CHAPTER II

PRELIMINARIES

In this chapter we will give some definitions and results from topology and group theory. The materials are standard and can be found in [1], [2], [3], [4], [5]. We shall assume that the reader is familiar with common terms used in set theory.

2.1 Cartesian product

Let $\{X_\alpha : \alpha \varepsilon A\}$ be a family of sets. The set of all mappings $x: A \rightarrow \bigcup_{\alpha \varepsilon A} X_\alpha$ such that $x(\alpha) \varepsilon X_\alpha$ for each $\alpha \varepsilon A$ is called the cartesian product or product of X_α' s. We shall denote this set by $\prod_{\alpha \varepsilon A} X_\alpha$. When A is finite, say $A = \{1,\ldots,n\}$ we also denote $\prod_{\alpha \varepsilon A} X_\alpha$ by $X_1 \times X_2 \times \ldots \times X_n$. For each $x \varepsilon \prod_{\alpha \varepsilon A} X_\alpha$ and $\alpha \varepsilon A$, $x(\alpha)$ is usually denoted by x_α and is called the α^{th} co-ordinate of x. The mapping $P_\alpha : \prod_{\alpha \varepsilon A} X_\alpha \longrightarrow X_\alpha$ defined by $P_\alpha(x) = x_\alpha$, is called the α^{th} projection.

2.2 Algebraic Concepts

By a group we mean an ordered pair $(G,\circ)$, where G is a non-empty set and $\circ$ is a binary operation on G satisfying the following conditions:

(i) The operation is associative, that is,

$x\circ(y\circ z) = (x\circ y)\circ z$ for all element x,y,z of G.

(ii) There exists an element e of G such that

$e \circ x = x \circ e = x$ for each x in G.

(iii) For each x in G, there is an element x^{-1} in G

such that $x \circ x^{-1} = e = x^{-1} \circ x$.

For convenience, we shall denote the group $(G, \circ)$ simply by G. It can be shown that the element e in (ii) is unique, it is known as the <u>identity</u> of G. For each x in G, the element x^{-1} in (iii) is also unique. It is known as the <u>inverse</u> of x.

A group G in <u>abelian</u> or <u>commutative</u>, if and only if $x \circ y = y \circ x$ for all elements x, y of G. By the <u>order of G</u>, denoted by $|G|$, we mean the cardinality of G. G is called <u>finite</u> or <u>infinite</u> as its order is finite or infinite. For any element x in G, <u>the order of x</u> is the least positive integer m such that $x^m = e$. If no such integer exists we say that x is of <u>infinite order</u>. A group H is a <u>subgroup</u> of G if and only if $H \subset G$ and the group operation of H is the restriction of that of G. It can be shown that any non-empty set H forms a subgroup of $(G, \circ)$ if and only if $x \circ y^{-1} \in H$ for any x, y in H. If S is a subset of G, the symbol $\langle S \rangle$ will denote the subgroup of G generated by S. This $\langle S \rangle$ consists of all product of the form $a_1^{n_1} \circ a_2^{n_2} \circ \dots \circ a_k^{n_k}$ with $a_i \in S$, n_i are integers, and k is a positive integer. If $\langle S \rangle = G$, S is said to be a <u>set of generators</u> of G; the element of S are <u>generators</u> of G. If the subset S consists of a single element a, then the subgroup $\langle S \rangle$, also denoted by $\langle a \rangle$, generated by it is called the <u>cyclic subgroup</u> of G. A group that coincides with one of its cyclic subgroup, is called a <u>cyclic group</u>.

If H is a subgroup of a group G and x, y are elements of G such that $x \circ y^{-1} \in H$, we say that x is right congruent to y modulo H and denoted by $x \tilde{H}R\, y$. If $x^{-1} \circ y \in H$, we say that x is left congruent to y modulo H and denoted by $x \tilde{H}L\, y$. If $\tilde{H}R$ and $\tilde{H}L$ are coincide we shall denote them by $\tilde{H}$. It can be shown that left (right) congruence modulo H is an equivalence relation on G. The equivalence class of $x \in G$ under left (right) congruence modulo H is the set $x \circ H = \{x \circ h : h \in H\}$ ($H \circ x = \{h \circ x : h \in H\}$), it is called a left (right) coset of H in G. It follows that $G = \cup(x_o H) = \cup(H \circ x)$ where the union is taken over all pairwise disjoint cosets. If H is a subgroup of G such that left and right congruence modulo H coincide, then H is said to be a normal subgroup of G. In an abelian group, each subgroup is normal. If H is a normal subgroup of a group G, then $G/_{\tilde{H}}$ is a group under the binary operation given by $(xH)(yH) = xyH$, this group is called the quotient group or factor group of G by $\tilde{H}$, and will be denoted by $G/_H$.

A mapping h on a group $(G, \circ)$ into group $(G', *)$ is said to be a homomorphism provided

$$h(x_o y) = h(x) * h(y), \quad \text{for all } x, y \text{ in } G.$$

If h is bijective, h is called an isomorphism. Two groups G and G' are isomorphic, denoted $G \cong G'$, if there is an isomorphism $h : G \longrightarrow G'$.

Let $h : G \longrightarrow H$ be a homomorphism. The kernel of h is the subset of G :

$$\text{kernel } h = \{x \in G : h(x) = e\} .$$

It can be shown that kernel of any homomorphism is a normal subgroup.

An abelian group $(G, \circ)$ is said to be the direct sum of its subgroups G_α, $\alpha \in I$, if for each $g \in G$, $g \neq e$, there is a unique expression (but for order) for g of the form

$$g = g_{\alpha_1} \circ g_{\alpha_2} \circ \dots \circ g_{\alpha_k}$$

where $g_{\alpha_j} \in G_{\alpha_j}$, with $\alpha_1, \dots, \alpha_k$ being distinct elements of I and no g_{α_j} is an identity. When G is the direct sum of its subgroups G_α, $\alpha \in I$ we write $G = \sum_{\alpha \in I} G_\alpha$, and say that each G_α is a direct summand of G. In case I is finite, say $I = \{1, 2, \dots, n\}$, we also write $G_1 \oplus \dots \oplus G_n$ for $\sum_{\alpha \in I} G_\alpha$.

Theorem 2.2.1 Let $\{G_\alpha : \alpha \in I\}$ be a family of abelian groups. Then there exists an abelian group G which is the direct sum of subgroups isomorphic to G_α.

For the proof of this theorem see [2], page 183.

A group F is said to be a free abelian group if it can be expressed as a direct sum of a number of infinite cyclic groups, i.e. F can be written as

$$F = \sum_{\nu} \langle x_\nu \rangle ,$$

where $\langle x_\nu \rangle$ denotes an infinite cyclic group with x_ν as a generator. The totality of generators x_ν of all these cyclic direct summands is called a basis of F. Every element of F can be written in one and only one way as a product, with integer exponents, of a finite number of elements of the basis.

Theorem 2.2.2 Given an abelian group $(G,\circ)$ with set $\mathcal{A} = \{a_\alpha : \alpha \varepsilon I\}$ of generators. Then there exist a free abelian group F, with a basis W, and a subgroup H of F such that

(1) there exists a bijection $\theta : \mathcal{A} \longrightarrow W$,

(2) there exists an isomorphism Ψ from $F/_H$ onto G such that

$$\Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ H\right) = \prod_{i=1}^{m} a_{\alpha_i}^{n_i} ,$$

for all a_{α_i} and for all integer n_i.

Proof For each $\alpha \varepsilon I$, let $G_\alpha = \langle x_\alpha \rangle$ be an infinite cyclic group with x_α as a generator. Then G_α is abelian for all α. Hence, by theorem 2.2.1, there exists a group which is the direct sum of subgroup isomorphic to G_α. Let $F = \sum_{\alpha \varepsilon I} \bar{G}_\alpha$, where $\bar{G}_\alpha$ is isomorphic to G_α. Then for each $\alpha \varepsilon I$, $\bar{G}_\alpha$ is an infinite cyclic group. For each $\alpha \varepsilon I$, let $\bar{x}_\alpha$ be a generator of $\bar{G}_\alpha$.

Let $W = \{\bar{x}_\alpha : \alpha \varepsilon I\}$.

Therefore F is free abelian group with basis W.

Define $\theta : \mathcal{A} \longrightarrow W$ by

$$\theta(a_\alpha) = \bar{x}_\alpha .$$

Since $a_\alpha \neq a_\beta$ and $\bar{x}_\alpha \neq \bar{x}_\beta$ for $\alpha \neq \beta$, hence θ is a bijection from $\mathcal{A}$ to W. That is (1) holds.

Define $h : F \longrightarrow G$ by

$$h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i}\right) = \prod_{i=1}^{m} a_{\alpha_i}^{n_i} .$$

If follows that h is a bijective homomorphism.

Let $H = \{ \prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} : \prod_{i=1}^{m} a_{\alpha_i}^{n_i} = e.\}$, i.e. H is the kernel of h.

Define $\Psi : F/_H \longrightarrow G$ by

$$\Psi\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ H\right) = h\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i}\right) = \prod_{i=1}^{m} a_{\alpha_i}^{n_i} .$$

Now Ψ is well defined, for if

$$\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ H = \prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n'_i} \circ H ,$$

then

$$\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ \left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n'_i}\right)^{-1} \varepsilon \; H,$$

$$h\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ \left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n'_i}\right)^{-1}\right) = e ,$$

and so

$$h\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i}\right) = h\left(\prod_{i=1}^{m}(\Theta(a_{\alpha_i}))^{n'_i}\right).$$

Ψ is a homomorphism, for

$$\Psi\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ H \circ \prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n'_i} \circ H\right) = \Psi\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i} \circ \prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n'_i} \circ H\right),$$

$$= \Psi\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i+n'_i} \circ H\right)$$

$$= h\left(\prod_{i=1}^{m} (\Theta(a_{\alpha_i}))^{n_i+n'_i}\right),$$

$$= h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i}\right) \circ h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i}\right),$$

$$= \Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ H\right) \circ \Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i} \circ H\right).$$

Ψ is one-to-one, for if

$$\Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ H\right) = \Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i} \circ H\right),$$

then

$$h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i}\right) = h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i}\right),$$

therefore

$$h\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ \left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i}\right)^{-1}\right) = e ,$$

$$\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ \left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i}\right)^{-1} \in H ,$$

and so

$$\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ H = \prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n'_i} \circ H.$$

Clearly, the image of Ψ is the image of h, i.e. H.

Therefore, Ψ is an isomorphism from $F/_H$ onto G such that

$$\Psi\left(\prod_{i=1}^{m} (\theta(a_{\alpha_i}))^{n_i} \circ H\right) = \prod_{i=1}^{m} a_{\alpha_i}^{n_i} ,$$

for all a_{α_i} and for all integer n_i, i.e. (2) holds.

In the proof of the above theorem, the homomorphism h maps each element $x_{\alpha_1}^{n_1} \circ x_{\alpha_2}^{n_2} \circ \cdots \circ x_{\alpha_k}^{n_k}$ of F to the element $a_{\alpha_1}^{n_1} \circ a_{\alpha_2}^{n_2} \circ \cdots \circ a_{\alpha_k}^{n_k}$ of G. Hence, for each $x_{\alpha_1}^{n_1} \circ \cdots \circ x_{\alpha_k}^{n_k}$ in H, we have

$$a_{\alpha_1}^{n_1} \circ \cdots \circ a_{\alpha_k}^{n_k} = e .$$

Such an equation will be called a relation between the elements of $\mathcal{A}$ in G. We shall say that it is a relation corresponding to the element $x_{\alpha_1}^{n_1} \circ x_{\alpha_2}^{n_2} \circ \cdots \circ x_{\alpha_k}^{n_k}$ in H. Let N be any subset of H that generates H. The system $\mathcal{R}$ of all relationsthat correspond to the elements in N is called a system of defining relations of G. An abelian group G with a given set of generators is completely determined by its defining relations, since the set N completely determines the normal subgroup H of the free abelian group F and therefore the factor group $F/_H$.

By a field we mean a triple (K,+,.), where +,. are binary operations on K, known as addition and multiplication respectively, such that the following hold :

(i) K forms a commutative group under addition.

(ii) $K^* = K - \{0\}$, where 0 is the additive identity forms a commutative group under multiplication.

(iii) For any $a,b,c \in K$, we have

$$a(b + c) = ab + ac.$$

For convenience, we shall denote a field (K,+,.) simply by K.

Let $(K,+,.)$ be a field and $(V,+)$ be a commutative group with a rule of multiplication which assigns to $a\varepsilon K$, $u\varepsilon V$, a product $au\varepsilon V$. Then V is called a vector space over K if the following axioms hold :

(1) For any $a\varepsilon K$ and any $u, v\varepsilon V$, $a(u+v) = au+av$.

(2) For any $a, b\varepsilon K$ and any $u\varepsilon V$, $(a+b)u = au+bu$.

(3) For any $a, b\varepsilon K$ and any $u\varepsilon V$, $a(bu) = (ab)u$.

(4) For $u\varepsilon V$, $1\cdot u = u$ where 1 is the multiplicative identity of K.

The elements of K and V will be refered as scalar and vector, respectively. If V is a vector space over the field K and $\{x_i\}(1\leq i\leq n)$ is a finite subset of V, then for $a_i \varepsilon K$, $1\leq i\leq n$, $\sum_{i=1}^{n} a_i x_i$ is called a linear combination of the x_i. The vectors $x_1,\ldots, x_n \varepsilon V$ are said to be linearly dependent, or simply dependent, if there exist scalars $a_1,\ldots, a_n \varepsilon K$, not all of them zero, such that $\sum_{i=1}^{n} a_i x_i = 0$. An arbitrary set A of vectors is said to be a linearly dependent set if some finite subset of A is linearly dependent. Otherwise, the set A is called a linearly independent or simply independent. If $\mathcal{B}$ is a linearly independent subset of V such that for every $v \varepsilon V$, v can be written as a linear combination of vectors in $\mathcal{B}$, we say that $\mathcal{B}$ is a basis of V. It can be shown that every vector in V has a unique representation as a linear combination of elements of $\mathcal{B}$. and that every basis of V has the same cardinal number. The cardinal number of a basis of a vector space is called its dimension. If the cardinal number of a basis of a vector space is finite, the vector space is called finite dimensional.

Observe that the set $\mathbb{R}$ of real numbers can be considered as a vector space over the field Q of rational numbers. It can be shown that $\mathbb{R}$ has a basis over Q. Such a basis is known as a Hamel basis. A proof of the existence of such a basis can be found in [6] .

2.3 Topological Concepts

By a topological space we mean an ordered paired (X,τ), where X is a set and τ is a family of subset of X satisfying the following conditions :

a) X and $\emptyset$ are elements of τ .

b) The intersection of any finite number of members of τ is in τ.

c) The arbitrary union of members of τ is in τ.

Any family τ satisfying these three conditions will be called a topology for X. We shall also denote a topological space (X,τ) simply by X. Each member of τ will be called τ-open set of X or simply open set of X. For any subset Y of a topological space (X,τ), it can be shown that the family $\tau_Y = \{T \cap Y : T \in \tau\}$ is a topology for Y. The topological space (Y, τ_Y) is called a subspace of (X,τ), the topology τ_Y is called the relative topology of Y induced by τ.

A subcollection $\mathcal{B}$ of a topology τ of X is said to be a base of τ provided the following condition hold : for each $T \in \tau$ and $x \in T$, the exists $B \in \mathcal{B}$ such that $x \in B \subset T$, or equivalently, each T in τ is a union of members of $\mathcal{B}$. It can be shown that if a family $\mathcal{B}$ of subsets of a set X has the properties ;

(i) the union of set in $\mathcal{B}$ is X,

(ii) for each B_1, $B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of members of $\mathcal{B}$,

then $\mathcal{B}$ is a base for some topology for X.

This topology consists of all sets that can be written as unions of sets in $\mathcal{B}$. Observe that the family of all open intervals form a base of a topology for the set $\mathbb{R}$ of real numbers. This topology is known as the usual topology for $\mathbb{R}$.

A subfamily $\mathcal{C}$ of a topology τ for X is a subbase if the set of all finite intersections of member of $\mathcal{C}$ form a base for τ.

If $\{(X_\alpha,\tau_\alpha) : \alpha \in A\}$ is a family of topological spaces, then the family $\mathcal{B} = \{P_\alpha^{-1}(T_\alpha) : T_\alpha \in \tau_\alpha, \alpha \in A\}$ forms a subbase of a topology τ for the cartesian product $\prod_{\alpha\in A} X_\alpha$. This topology τ is known as the product topology. The topological space $(\prod_{\alpha\in A} X_\alpha,\tau)$ will be called the product space of $\{(X_\alpha,\tau_\alpha): \alpha \in A\}$.

By a neighborhood of a point x in a topological space X, we shall mean a subset N of X for which there exists an open set T of X such that $x \in T \subset N$.

A function f of a topological space (X,τ) into a topological space $(Y,\mathcal{U})$ is continuous at a point $x \in X$ if, given any neighborhood V_y of the point $y = f(x)$, there is a neighborhood U_x of the point x such that $f(U_x) \subset V_y$. The mapping f is said to be continuous on X if it is continuous at every point of X.

Theorem 2.3.1 If X and Y an topological spaces and f is a function on X to Y, then the following statements are equivalent

a) The function f is continuous.

b) For any open set $V \subset Y$, $f^{-1}[V] = \{x \in X : f(x) \in V\}$ is open set of X.

For the proof of this theorem see [4].

A mapping f of a topological space X into a topological space Y is said to be open if for each open set U in X, $f(U) = \{f(x) : x \in U\}$ is an open set of Y.

A sequence $\{x_n\}$ of points in a topological space X is said to converge to x (written $\lim_{n \to \infty} x_n = x$) if for each neighborhood U of x, there is a natural number n_o such that for any natural number n, $n \geq n_o$ implies $x_n \in U$.

Theorem 2.3.2 Let X, Y be topological spaces. If $f : X \longrightarrow Y$ is continuous at x and $\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} f(x_n) = f(x)$. For the proof of this theorem see [4].

Let X be a topological space, R be an equivalence relation on X and $Y = X/_R$ be the quoitent set of X with respect to the relation R. The mapping $\Psi : X \longrightarrow Y$ defined by $\Psi(x) = \bar{x}$, the equivalence class of x, will be called the canonical mapping. It can be shown that the family $\tau_\Psi = \{V \subset Y : \Psi^{-1}(V) \text{ is open}\}$ is a topology for Y; it is called the quotient topology and (Y, τ_Ψ) is called the quotient space of X by R.

Theorem 2.3.3 Let X be a topological space, R an equivalence relation on X, Ψ the canonical mapping of X onto $X/_R$, then a mapping g of $X/_R$ into a topological space Y is continuous if and only if $g \circ \Psi$ is continuous on X.

For the proof of this theorem see [4].

2.4 Topological Groups

A triple $(G,\circ,\tau)$ is a topological group if $(G,\circ)$ is a group, (G,τ) is a topological space and the function whose value at a member (x,y) of $G \times G$ is $x \circ y^{-1}$ is continuous relative to the product topology for $G \times G$. We sometimes denote a topological group $(G,\circ,\tau)$ simply by G.

The following are examples of topological groups :

a) The set $\mathbb{R}$ of real numbers with addition as the group operation and the usual topology form a topological group.

b) The set $\mathbb{R}^*$ of nonzero real numbers with multiplication as the group operation and the relative topology of the usual topology for $\mathbb{R}$ form a topological group.

c) The set $\mathbb{R}^+$ of positive real numbers with multiplication as the group operation and the relative topology of the usual topology for $\mathbb{R}$ form a topological group.

d) The set $\mathbb{R}^n$ of all real n-tuples with an addition +, defined by $(x_1,\dots, x_n)+(y_1,\dots, y_n) = (x_1+ y_1,\dots, x_n+ y_n)$, as a group operation and the usual topology for $\mathbb{R}^n$ form a topological group.

e) The set $\mathbb{C}^*$ of nonzero complex numbers with complex multiplication defined by $(x,y).(z,w) = (xz-yw, yz+xw)$, where $(x,y),(z,w) \in \mathbb{C}^*$, as a group operation, and the relative topology of the usual topology for $\mathbb{R}^2$ form a topological group.

If H is a subgroup of G, H endowed with the relative topology is a topological group ; it is called a topological subgroup or simply a subgroup of G. If H is a normal subgroup of G, then $G/_H$, the quotient group with respect to the equivalence relation $\tilde{H}$, and the quotient topology form a topological group; it is called the quotient group of G by $\tilde{H}$.

A function f on a topological group $(G,_o,\tau)$ onto a topological group $(G',+,\tau')$ is an isomorphism if

1) f is bijective ;
2) $f(x_oy) = f(x)+f(y)$ for all x, y in G, and
3) f and its inverse, f^{-1}, are continuous.

2.5 Vector Group

A vector group is the vector space V over the field $\mathbb{R}$ of real numbers and a topology τ on V such that addition and scalar multiplication are continuous, where the topology on $\mathbb{R}$ is the usual topology.