \\ &= f(\dot{x} + \dot{h}) - f_{n_0}(\dot{x} + \dot{h}) \end{aligned}$$

$$= \quad U_{\dot{h}}f(\dot{x}) - U_{\dot{h}}f_{n_o}(\dot{x})$$

$$= \quad (U_{\dot{h}}f - U_{\dot{h}}f_{n_o})\,(\dot{x})$$

so that $\|U_{\dot{h}}f - U_{\dot{h}}f_{n_o}\|_p = \|U_{\dot{h}}(f-f_{n_o})\|_p$.

By the triangle inequality, it follows that

$$\|U_{\dot{h}}f - U_{\dot{h}_o}f\|_p \leq \|U_{\dot{h}}f - U_{\dot{h}}\,f_{n_o}\|_p + \|U_{\dot{h}}\,f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_p$$
$$+ \|U_{\dot{h}_o}\,f_{n_o} + U_{\dot{h}_o}\,f\|_p$$
$$\leq \quad 2\,\|f - f_{n_o}\|_p + \|U_{\dot{h}}\,f_{n_o} - U_{\dot{h}_o}\,f_{n_o}\|_p \ .$$

Since $f_{n_o} \in C(\mathbb{T})$, it follows by Theorem 2.4 that $\lim_{\dot{h}\to\dot{h}_o} \|U_{\dot{h}}\,f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_\infty = 0$. Using the property of L^p - normed and property of integration, we obtain

$$\|U_{\dot{h}}f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_p^p = \int_{\mathbb{T}} |U_{\dot{h}}f_{n_o}(\dot{x}) - U_{\dot{h}_o}f_{n_o}(\dot{x})|^p d\dot{x}$$
$$\leq \sup_{\dot{x}\in\mathbb{T}} |U_{\dot{h}}f_{n_o}(\dot{x}) - U_{\dot{h}_o}f_{n_o}(\dot{x})|^p \int_{\mathbb{T}} d\dot{x}$$
$$= \|U_{\dot{h}}f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_\infty \ .$$

Hence $\lim_{\dot{h}\to\dot{h}_o} \|U_{\dot{h}}f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_p = 0.$

That is, there exists $\eta > 0$ such that for all $\dot{h} \in \mathbb{T}$, $|\dot{h} - \dot{h}_o| < \eta$ implies $\|U_{\dot{h}}f_{n_o} - U_{\dot{h}_o}f_{n_o}\|_p < \varepsilon/3$. So that for any $\varepsilon > 0$, we can choose $\eta > 0$ such that for all $\dot{h} \in \mathbb{T}$, $|\dot{h} - \dot{h}_o| < \eta$ implies $\|U_{\dot{h}}f - U_{\dot{h}_o}f\|_p < \varepsilon$.

Therefore $\lim_{h \to h_o} \|U_h f - U_{h_o} f\|_p = 0$. Thus the Theorem is now proved.

6. Theorem. $L^\infty(\mathbb{T})$ is not homogeneous under the norm $\|\cdot\|_\infty$

Proof. By counter example, let

$$F : \mathbb{R} \longrightarrow \{0 , 1\}$$

defined by

$$F(x) = \begin{cases} 1 & \text{if } x - [x] \in [0 , \tfrac{1}{2}) \\ 0 & \text{if } x - [x] \in (\tfrac{1}{2} , 1) \end{cases}$$

Then F is a 1 - periodic function over $\mathbb{R}$. Observe that $\lim_{h \to 0} \|U_h f - f\|_\infty$ does not converge to 0. Hence $L\ (\mathbb{T})$ cannot be homogeneous.

7. Definition Let $A(\mathbb{T})$ be the set of all continuous complex valued function f over such that $\sum_{n \in \mathbb{Z}} |c_n(f)|$ is a convergent series, where $c_n(f)$ is the fourier coefficient of f.

8. Theorem. $A(\mathbb{T})$ is a linear space under the natural pointwise addition and scalar multiplication.

Proof. For any f, g in $A(\mathbb{T})$, $\left\{ \sum_{n=-k}^{k} |c_n(f)| \right\}$ and $\left\{ \sum_{n=-k}^{k} |c_n(g)| \right\}$ are bounded sequences. And so $\left\{ \sum_{n=-k}^{k} |c_n(f+g)| \right\}$ is a bounded sequence.

Which implies $\sum_{n\in\mathbb{Z}} |c_n(f+g)| \leq \sum_{n\in\mathbb{Z}} |c_n(f)| + \sum_{n\in\mathbb{Z}} |c_n(g)|$ is a convergent series. Thus $f+g$ is in $A(\mathbb{T})$.

For any complex number α and for any f in $A(\mathbb{T})$, αf is in $A(\mathbb{T})$. In fact, $\sum_{n\in\mathbb{Z}} |c_n(\alpha f)| = |\alpha| \sum_{n\in\mathbb{Z}} c_n(f)|$. Since $\sum_{n\in\mathbb{Z}} |c_n(f)|$ is a convergent series, $|\alpha| \sum_{n\in\mathbb{Z}} |c_n(f)|$ is a convergent series. Hence it now follows easily that $A(\mathbb{T})$ is a linear space .

$$\text{Set } \|f\|_{A(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |c_n(f)| .$$

Then it can be easily seen that the mapping $f \longmapsto \|f\|_{A(\mathbb{T})}$ defines a norm over $A(\mathbb{T})$.

9. <u>Theorem</u>. Under the above norm, $A(\mathbb{T})$ is a homogeneous Banach space .

<u>Proof</u>. First we will show that $A(\mathbb{T})$ is a Banach space. Given any $\varepsilon > 0$, Let $\{f^{(k)}\}$ be a Cauchy sequence in $A(\mathbb{T})$. There exists an integer N such that, for all $k, k' \geq N$, $\|f^{(k)} - f^{(k')}\| < \varepsilon$. Fix any n in $\mathbb{Z}$, since

$$|c_n(f^{(k)}) - c_n(f^{(k')})| \leq \|f^{(k)} - f^{(k')}\|_{A(\mathbb{T})} < \varepsilon,$$

for all $k, k' \geq N$, $\{c_n(f^{(k)})\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete. There exists $c_n(f) \in \mathbb{C}$ such that $c_n(f^{(k)}) \longrightarrow c_n(f)$ as $k \to \infty$. By uniqueness, it follows that $c_n(f)$ is the n-th Fourier coefficient of f in $A(\mathbb{T})$. Next we proceed to show that $\lim_{k\to\infty} \|f_k - f\|_{A(\mathbb{T})} = 0$.

Since for any m, we have $\sum_{n=-m}^{m} |c_n(f^{(k)}) - c_n(f^{(k')})| < \varepsilon$. By letting k' tend to ∞, we get $\sum_{n=-m}^{m} |c_n(f^{(k)}) - c_n(f)| < \varepsilon$; so that $\|f^{(k)} - f\| = \sum_{n=-\infty}^{\infty} |c_n(f^{(k)}) - c_n(f)| < \varepsilon$ for all $k \geqslant N$. This completes the proof .

Now we will show that $A(\mathbb{T})$ is a homogeneous space. Let $f \in A(\mathbb{T})$ and $\dot{h} \in \mathbb{T}$. Consider

$$
\begin{aligned}
c_n(U_{\dot{h}}f) &= \langle U_{\dot{h}}f , E_n \rangle \\
&= \int_0^1 f(x+h)\, e^{-2i\pi nx} dx \\
&= \int_0^1 f(x)\, e^{-2i\pi n(x-h)} dx \\
&= \langle f, U_{-\dot{h}} E_n \rangle \\
&= \langle f, E_n(-\dot{h})\, E_n \rangle \\
&= \langle f, E_n \rangle\, E_n(\dot{h}) \\
&= c_n(f)\, E_n(\dot{h}) \quad .
\end{aligned}
$$

And so

$$
\begin{aligned}
\| U_{\dot{h}} f \|_{A(\mathbb{T})} &= \sum_{n \in \mathbb{Z}} |c_n(U_{\dot{h}}f)| \\
&= \sum_{n \in \mathbb{Z}} |c_n(f)| = \|f\|_{A(\mathbb{T})} \quad .
\end{aligned}
$$

Hence $A(\mathbb{T})$ satisfies conditions (1) and (2) of being a homogeneous spaces.

To prove the remaining condition, given any $\varepsilon > 0$, since $\sum_{n \in \mathbb{Z}} |c_n(f)|$ converges, there exists N such that $\sum_{|k| \geq N} |c_k(f)| < \varepsilon/2$. For any n, E_n is continuous. And so for each $|n| < N$, there exists $\eta_n > 0$ such that for $|\dot{h} - \dot{h}_0| < \eta_n$, implies that $|E_n(\dot{h}) - E_n(\dot{h}_0)| < \frac{\varepsilon}{2\|f\|_{A(\mathcal{T})}}$

Let $\eta = \min\{\eta_n \mid |n| < N\}$. Then for any $\dot{h} \in \mathcal{T}$, $|\dot{h} - \dot{h}_0| < \eta$ implies $|E_n(\dot{h}) - E_n(\dot{h}_0)| < \frac{\varepsilon}{2\|f\|_{A(\mathcal{T})}}$ for any $|n| < N$. And so

$$\sum_{|k|<N} |c_k(U_{\dot{h}}f - U_{\dot{h}_0}f)| = \sum_{|k|<N} |c_k(f)(E_k(\dot{h}) - E_k(\dot{h}_0))|$$
$$\leq \frac{\varepsilon}{2\|f\|_{A(\mathcal{T})}} \cdot \|f\|_{A(\mathcal{T})}$$
$$= \frac{\varepsilon}{2} .$$

Consequently, for any $\dot{h} \in \mathcal{T}$, $|\dot{h} - \dot{h}_0| < \eta$, we have

$$\|U_{\dot{h}}f - U_{\dot{h}_0}f\|_{A(\mathcal{T})} = \sum_{n \in \mathbb{Z}} |c_n(U_{\dot{h}}f - U_{\dot{h}_0}f)|$$
$$= \sum_{|k|<N} |c_k(U_{\dot{h}}f - U_{\dot{h}_0}f)| + \sum_{|k| \geq N} |c_k(U_{\dot{h}}f - U_{\dot{h}_0}f)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Therefore $A(\mathcal{T})$ is a homogeneous space .

This completes the proof .