CHAPTER III

CLIQUE PARAMETERS OF THE K-POWER OF PYRAMIDS

In this chapter, we investigate the values or bounds of the clique covering numbers and the clique partition numbers of the k-power of pyramids. The first section contains results of the clique covering numbers of the k-power of pyramids and the other contains results of the clique partition numbers of the square of pyramids.

3.1 Clique Coverings of the k-power of Pyramids

Definition 3.1.1. A *pyramid* or *pyramid graph* of order $\frac{n(n+1)}{2}$, PG_n, is a graph of $\frac{n(n+1)}{2}$ vertices if the vertex set $V(PG_n) = \bigcup_{i=1}^{n}\{(i,j) \mid j = 1,2,...,i\}$, then for $(a,b),(c,d) \in V(PG_n)$, vertices (a,b) and (c,d) are adjacent in PG_n if and only if ($a = c$ and $|b-d| = 1$) or ($|a-c| = 1$ and $b = d$) or ($|a-c| = 1$ and $|b-d| = 1$).

Throughout this chapter, we use this vertex set to label vertices in PG_n.

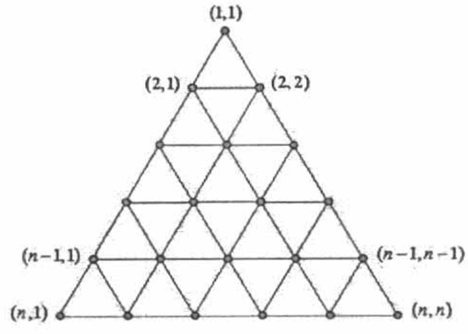

Figure 3.1: Pyramid of $\frac{n(n+1)}{2}$ vertices (PG_n)

Next, we find lower bounds of the clique covering numbers of the k-power of pyramids.

Lemma 3.1.2. *For $n, k \in \boldsymbol{N}$ where $1 \leq k < n-1$,*

$$cc(PG_n^k) \geq \frac{(n-k)(n-k+1)}{2}.$$

Proof. Let $I_k = \bigcup_{i=1}^{n-k}\{(i,j)(i+k,j) \mid j = 1,2,...,i\}$. Then I_k is a subset of $E(PG_n^k)$ and $|I_k| = \sum_{i=1}^{n-k} i = \frac{(n-k)(n-k+1)}{2}$.

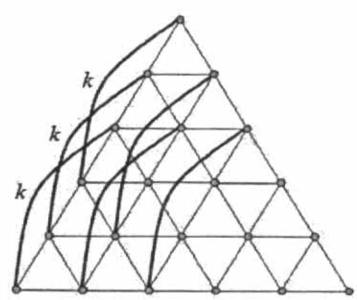

Figure 3.2: I_k in Lemma 3.1.2

We claim that I_k is a clique-independent set of PG_n^k. Let $e_1, e_2 \in I_k$ where $e_1 \neq e_2$. Then $e_1 = (i_1, j_1)(i_1+k, j_1)$ and $e_2 = (i_2, j_2)(i_2+k, j_2)$ for some $i_1, i_2 \in \{1,2,...,n-k\}$ and $j_1 \in \{1,2,...,i_1\}$ and $j_2 \in \{1,2,...,i_2\}$.

Case 1 : $i_1 \neq i_2$.

WLOG, assume $i_1 < i_2$. Then $d_{PG_n}((i_1,j_1),(i_2+k,j_2)) =$
$d_{PG_n}((i_1,j_1),(i_1+k,j_1)) + d_{PG_n}((i_1+k,j_1),(i_2,j_2)) \geq k+1 > k$. Thus (i_1,j_1) is not adjacent to (i_2+k,j_2) in PG_n^k. Hence, e_1 and e_2 are clique-independent edges of PG_n^k.

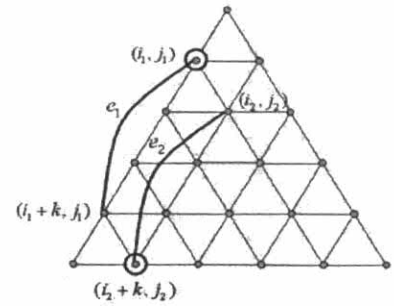

Figure 3.3: Case 1 in Lemma 3.1.2

Case 2 : $i_1 = i_2$.

WLOG, assume $j_1 < j_2$. Then $d_{PG_n}((i_1 + k, j_1), (i_2, j_2)) =$ $d_{PG_n}((i_1+k, j_1), (i_2+k, j_2))+d_{PG_n}((i_2+k, j_2), (i_2, j_2)) \geq k+1 > k$. Thus (i_1+k, j_1) is not adjacent to (i_2, j_2) in PG_n^k. Hence, e_1 and e_2 are clique-independent edges of PG_n^k.

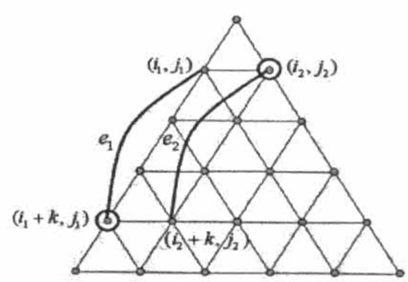

Figure 3.4: Case 2 in Lemma 3.1.2

By two cases, we can conclude that I_k is a clique-independent set of PG_n^k. Therefore, $cc(PG_n^k) \geq |I_k| = \frac{(n-k)(n-k+1)}{2}$. □

We next find upper bounds of the clique covering numbers of the k-power of pyramids.

Lemma 3.1.3. *For $l = 1, 2, ..., k,\ \ i = 1, 2, .., n - l$ and $j = 1, 2, ..., i$, let $V_l(i, j)$ be a subset of $V(PG_n)$ defined by*

$$
\begin{aligned}
V_l(i,j) = \{ \quad & (i,j), \\
& (i+1,j),(i+1,j+1), \\
& (i+2,j),(i+2,j+1),(i+2,j+2), \\
& \vdots \\
& (i+l,j),(i+l,j+1),(i+l,j+2),...,(i+l,j+l) \quad \}.
\end{aligned}
$$

Let $C_l(i, j) = PG_n^k[V_l(i, j)]$. Then $C_l(i, j)$ is a subgraph of $C_k(i', j')$ for some $i' \in \{1, 2, ..., n-k\}$ and $j' \in \{1, 2, ..., i'\}$.

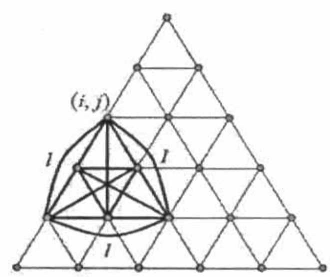

Figure 3.5: $C_l(i, j)$ in Lemma 3.1.3

Proof. If $i \leq n - k$, then $C_l(i, j)$ is a subgraph of $C_k(i, j)$. Assume $i > n - k$. If $j \leq n - k$ then $C_l(i, j)$ is a subgraph of $C_k(n - k, j)$. Otherwise $j > n - k$, then $C_l(i, j)$ is a subgraph of $C_k(n - k, n - k)$. □

In the next lemma, we show upper bounds of the clique covering numbers of the k-power of pyramids.

Lemma 3.1.4. *For $n, k \in \boldsymbol{N}$ where $1 \leq k < n - 1$,*

$$cc(PG_n^k) \leq \frac{(n-k)(n-k+1)}{2}.$$

Proof. For $l = 1, 2, ..., k,\ \ i = 1, 2, .., n - l$ and $j = 1, 2, ..., i$, let $V_l(i, j)$ be a subset of $V(PG_n)$ defined as in the previous lemma. Note that the distance between two vertices of $V_l(i, j)$ in PG_n is at most k. Thus $C_l(i, j) := PG_n^k[V_l(i, j)]$, an induced subgraph of PG_n^k, is a clique in PG_n^k. Let $\mathcal{C} = \bigcup_{i=1}^{n-k} \{C_k(i, j) \mid j = 1, 2, ..., i\}$. Then $|\mathcal{C}| = \sum_{i=1}^{n-k} i = \frac{(n-k)(n-k+1)}{2}$. We claim that $\mathcal{C}$ is a clique covering of PG_n^k. Let $e \in E(PG_n^k)$. Then $e = (i_1, j_1)(i_2, j_2)$ for some $i_1, j_1, i_2, j_2 \in \{1, 2, ..., n\}$. Assume that the distance between (i_1, j_1) and (i_2, j_2) in PG_n is d. WLOG, assume $i_1 \leq i_2$.

Case 1 : $|j_2 - j_1| \leq i_2 - i_1$. Then $e \in E(C_d(i_1, j_1))$.

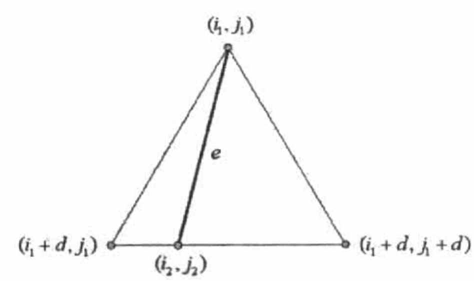

Figure 3.6: Case 1 in Lemma 3.1.4

Case 2 : $j_1 - j_2 > i_2 - i_1$. Then $e \in E(C_d(i_2 - d, j_2))$.

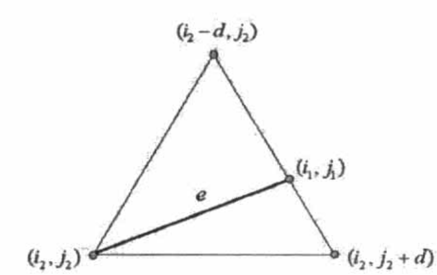

Figure 3.7: Case 2 in Lemma 3.1.4

Case 3 : $j_2 - j_1 > i_2 - i_1$. Then $e \in E(C_d(i_2 - d, j_2 - d))$.

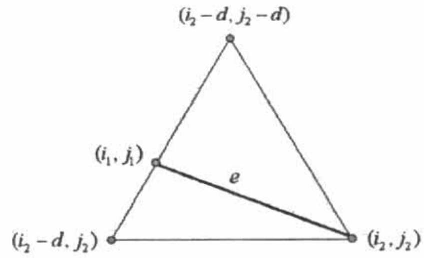

Figure 3.8: Case 3 in Lemma 3.1.4

By Lemma 3.1.3, $e \in E(C_k(i', j'))$ for some $i' \in \{1, 2, ..., n-k\}$ and $j' \in \{1, 2, ..., i'\}$. Hence $\mathcal{C}$ is a clique covering of PG_n^k.

Therefore, $cc(PG_n^k) \leq |\mathcal{C}| = \frac{(n-k)(n-k+1)}{2}$. □

Next, we conclude the values of the clique covering numbers of the k-power of pyramids in Theorem 3.1.5.

Theorem 3.1.5. *For $n, k \in \boldsymbol{N}$,*

$$cc(PG_n^k) = \begin{cases} 1 & \text{if } \; k \geq n-1, \\ \frac{(n-k)(n-k+1)}{2} & \text{if } \; 1 \leq k < n-1. \end{cases}$$

Proof. **Case 1** : $k \geq n-1$.

Since $diam(PG_n^k) = n-1$, we have that PG_n^k is a complete graph. Hence $cc(PG_n^k) = 1$.

Case 2 : $1 \leq k < n-1$.

By Lemma 3.1.2 and Lemma 3.1.4, $cc(PG_n^k) = \frac{(n-k)(n-k+1)}{2}$. □

We investigate values of the clique partition numbers of the square of pyramids in the next section.

3.2 Clique Partitions of the Square of Pyramids

First, we find the number of edges of the square of pyramids.

Proposition 3.2.1. *For $n \in \boldsymbol{N}$ where $n \geq 4$,*

$$|E(PG_n^2)| = \frac{3}{2}(n-1)(3n-4).$$

Proof. Recall that $V(PG_n^2) = V(PG_n) = \bigcup_{i=1}^{n}\{(i,j) \mid j = 1,2,...,i\}$. Thus $|V(PG_n^2)| = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. For $v \in V(PG_n^2)$,

$$d_{PG_n^2}(v) = \begin{cases} 5, & \text{if } v \in \{(1,1),(n,1),(n,n)\}, \\ 8, & \text{if } v \in \{(2,1),(2,2),(n-1,1),(n,2),(n,n-1),(n-1,n-1)\}, \\ 12, & \text{if } v \in \{(3,2),(n-1,2),(n-1,n-2)\}, \\ 11, & \text{if } v \in \{(i,1),(i,i),(n,i) \mid i = 3,4,...,n-2\}, \\ 15, & \text{if } v \in \{(i,2),(i,i-1),(n-1,i-1) \mid i = 4,5,...,n-2\}, \\ 18, & \text{otherwise.} \end{cases}$$

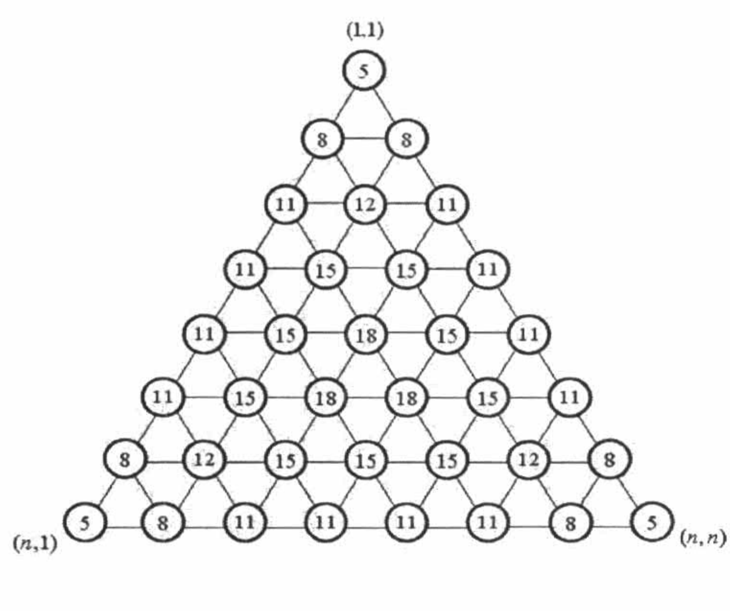

Figure 3.9: Degrees of vertices of PG_n^2

Thus

$$\begin{aligned}\sum_{v\in V(PG_n^2)} d(v) &= 5(3)+8(6)+12(3)+11[3(n-4)]+15[3(n-5)]+18\sum_{i=1}^{n-6} i\\ &= 15+48+36+33(n-4)+45(n-5)+\frac{18(n-6)(n-5)}{2}\\ &= 78n-258+9(n^2-11n+30)\\ &= 9n^2-21n+12\\ &= 3(n-1)(3n-4).\end{aligned}$$

Hence

$$|E(PG_n^2)| = \frac{\sum d(v)}{2} = \frac{3}{2}(n-1)(3n-4).$$

□

In the next theorem, we show bounds of the clique partition numbers of the square of pyramids.

Theorem 3.2.2. *For $n \in \boldsymbol{N}$.*

(i) If $n = 1, 2$ or 3, then $cp(PG_n^2) = 1$.

(ii) If $n = 2r + 1$ where $r \geq 2$, then

$$\frac{(n-2)(n-1)}{2} \leq cp(PG_n^2) \leq \frac{(n-1)(7n-19)}{4}.$$

(iii) If $n = 2r$ where $r \geq 2$, then

$$\frac{(n-2)(n-1)}{2} \leq cp(PG_n^2) \leq \frac{7n^2}{4} - 5n + 4.$$

Proof. (i) Let $n = 1,2$ or 3.

Then PG_n^2 is a complete graph. Hence $cp(PG_n^2) = 1$.

(ii) Let $n = 2r+1$ where $r \geq 2$.

For $i = 1, 3, ..., 2r-1$ and $j = 1, 3, ..., i$, let

$A_{i,j} = PG_n^2[\{(i,j), (i+1,j), (i+1,j+1), (i+2,j), (i+2,j+1), (i+2,j+2)\}]$.

We have that $A_{i,j}$ is a copy of K_6 and $|E(A_{i,j})| = 15$.

For $i = 3, 5, ..., 2r-1$ and $j = 2, 4, ..., i-1$, let

$B_{i,j} = PG_n^2[\{(i,j), (i+1,j), (i+1,j+1)\}]$ and

$C_{i,j} = PG_n^2[\{(i,j), (i+2,j), (i+2,j+2)\}]$.

We have that $B_{i,j}$ and $C_{i,j}$ are copies of K_3 and $|E(B_{i,j})| = |E(C_{i,j})| = 3$.

For $i = 2, 4, ..., 2r-2$ and $j = 1, 2, ..., i$, let

$D_{i,j} = PG_n^2[\{(i,j), (i+2,j), (i+2,j+2)\}]$.

We have that $D_{i,j}$ is a copy of K_3 and $|E(D_{i,j})| = 3$.

By Proposition 3.2.1,

$$\begin{aligned}
|E(PG_n^2)| &= \frac{3}{2}(n-1)(3n-4) \\
&= \frac{3}{2}((2r+1)-1)(3(2r+1)-4) \qquad (\text{ since } n = 2r+1 \text{ }) \\
&= \frac{3}{2}(2r)(6r-1) \;=\; 18r^2 - 3r.
\end{aligned}$$

Let

$$H \;=\; PG_n^2 \setminus [\bigcup_{\substack{i=1,3,...,2r-1 \\ j=1,3,...,i}} A_{i,j} + \bigcup_{\substack{i=3,5,...,2r-1 \\ j=2,4,...,i-1}} B_{i,j} + \bigcup_{\substack{i=3,5,...,2r-1 \\ j=2,4,...,i-1}} C_{i,j} + \bigcup_{\substack{i=2,4,...,2r-2 \\ j=1,2,...,i}} D_{i,j} \;].$$

Then

$$\begin{aligned}
|E(H)| &= (18r^2 - 3r) - [\; 15\sum_{i=1}^{r} i + 3\sum_{i=1}^{r-1} i + 3\sum_{i=1}^{r-1} i + 3\sum_{i=1}^{r-1} 2i \;] \\
&= 18r^2 - 3r - \frac{15(r)(r+1)}{2} - 3(r-1)(r) - 3(r-1)(r) \\
&= 12r^2 + 3r - \frac{15(r)(r+1)}{2} \\
&= \frac{9r^2 - 9r}{2}.
\end{aligned}$$

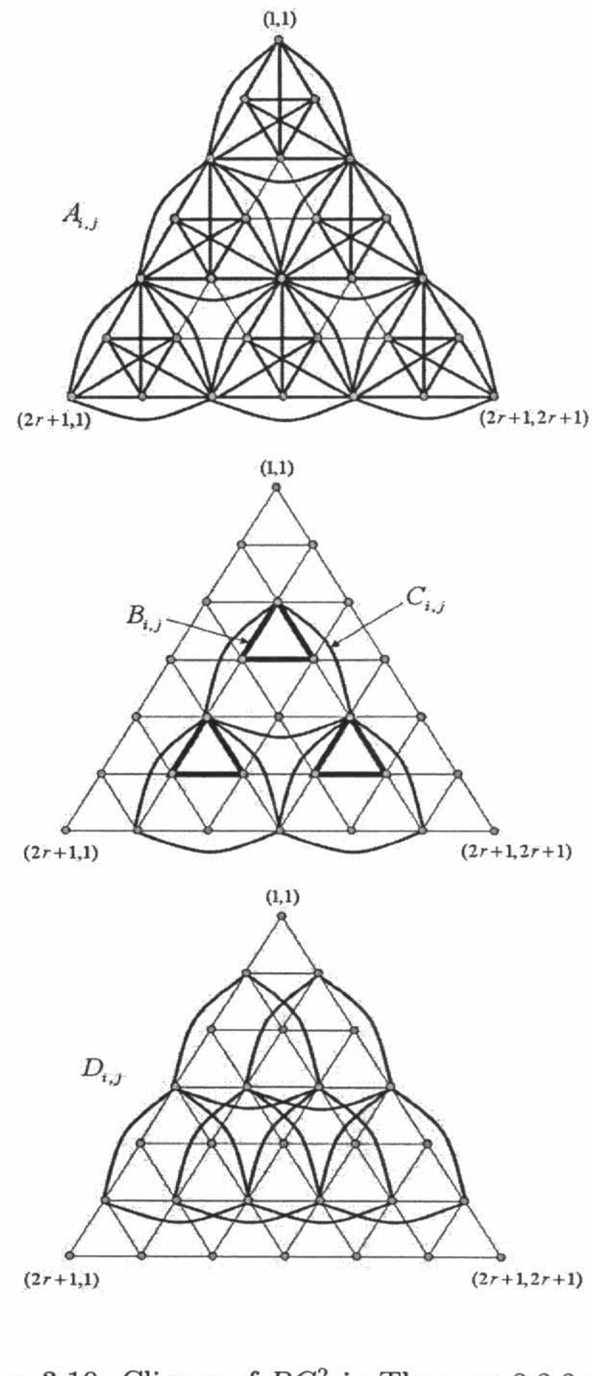

Figure 3.10: Cliques of PG_n^2 in Theorem 3.2.2 (ii)

We have that

$$\bigcup_{\substack{i=1,3,...,2r-1\\ j=1,3,...,i}} \{A_{i,j}\} \cup \bigcup_{\substack{i=3,5,...,2r-1\\ j=2,4,...,i-1}} \{B_{i,j}\} \cup \bigcup_{\substack{i=3,5,...,2r-1\\ j=2,4,...,i-1}} \{C_{i,j}, D_{i,j}\} \cup E(H)$$

forms a clique partition $\mathcal{P}$ of PG_n^2 such that

$$\begin{aligned}|\mathcal{P}| &= \sum_{i=1}^{r} i + 2\sum_{i=1}^{r-1} i + \sum_{i=1}^{r-1} 2i + \frac{9r^2-9r}{2}\\ &= \frac{r(r+1)}{2} + (r-1)(r) + (r-1)(r) + \frac{9r^2-9r}{2}\\ &= \frac{r(r+1)}{2} + \frac{13r(r-1)}{2}\\ &= \frac{14r^2-12r}{2} = r(7r-6).\end{aligned}$$

Since $r = \frac{n-1}{2}$,

$$|\mathcal{P}| = (\frac{n-1}{2})(7(\frac{n-1}{2}) - 6) = \frac{(n-1)(7n-19)}{4}.$$

Thus $cp(PG_n^2) \leq \frac{(n-1)(7n-19)}{4}$. By Theorem 3.1.5, $cc(PG_n^2) = \frac{(n-2)(n-1)}{2}$. Hence $cp(PG_n^2) \geq \frac{(n-2)(n-1)}{2}$. Therefore, $\frac{(n-2)(n-1)}{2} \leq cp(PG_n^2) \leq \frac{(n-1)(7n-19)}{4}$.

(iii) Let $n = 2r$ where $r \geq 2$.

For $i = 1, 3, ..., 2r-3$ and $j = 1, 3, ..., i$, let

$A_{i,j} = PG_n^2[\{(i,j), (i+1,j), (i+1,j+1), (i+2,j), (i+2,j+1), (i+2,j+2)\}]$.

We have that $A_{i,j}$ is a copy of K_6 and $|E(A_{i,j})| = 15$.

For $i = 3, 5, ..., 2r-3$ and $j = 2, 4, ..., i-1$, let

$B_{i,j} = PG_n^2[\{(i,j), (i+1,j), (i+1,j+1)\}]$ and

$C_{i,j} = PG_n^2[\{(i,j), (i+2,j), (i+2,j+2)\}]$.

We have that $B_{i,j}$ and $C_{i,j}$ are copies of K_3 and $|E(B_{i,j})| = |E(C_{i,j})| = 3$.

For $i = 2, 4, ..., 2r-2$ and $j = 1, 2, ..., i$, let

$D_{i,j} = PG_n^2[\{(i,j), (i+2,j), (i+2,j+2)\}]$.

We have that $D_{i,j}$ is a copy of K_3 and $|E(D_{i,j})| = 3$.

For $j = 1, 2, ..., 2r-1$, let

$F_j = PG_n^2[\{(2r-1, j), (2r, j), (2r, j+1)\}]$.

We have that F_j is a copy of K_3 and $|E(F_j)| = 3$.

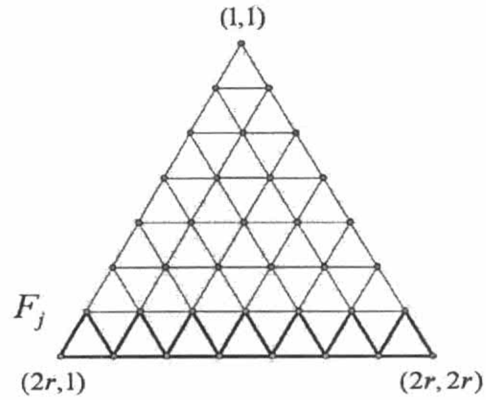

Figure 3.11: F_j in Theorem 3.2.2 (iii)

By Proposition 3.2.1,

$$\begin{aligned} |E(PG_n^2)| &= \frac{3}{2}(n-1)(3n-4) \\ &= \frac{3}{2}((2r)-1)(3(2r)-4) \qquad (\text{ since } n = 2r\ ) \\ &= 18r^2 - 21r + 6. \end{aligned}$$

Let

$$H = PG_n^2 \setminus [\bigcup_{\substack{i=1,3,...,2r-3 \\ j=1,3,...,i}} A_{i,j} + \bigcup_{\substack{i=3,5,...,2r-3 \\ j=2,4,...,i-1}} B_{i,j} + \bigcup_{\substack{i=3,5,...,2r-3 \\ j=2,4,...,i-1}} C_{i,j} + \bigcup_{\substack{i=2,4,...,2r-2 \\ j=1,2,...,i}} D_{i,j} + \bigcup_{j=1,2,...,2r-1} F_j\ ].$$

Then

$$\begin{aligned} |E(H)| &= (18r^2 - 21r + 6) - [\ 15\sum_{i=1}^{r-1} i + 3\sum_{i=1}^{r-2} i + 3\sum_{i=1}^{r-2} i + 3\sum_{i=1}^{r-1} 2i + 3(2r-1)\ ] \\ &= 18r^2 - 21r + 6 - \frac{15(r-1)(r)}{2} - 3(r-2)(r-1) - 3(r-1)(r) - 3(2r-1) \\ &= 12r^2 - 15r + 3 - \frac{15(r-1)(r)}{2} \\ &= \frac{9r^2 - 15r + 6}{2}. \end{aligned}$$

We have that

$$\bigcup_{\substack{i=1,3,\ldots,2r-3\\ j=1,3,\ldots,i}} \{A_{i,j}\} \cup \bigcup_{\substack{i=3,5,\ldots,2r-3\\ j=2,4,\ldots,i-1}} \{B_{i,j}, C_{i,j}\} \cup \bigcup_{\substack{i=2,4,\ldots,2r-2\\ j=1,2,\ldots,i}} \{D_{i,j}\} \cup \bigcup_{j=1,2,\ldots,2r-1} \{F_j\} \cup E(H)$$

forms a clique partition $\mathcal{P}$ of PG_n^2 such that

$$\begin{aligned} |\mathcal{P}| &= \sum_{i=1}^{r-1} i + 2\sum_{i=1}^{r-2} i + \sum_{i=1}^{r-1} 2i + (2r-1) + \frac{9r^2-15r+6}{2} \\ &= \frac{(r-1)(r)}{2} + (r-2)(r-1) + (r-1)(r) + (2r-1) + \frac{9r^2-15r+6}{2} \\ &= \frac{14r^2-20r+8}{2} \\ &= 7r^2 - 10r + 4. \end{aligned}$$

Since $r = \frac{n}{2}$, $|\mathcal{P}| = \frac{7n^2}{4} - 5n + 4$. Thus $cp(PG_n^2) \leq \frac{7n^2}{4} - 5n + 4$.
By Theorem 3.1.5, $cc(PG_n^2) = \frac{(n-2)(n-1)}{2}$. Hence, $cp(PG_n^2) \geq \frac{(n-2)(n-1)}{2}$.
Therefore, $\frac{(n-2)(n-1)}{2} \leq cp(PG_n^2) \leq \frac{7n^2}{4} - 5n + 4$. □

We have obtained the complete results of the values of the clique covering numbers of the k-power of pyramids for all $k \in \mathbf{N}$. For the values of the clique partition numbers of the k-power of pyramids, we get bounds of the clique partition numbers of the square of pyramids. Open problems for future work are to improve bounds of the clique partition numbers of the square of pyramids, and find the values of the clique partition numbers of the k-power of pyramids where $k \geq 3$.