CHAPTER III

CONGRUENCES ON ORTHODOX SEMIGROUPS

Hall has given an explicit form of the minimum inverse congruence on any orthodox semigroup in [4]. We show in this chapter that the restriction of the minimum inverse congruence on orthodox semigroup S to E(S) is the minimum semilattice congruence on E(S). An explicit form of the minimum right-inverse congruence on a generalized inverse semigroup is given. It is shown that the restriction of the minimum inverse congruence on an orthodox semigroup S to any regular subsemigroup T of S is the minimum inverse congruence on T.

Let S be a regular semigroup. Let δ be the congruence on S generated by the relation $\{(ef, fe) \mid e, f \in E(S)\}$. Then δ is the minimum inverse congruence on S. Because a homomorphic image of a regular semigroup is regular, S/δ is regular, so δ is a regular congruence on S. Since S is regular, $E(S/\delta) = \{e\delta \mid e \in E(S)\}$. Let $e, f \in E(S)$. Then $(ef, fe) \in \delta$, so $(e\delta)(f\delta) = (ef)\delta = (fe)\delta = (f\delta)(e\delta)$. Hence, any two idempotents of S/δ commute. Therefore S/δ is inverse, and so δ is an inverse congruence on S. Next, let ρ be an inverse congruence on S. Let $e, f \in E(S)$. Then $e\rho, f\rho \in E(S/\rho)$. But S/ρ is inverse. Then $(ef)\rho = (e\rho)(f\rho) = (f\rho)(e\rho) = (fe)\rho$, that is; $(ef, fe) \in \rho$. This shows that $\{(ef, fe) \mid e, f \in E(S)\} \subseteq \rho$. But δ is the smallest congruence on S containing $\{(ef, fe) \mid e, f \in E(S)\}$.

Therefore $\delta \subseteq \rho$. Hence, δ is the minimum inverse congruence on S.

Recall the following : A regular semigroup S is a right-inverse semigroup if efe = fe for all e, f $\in$ E(S). A regular semigroup S is a generalized inverse semigroup if for any e, f, g, h $\in$ E(S), efgh = egfh. Right-inverse semigroups and generalized inverse semigroups are generalizations of inverse semigroups.

Let S be a regular semigroup. The similar proof as above, if ν is the congruence on S generated by $\{(efe, fe) \mid e, f \in E(S)\}$, then ν is the minimum right-inverse congruence on S, and if τ is the congruence on S generated by $\{(efgh, egfh) \mid e, f, g, h \in E(S)\}$, then τ is the minimum generalized inverse congruence on S.

For the remaining of this thesis, in any regular semigroup, the following notation will be used :

δ = the minimum inverse congruence,

ν = the minimum right-inverse congruence,

τ = the minimum generalized inverse congruence on S.

If the emphasis of the semigroup S is needed, we used $\delta(S)$, $\nu(S)$ and $\tau(S)$ for δ, ν and τ; respectively.

Because every inverse semigroup is right-inverse and generalized inverse, the following relationships follow : In any regular semigroup, $\nu \subseteq \delta$ and $\tau \subseteq \delta$.

In a semigroup S, for a $\in$ S, recall that the notation V(a) denotes the set of all inverses of a in S, that is ;

$$V(a) = \{x \in S \mid a = axa \text{ and } x = xax\}.$$

The following theorem characterizes an orthodox semigroup in terms of the sets of inverses of its elements :

3.1 Theorem [4, Theorem 2]. A regular semigroup S is orthodox if and only if, for any elements a, b in S, $V(a) \cap V(b) \neq \phi$ implies $V(a) = V(b)$. In fact, a regular semigroup S is orthodox if for any idempotents e, f in S, $V(e) \cap V(f) \neq \phi$ implies $V(e) = V(f)$.

Hall has given an explicit form of the minimum inverse congruence on an orthodox semigroup in [4] as follows :

3.2 Theorem [4, Theorem 3]. If S is an orthodox semigroup, then the relation $\delta = \{(x, y) \in S \times S \mid V(x) = V(y)\}$ is the minimum inverse congruence on S.

Moreover, if S is a regular semigroup and the relation $\delta = \{(x, y) \in S \times S \mid V(x) = V(y)\}$ is an inverse congruence on S, then S is an orthodox semigroup.

Let S be an orthodox semigroup. Then E(S) is a band, so it is also orthodox. By Theorem 1.1, for each $e \in E(S)$, $V(e) \subseteq E(S)$. Let δ and $\delta(E(S))$ denote the minimum inverse congruences on S and on E(S); respectively. Hence, we can easily see that $\delta(E(S)) = \delta \cap (E(S) \times E(S))$. Since E(S) is a band, $E(S) / \delta(E(S))$ is also a band. Because $E(S) / \delta(E(S))$ is also an inverse semigroup, it follows that $E(S) / \delta(E(S))$ is a semilattice, and hence $\delta(E(S))$ is a semilattice congruence. Hence, we can conclude the following :

3.3 Proposition. Let S be an orthodox semigroup. Then

$$\delta(E(S)) = \delta \cap (E(S) \times E(S))$$

and it is a semilattice congruence on S.

A semigroup S is called a rectangular band if x = xyx for each x, y $\in$ S.

Let S be an orthodox semigroup and δ be the minimum inverse congruence on S. Then for a $\in$ S, $a\delta = V(a')$ for any inverse a' of a. Hence, if e $\in$ E(S), then $e\delta = V(e)$ which implies V(e) is a sub-semigroup of S and so of E(S) because $(e\delta)(e\delta) = e\delta$. Then, if e, f, g $\in$ E(S) such that e, f $\in$ V(g), then V(e) = V(f) [Theorem 3.1], so e = efe. Therefore, we get

3.4 Proposition. Let S be an orthodox semigroup. Then for any e $\in$ E(S), V(e) is a rectangular band.

By Proposition 3.3 and Proposition 3.4, the following clearly follows :

3.5 Proposition. Let S be an orthodox semigroup. Then $\delta(E(S))$ decomposes E(S) to a semilattice of rectangular bands, that is; E(S) is a semilattice E(S) / $\delta(E(S))$ of rectangular bands.

Let S be a semigroup and ρ be a congruence on S. Then ρ is a semilattice congruence on S if and only if $a^2 \rho a$ and $ab\rho ba$ for all a, b $\in$ S. Hence, arbitrary intersection of semilattice congruences on S is a semilattice congruence on S, so that the intersection of all semilattice congruences on S is the minimum semilattice congruence on S.

For any semigroup S, let η or $\eta(S)$ if emphasis is needed, denote the minimum semilattice congruence on S.

Let S be an orthodox semigroup. Then by Proposition 3.3, $\eta(E(S)) \subseteq \delta(E(S))$. Because $E(S) / \eta(E(S))$ is a semilattice, it follows that $E(S) / \eta(E(S))$ is an inverse semigroup, so $\eta(E(S))$ is an inverse congruence on E(S). Therefore $\delta(E(S)) \subseteq \eta(E(S))$.

Hence by Proposition 3.3 and the above proof, the following theorem follows directly :

3.6 <u>Theorem</u>. Let S be an orthodox semigroup. Let $\delta(S)$, $\delta(E(S))$ and $\eta(E(S))$ be the minimum inverse congruence on S, the minimum inverse congruence on E(S) and the minimum semilattice congruence on E(S); respectively. Then

$$\delta(E(S)) = \delta(S) \cap (E(S) \times E(S)) = \eta(E(S)).$$

Let ρ be a semilattice congruence on a semigroup S. Let G be a subgroup of S having e as its identity. Let $g \in G$. Then

$$g\rho = (ge)\rho = (gg^{-1}g)\rho = (g\rho)^2(g^{-1}\rho) = (g\rho)(g^{-1}\rho) = (gg^{-1})\rho = e\rho.$$

Hence $G \subseteq e\rho$. This proves that any subgroup G of S, $G \subseteq a\rho$ for some $a \in S$, that is; any two elements of G are ρ-related.

Let S be an orthodox semigroup. From Theorem 3.6, if $\eta(S) = \delta(S)$, then $\eta(S) \cap (E(S) \times E(S)) = \eta(E(S))$. The converse is not generally true. An example is as follows :

Example. Let G be a nontrivial group with identity 1. Then G is orthodox and $E(G) = \{1\}$. Because G is a group, G is a $\eta(G)$-class, so $\eta(G) = G \times G$. Because G is a group, G is an inverse semigroup, so $\delta(G)$ is the identity congruence on G. Hence $\eta(G) \cap (E(G) \times E(G)) = \{(1, 1)\} = \eta(E(G))$ but $\eta(G) \neq \delta(G)$.

We have mentioned that every regular semigroup has the minimum right-inverse congruence. We give in the next theorem an explicit form of the minimum right-inverse congruence on a generalized inverse semigroup.

3.7 Theorem. Let S be a generalized inverse semigroup. Then $\nu = \{(a,b) \in S \times S \mid V(a) = V(b) \text{ and } a'a = b'b \text{ for some } a' \in V(a),\ b' \in V(b)\}$ is the minimum right-inverse congruence on S.

Proof : It is clear that ν is reflexive and symmetric. Next, we show that ν is transitive. Let $(a, b), (b, c) \in \nu$. Then $V(a) = V(b) = V(c)$ and $a'a = b'b$, $b''b = c'c$ forsome $a' \in V(a)$, $b', b'' \in V(b)$ and $c' \in V(c)$. Thus $a'a = b'b = b'bb''b = b'bc'c = (b'bc')c$. Because

$$\begin{aligned} (b'bc')b(b'bc') &= b'(bc'b)b'bc' \\ &= b'bb'bc' \\ &= b'bc' \end{aligned}$$

and

$$\begin{aligned} b(b'bc')b &= (bb'b)c'b \\ &= bc'b \\ &= b\ , \end{aligned}$$

$b'bc' \in V(b) = V(c)$. Hence $(a, c) \in \nu$. Therefore ν is transitive.

Let $a, b, c \in S$ such that $(a, b) \in \nu$. Then $V(a) = V(b)$ and

$a'a = b'b$ for some $a' \in V(a)$, $b' \in V(b)$. Because S is orthodox, by Theorem 3.3, $V(ac) = V(bc)$ and $V(ca) = V(cb)$. Let $c' \in V(c)$. Then $c'a'ac = c'b'bc$. Now, we have $V(ac) = V(bc)$ and $c'a'ac = c'b'bc$. By Theorem 1.1, $c'a' \in V(ac)$ and $c'b' \in V(bc)$ because S is orthodox. Therefore $ac \nu bc$. Since $a'a = b'b$ and S is generalized inverse, it follows that

$$
\begin{aligned}
a'c'ca &= a'c'caa'a \\
&= a'c'cab'b \\
&= a'aa'c'cab'bb'b \\
&= a'(aa')(c'c)(ab')(bb')b \\
&= a'(aa')(ab')(c'c)(bb')b \\
&\quad \text{(because } aa',\ c'c,\ ab',\ bb' \in E(S)) \\
&= a'ab'c'cb.
\end{aligned}
$$

We will show $a'ab' \in V(b)$. Since

$$(a'ab')a(a'ab') = a'(ab'a)a'ab' = a'aa'ab' = a'ab'$$

and

$$a(a'ab')a = (aa'a)b'a = ab'a = a.$$

It follows that $a'ab' \in V(b)$, and hence $a'ab'c' \in V(cb)$ because S is orthodox [Theorem 1.1]. Therefore $V(ca) = V(cb)$, $(a'c')(ca) = (a'ab'c')(cb)$, $a'c' \in V(ca)$ and $a'ab'c' \in V(cb)$. Hence $ca \nu cb$.

Therefore ν is a congruence on S.

Because S is regular, S/ν is regular. To show ν is a right-inverse congruence on S, let $e, f \in E(S)$. Then $ef, fe \in E(S)$ and

$$(efe)(fe)(efe) = efefefe = efe$$

$$(fe)(efe)(fe) = fefefe = fe.$$

Hence $efe \in V(efe) \cap V(fe)$, so $V(efe) = V(fe) \subseteq E(S)$ by Theorem 3.1 .

and Proposition 3.4. But $(efe)(efe) = (efe)(fe)$. Then $(efe, fe) \in \nu$. Because S is regular, $E(S/\nu) = \{e\nu \mid e \in E(S)\}$. Therefore ν is a right-inverse congruence on S.

To show ν is the minimum right-inverse congruence on S, let ρ be a right-inverse congruence on S. Let $(a, b) \in \nu$. Then $V(a) = V(b)$ and $a'a = b'b$ forsome $a' \in V(a)$, $b' \in V(b)$. Therefore,

$$\begin{aligned} a\rho &= (aa'a)\rho \\ &= (aa'ba'a)\rho \\ &= (aa')\rho(ba')\rho a\rho \\ &= (ba')\rho(aa')\rho(ba')\rho a\rho \quad \text{(because } ba', \\ &\qquad aa' \in E(S) \text{ and } \rho \text{ is a right-inverse} \\ &\qquad \text{congruence on } S) \\ &= (ba'aa'ba')\rho a\rho \\ &= (ba'ba')\rho a\rho \\ &= (ba')\rho a\rho \\ &= (b\rho)(a'a)\rho \\ &= (b\rho)(b'b)\rho \\ &= b\rho \quad , \end{aligned}$$

so $(a, b) \in \rho$. Hence $\nu \subseteq \rho$.

Therefore, the theorem is completely proved. #

The following notation will be used : If T is a subsemigroup of a semigroup S, for each $a \in T$, let $V_T(a)$ denote the set of all inverses of a in T. It is clear that if T is a subsemigroup of a semigroup S, then $V_T(a) = V_S(a) \cap T$ for any $a \in T$.

For any orthodox semigroup, we have the following property :

3.8 Proposition. Let T be a regular subsemigroup of an orthodox semigroup S. Then for any a, b $\in$ T, $V_T(a) = V_T(b)$ if and only if $V_S(a) = V_S(b)$.

Proof : Let a, b $\in$ T such that $V_T(a) = V_T(b)$. Then $V_S(a) \cap T = V_S(b) \cap T$, hence $V_S(a) \cap V_S(b) \neq \phi$. By Theorem 3.1, $V_S(a) = V_S(b)$ because S is orthodox.

The converse is obvious. #

Let S be an orthodox semigroup. Then E(S) is a regular subsemigroup of S. We have shown in Proposition 3.3 that the minimum inverse congruence on E(S) is the restriction of the minimum inverse congruence on S to E(S). We end this chapter by showing that the minimum inverse congruence on a regular subsemigroup T of an orthodox semigroup S is the restriction of the minimum inverse congruence of S to T.

3.9 Theorem. Let T be a regular subsemigroup of an orthodox semigroup S. Then

$$\delta(T) = \delta(S) \cap (T \times T).$$

Hence, if A is an ideal of an orthodox semigroup S, then $\delta(A) = \delta(S) \cap (A \times A)$.

Proof : Let a, b $\in$ S such that $(a, b) \in \delta(T)$. Then a, b $\in$ T. Because T is orthodox [Proposition 1.2], $V_T(a) = V_T(b)$. By Proposition 3.8, $V_S(a) = V_S(b)$, and hence $(a, b) \in \delta(S)$ because S is orthodox.

Conversely, let $(a, b) \in \delta(S) \cap (T \times T)$. Then $a, b \in T$ and $V_S(a) = V_S(b)$. By Proposition 3.8, $V_T(a) = V_T(b)$. Because T is orthodox, $(a, b) \in \delta(T)$. #