CHAPTER IV

ANTICENTERS OF GROUPS

The materials of this chapter are drawn from references [2], [3], [7], [8], [9], [11]

In this chapter , we introduce the notion of anticenter and study some of its elementary properties . However , the main theorem is Hill's theorem which characterized the anticenters of abelian groups .

<u>Definition 4.1</u>. Let G be a group . <u>The rim of G</u> , denoted by <u>R(G)</u> , is defined to be the set

$$\{ a \in G \ / \ \text{for all } b \in G \text{ such that } ab = ba, [a,b] \text{ is cyclic.} \} .$$

The subgroup of G generated by R(G) is called the <u>anticenter of G</u> , denoted by <u>AC(G)</u> .

Let us list some elementary properties of the rim of a group G .

<u>Property (a)</u>. The identity e of G belongs to R(G) .

<u>Proof</u>. For any $b \in G$, $eb = be$ and e , $b \in [b]$. Hence $e \in R(G)$.

<u>Property (b)</u>. If an element $a \in R(G)$, then so is its inverse .

<u>Proof</u>. For any $b \in G$ such that $a^{-1}b = ba^{-1}$, then

$$a(a^{-1}b)a = a(ba^{-1})a ,$$

so that

$$ba = ab .$$

Since $a \in R(G)$, a , $b \in [c]$ for some $c \in G$, and so a^{-1} , $b \in [c]$. Hence $a^{-1} \in R(G)$.

<u>Property (c)</u>. If an element $a \in R(G)$, then $b^{-1}ab \in R(G)$ for all $b \in G$.

<u>Proof</u>. For any b , $x \in G$ such that

$$(b^{-1}ab)x = x(b^{-1}ab) ,$$

then

$$b(b^{-1}ab)xb^{-1} = bx(b^{-1}ab)b^{-1} ,$$

so that

$$a(bxb^{-1}) = (bxb^{-1})a .$$

Since $a \in R(G)$, a , $bxb^{-1} \in [c]$ for some $c \in G$. Let $a = c^m$ and $bxb^{-1} = c^n$ for some integers m , n .
Then

$$x = b^{-1}(c^n)b = (b^{-1}cb)^n$$

and

$$b^{-1}ab = b^{-1}(c^m)b = (b^{-1}cb)^m ,$$

so that x , $b^{-1}ab \in [b^{-1}cb]$. Hence $b^{-1}ab \in R(G)$.

<u>Theorem 4.2</u>. The anticenter of a group G is a normal subgroup of G .

<u>Proof</u>. Since $R(G)$ is non-empty , by Property (a) , $AC(G)$ is non-empty .

For any element $b \in G$, let $x \in b^{-1}AC(G)b$;
then

$$x = b^{-1}yb$$

for some $y \in AC(G)$. Since $AC(G) = [R(G)]$, it follows from property (b) that

$$y = a_1a_2 \ldots\ldots a_k$$

for some $a_i \in R(G)$, $i = 1, 2, \ldots, k$. Thus

$$x = b^{-1}yb = b^{-1}(a_1a_2 \ldots\ldots a_k)b$$

$$= (b^{-1}a_1b)(b^{-1}a_2b) \ldots\ldots (b^{-1}a_kb) .$$

It follows from property (c) that $x \in AC(G)$; i.e , $b^{-1}AC(G)b \subseteq AC(G)$. Hence $AC(G)$ is normal .

Convention. For the remainder of this thesis , group means additive abelian group .

Theorem 4.3. If G is a group , then $AC(G) = R(G)$.

Proof. It is clear that $R(G) \subseteq AC(G)$.

To prove the reverse inclusion , it suffices to show that $R(G)$ is closed under the group operation since $AC(G)$ is the subgroup of G generated by $R(G)$.

Let a , b be in $R(G)$. For any x in G , a and b + x are in $[c]$ for some c in G and also b and c are in $[d]$ for some d in G . (Note : G is commutative) . Hence we can find integers m , n , s , and t so that

$$a = mc \quad , \quad b + x = nc ,$$

$$b = sd \quad , \quad c = td .$$

Now

$$a + b = mc + sd = m(td) + sd = (mt + s)d$$

and

$$x = nc - b = n(td) - sd = (nt - s)d$$

so that a + b and x are in $[d]$. It follows that a + b is in $R(G)$, as to be proved .

In particular , if G is a locally cyclic group , then every pair x , y of elements of G belong to a cyclic subgroup so that $G \subseteq R(G)$. It then follows from Theorem 4.3 that we obtain the following .

<u>Corollary 4.4</u>. If G is a locally cyclic group, then $AC(G) = R(G) = G$.

<u>Lemma 4.5</u>. Let G be a torsion group and let

$$G = \sum_{p \in \mathbb{P}} G_p$$

be the p-primary decomposition of G, where $\mathbb{P}$ is the set of prime numbers (Cf. Theorem 3.2). Then for each $p \in \mathbb{P}$,

$$G_p \cap AC(G) = AC(G_p) .$$

<u>Proof</u>. Let x be any non-zero element in $G_p \cap AC(G)$. For any y in G_p, the subgroup $[x,y]$ of G generated by x and y is contained in G_p since both x, $y \in G_p$. But $x \in AC(G)$ so that $[x,y]$ is a cyclic subgroup of G_p. Hence $x \in AC(G_p)$, and $G_p \cap AC(G) \subseteq AC(G_p)$.

Conversely, let g be any non-zero element of $AC(G_p)$. For any non-zero y in G,

$$y = \sum_{i=1}^{n} y_i ,$$

where $y_i \in G_{p_i}$ with the order $O(y_i) = p_i^{r_i}$ $(i=1,2,\ldots,n)$.

Let

$$m = \prod_{i=1}^{n} p_i^{r_i} ;$$

we will show that $m = O(y)$, the order of y. Since $my = 0$, $O(y)$ divides m. On the other hand, since

$$\sum_{i=1}^{n} O(y)y_i = O(y)y = 0$$

and $G = \sum_{p \in \mathcal{P}} G_p$ is a direct sum, we have

$$O(y)y_i = 0$$

for each $i = 1, 2, \ldots, n$. Hence $O(y_i)$ divides $O(y)$ for each $i = 1, 2, \ldots, n$, so that m divides $O(y)$. It now follows that $m = O(y)$.

Let $O(g) = p^t$.

If p is not in $\{p_1, p_2, \ldots, p_n\}$, then the greatest common divisor of m and p^t, (m, p^t), is 1 so that

$$\alpha m + \beta p^t = 1$$

for some non-zero integers α and β. Then

$$\alpha my + \beta p^t y = y$$

and

$$\alpha mg + \beta p^t g = g,$$

so that

$$\beta p^t y = y$$

and

$$\alpha mg = g.$$

Let $c = \alpha g + \beta y$; then c is in G. We have

$$mc = \alpha mg + \beta my = g$$

and

$$p^t c = \alpha p^t g + \beta p^t y = y,$$

and hence g and y belong to the cyclic subgroup $[c]$ of G. Thus g is in $AC(G)$.

If p is in $\{p_1, p_2, \ldots, p_n\}$, say $p = p_1$, then g and y_1 belong to a cyclic subgroup $[c_p]$ of G_p.

By the above case , c_p and $\sum_{i=2}^{n} y_i$ belong to a cyclic subgroup $[z]$ of G . We have

$$\sum_{i=2}^{n} y_i = \alpha z$$

and

$$c_p = \beta z$$

for some integers α, β and , therefore ,

$$y = \sum_{i=1}^{n} y_i = y_1 + \alpha z = sc_p + \alpha z$$

for some integer s , so that

$$y = s(\beta z) + \alpha z = (s\beta + \alpha) z .$$

Thus g and y are in $[z]$, and hence g is in AC(G) .

Hence , in any case , we have g is in AC(G) . Since $AC(G_p) \subseteq G_p$, g is in $G_p \cap AC(G)$.

Now the lemma is completely proved .

<u>Lemma 4.6</u>. Let G be a torsion group and

$$G = \sum_{p \in \mathbb{P}} G_p$$

be its p-primary decomposition . Then

$$AC(G) = \sum_{p \in \mathbb{P}} AC(G_p) .$$

<u>Proof</u>. Since AC(G) is clearly torsion , we have

$$AC(G) = \sum_{p \in \mathbb{P}} [AC(G)]_p$$

by Theorem 3.2. But

$$[AC(G)]_p = AC(G) \cap G_p$$

so that

$$[AC(G)]_p = AC(G_p)$$

by Lemma 4.5 and the lemma followed .

<u>Definition 4.7</u>. Let A be a subgroup of a group G . A is said to be a <u>direct summand of G</u> if there exists a subgroup B of G such that $G = A \oplus B$.

<u>Lemma 4.8.</u> Let G be a decomposable p-group , for some prime p . Then no two elements , different from 0 , from distinct summands (of the same direct sum decomposition) of G can belong to a common cyclic subgroup of G .

<u>Proof</u>. Let A and B be distinct summands of G and $G = A \oplus B \oplus C$.

Suppose that there exist non-zero elements a in A , b in B such that a and b are in a cyclic subgroup $[g]$ of G . Then

$$g = a' + b' + c$$

for some a' in A , b' in B and c in C . Since a and b are in $[g]$,

$$a = mg$$

and

$$b = ng$$

for some non-zero integers m , n , so that

$$a = m(a' + b' + c) = ma' + mb' + mc$$

and

$$b = n(a' + b' + c) = na' + nb' + nc .$$

Since a is in A , b is in B and A , B are distinct summands of G ,

$$a = ma'$$

and

$$b = nb' .$$

Then $mb' = 0$ and $na' = 0$, and hence the order of b', $O(b')$, divides m and $O(a')$ divides n .

If $O(a') = O(b')$, then $a = 0 = b$, contradicting the choice of a and b .

If $O(a') \neq O(b')$, then we may assume that $O(a') < O(b')$. Since both $O(a')$ and $O(b')$ are powers of the prime p , $O(a')$ divides m also , and so $a = 0$, contradicting the choice of a .

Hence , in any case , we have a contradiction and , therefore , the lemma is proved .

<u>Lemma 4.9.</u> If G is a group and if $AC(G) \neq 0$, then G is either torsion or torsion-free .

<u>Proof</u>. Since $AC(G) \neq 0$, there exists a non-zero g in AC(G) . For any x in G , g and x generate a cyclic subgroup [c] of G .

If the order of g is finite , then [c] is the finite cyclic subgroup of G , by Lemma 3.7 and , therefore , the order of x is finite , by Lemma 3.8 .

If the order of g is infinite , then [c] is the infinite cyclic subgroup of G , by Lemma 3.8 and , therefore , the order of x is infinite , by Lemma 3.7 .

Hence G is either torsion or torsion-free .

We are now in position to prove Hill's theorem .

<u>Theorem 4.10</u>. Let G be a non-zero group . Suppose $AC(G) \neq 0$. Then either

(a) G is torsion-free and , in this case , G is isomorphic to a subgroup of the additive abelian of the rationals $\mathcal{Q}$, and therefore AC(G) = G . or

(b) G is torsion and , in this case ,

$$AC(G) = \sum_{p \in \Pi} AC(G_p) ,$$

where $G = \sum_{p \in \Pi} G_p$ is the p-primary decomposition and where

$$AC(G_p) = \begin{cases} G_p \text{ if } G_p \text{ is cyclic or of type } p^{\infty} . \\ 0 \text{ otherwise} . \end{cases}$$

<u>Proof</u>. Since AC(G) ≠ 0 , G is either torsion or torsion-free , by Lemma 4.9 .

(a) Suppose that G is torsion-free . Choose a non-zero g in AC(G) , for any non-zero x in G , g and x generate a cyclic subgroup [c] of G . Then there exist non-zero integers m , n such that

$$g = mc$$

and

$$x = nc ,$$

so that

$$mx = ng .$$

Hence , for any non-zero x in G , there exist non-zero integers m , n such that mx = ng . Define

$$\varphi(x) = n/m$$

for non-zero x in G and

$$\varphi(0) = 0 .$$

By the same proof in Lemma 3.10 , φ is an isomorphism from G onto an additive subgroup of $\mathcal{Q}$.

Since $\mathcal{Q}$ is locally cyclic and a subgroup of locally cyclic group is locally cyclic , a subgroup of $\mathcal{Q}$ coincides with its anticenter , by Corollary 4.4 so that AC(G) = G .

(b) If G is torsion , then

$$G = \sum_{p \in \mathbb{P}} G_p$$

where $\mathbb{P}$ denotes the set of all prime numbers , by Theorem 3.2. By Lemma 4.6 ,

$$AC(G) = \sum_{p \in \mathbb{P}} AC(G_p) .$$

If G_q is decomposable for some prime q , let $G_q = A \oplus B$ be a decomposition of G_q into non-zero summands . If $AC(G_q) \neq 0$, then there exists a non-zero g in $AC(G_q)$ such that

$$g = a + b$$

for some a in A , b in B . Suppose neither a nor b is 0 . Then g and a generate a cyclic subgroup $[c_q]$ of G_q and , therefore ,

$$g - a = b$$

is in $[c_q]$, contradicting Lemma 4.8 . Suppose one of the a and b , say b , is 0 . Then the subgroup generate by g = a and any non-zero element in B is cyclic , which leads to a contradiction as before . Now , in any case , we have a contradiction , and hence $AC(G_q) = 0$. Thus , if G_q is decomposable , then $AC(G_q) = 0$.

If G_p is indecomposable for some prime p , then G_p is locally cyclic , by Lemma 3.5 , and hence it coincides with its anticenter , by Corollary 4.4 . Since G_p is cyclic or of type p^{∞} if and only if G_p is indecomposable , (Cf. [8]) , $AC(G_p) = G_p$.

Hence $AC(G) = \sum_{p \in \mathbb{P}} AC(G_p)$, where $AC(G_p) = G_p$ if G_p is cyclic or of type p^∞ , and in all other cases $AC(G_p) = 0$.

The theorem is completely proved .