CHAPTER I

INTRODUCTION

1.1 Definitions

Definition 1.1.1. $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' of G is an *induced subgraph* if G' contains all edges of G that join two vertices in V', and is denoted by $G[V']$. Let H be a subgraph of a graph G. We write $G \backslash H$ for the subgraph of G obtained by deleting the set of edges $E(H)$. A graph G is *H-free* if G does not contain H as a subgraph.

Definition 1.1.2. The *union* of graphs $G_1, G_2, ..., G_k$, written $G_1 \cup G_2 \cup ... \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. The graph obtained by taking the union of graphs $G_1, G_2, ..., G_k$ with pairwise disjoint edge sets is the *disjoint union*, written $G_1 + G_2 + ... + G_k$.

Definition 1.1.3. A *complete graph* is a graph in which each pair of vertices is joined by an edge. The complete graph with n vertices is denoted by K_n.

Example 1.1.4. Complete graphs.

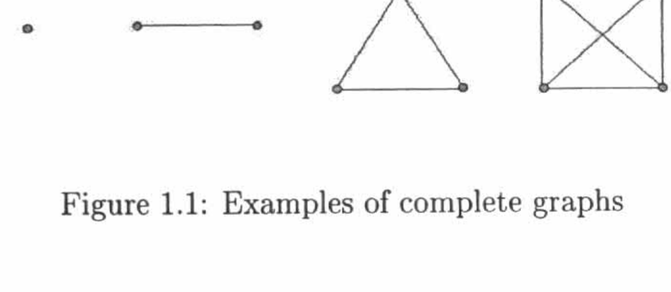

Figure 1.1: Examples of complete graphs

Definition 1.1.5. A *clique* of G is a complete subgraph of G.

Example 1.1.6. Cliques of $K_4\backslash e$.

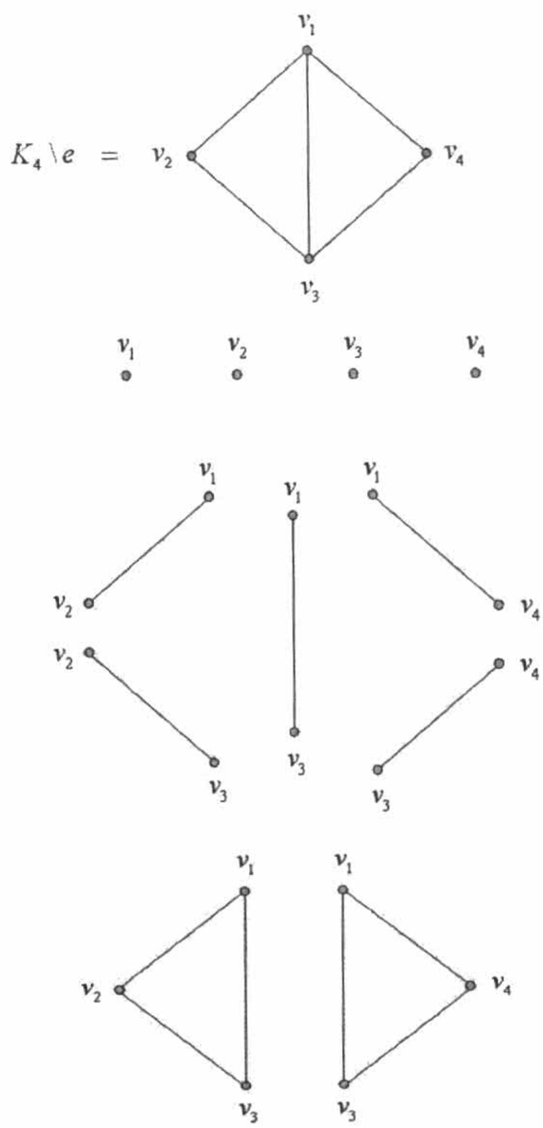

Figure 1.2: All cliques of $K_4\backslash e$

Definition 1.1.7. A *clique covering* of G is a set of cliques of G, which together contain each edge of G at least once. The smallest cardinality of clique coverings of G is called the *clique covering number* of G, and is denoted by $cc(G)$.
A *minimum clique covering* of G is a clique covering of G having size $cc(G)$.

Example 1.1.8. $cc(K_4\backslash e) = 2$.

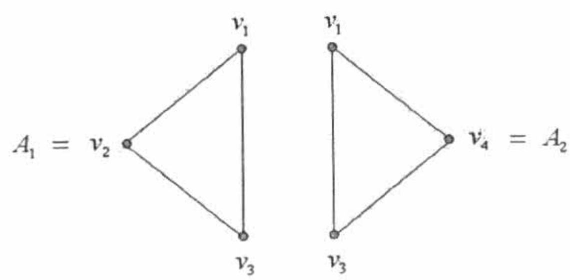

Figure 1.3: $\{A_1, A_2\}$ is a minimum clique covering of $K_4\backslash e$ in Figure 1.2

Definition 1.1.9. A *clique partition* of G is a set of cliques of G, which together contain each edge of G exactly once. The smallest cardinality of clique partitions of G is called the *clique partition number* of G, and is denoted by $cp(G)$.
A *minimum clique partition* of G is a clique partition of G having size $cp(G)$.

Example 1.1.10. $cp(K_4\backslash e) = 3$.

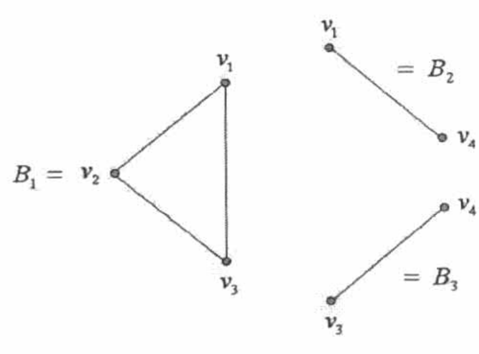

Figure 1.4: $\{B_1, B_2, B_3\}$ is a minimum clique partition of $K_4\backslash e$ in Figure 1.2

Remark 1.1.11. For any graph G, $cc(G)$ and $cp(G)$ always exist because every graph has at least one clique covering and clique partition, namely the edge set of the graph.

Remark 1.1.12. For any graph G, $cc(G) \leq cp(G)$ because clique partitions of G are clique coverings of G.

Definition 1.1.13. Let e and f be any two edges in a graph G. If there is no clique in G containing both e and f, then e and f are *clique-independent edges* of G. A set of pairwise clique-independent edges is called a *clique-independent set.*

Example 1.1.14. Clique-independent set of a graph.

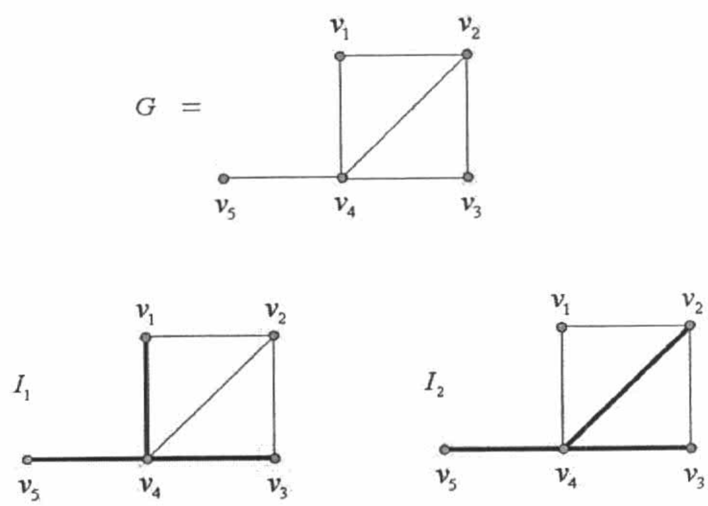

Figure 1.5: Example of a clique-independent set of a graph

Let G be the graph shown in Figure 1.5. Let $I_1 = \{v_1v_4, v_5v_4, v_4v_3\}$ and $I_2 = \{v_2v_4, v_5v_4, v_4v_3\}$ be subsets of the edge set of G. We have that I_1 is a clique-independent set of G, but I_2 is not a clique-independent set of G because there is a clique in G containing both v_2v_4 and v_4v_3.

Remark 1.1.15. Let e and f be any two edges in a graph G. If there exist two endpoints of e and f that are not adjacent in G, then e and f are *clique-independent edges* of G.

Remark 1.1.16. Let I be a clique-independent set of G. Since differrent elements in I must be covered by differrent cliques of G, $cc(G) \geq |I|$.

Definition 1.1.17. The graph G^k is the *k-power* of a graph G if $V(G^k) = V(G)$ and there is an edge between vertices u and v in G^k if and only if there is a path of length at most k between u and v in G.

Example 1.1.18. The k-power of graphs.

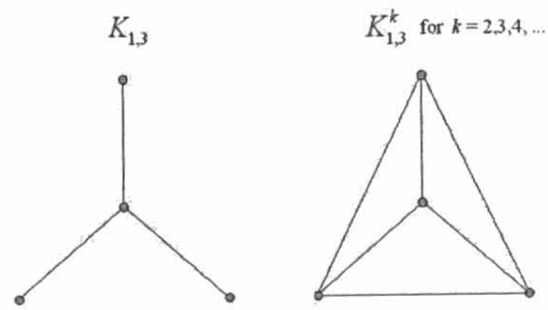

Figure 1.6: Example of the k-power of a star

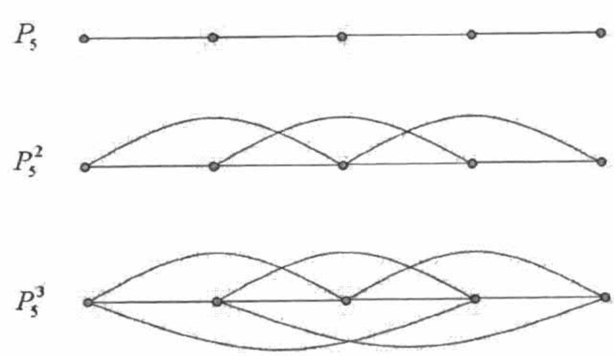

Figure 1.7: Example of the square and cube of a path

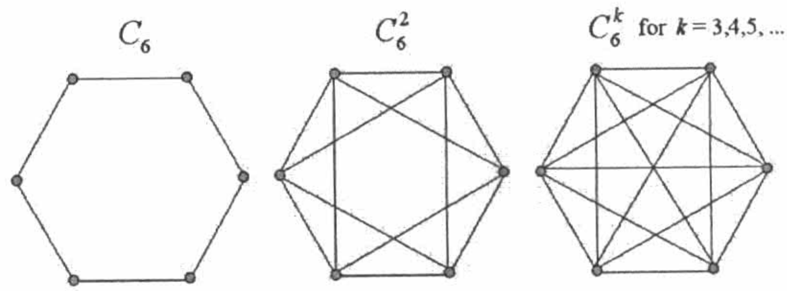

Figure 1.8: Example of the k-power of a cycle

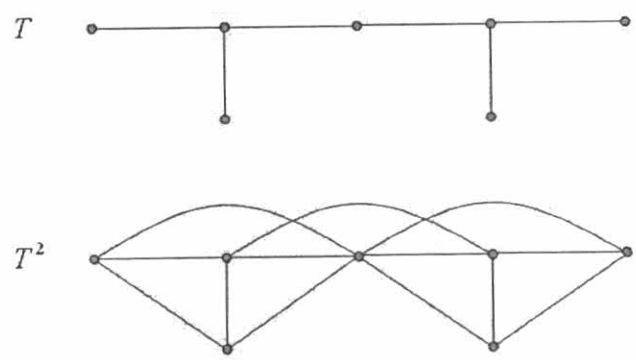

Figure 1.9: Example of the square of a tree

Definition 1.1.19. The *distance* between two vertices u and v of graph G, written $d_G(u,v)$ or simply $d(u,v)$, is the minimum length of the paths connecting them. The *diameter* of G, written $diam(G)$, is $max_{u,v \in V(G)} d(u,v)$.

Remark 1.1.20. Two vertices in G^k are adjacent if and only if the distance between them in G is at most k. If $diam(G) = m$, then G^k is a complete graph for all $k \geq m$.

1.2 History and Overview

The clique coverings and the clique partitions of graphs have been studied since 1941. Varities of literatures are involved in this topic. Hall [7] proved that the edge set of any graph G on n vertices can be covered using at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ cliques, none of which need to be larger than a triangle. In 1948, de Bruijn and Erdős [3] proved that for $n \geq 3$, if C is a non-trivial clique partition of K_n, then $|C| \leq n$. Erdős, Goodman and Pósa [4] showed that the edge set of any simple graph G with n vertices can be partitioned using at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ cliques, and that the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ gives equality. Pullman and de Caen in 1981 [12] studied clique partitions of regular graphs, the cartesian product of graphs and graphs of maximal degree at most three.

In addition, Gregory and Pullman [6] found the clique covering number of the complement of a perfect matching. Gregory, McGuinness and Wallis [5] proved that for n sufficiently large, the complement of a perfect matching on n vertices can be partitioned into about $n\log_2\log_2 n$ cliques. Shaohan and Wallis [13] studied clique partitions of triangulated graphs. Wallis and Wu [16] also investigated clique partitions of split graphs in 1991.

In 1996, Monson [11] listed results that the effect vertex and edge deletion have on the clique partition number of a graph. More recently, maximal-clique partitions have been studied by Uiyyasathian [14]. Cavers [2] reviewed the clique partition numbers and the clique covering numbers of graphs and introduced some new results in 2005.

Next, the square and the power of graphs are investigated boardly. In 1991, Wallis and Wu [15] formulated the clique partition numbers of the square of trees. Harary and McKee [8] studied the square of a chordal graph.

In 2003, Lih, Wang and Zhu [9] found the chromatic number of the square of a K_4-minor free graph. Molloy and Salavatipour [10] found a bound of the chromatic number of the square of a planar graph.

Agnarsson, Damaschke and Halldórsson in 2004 [1] investigated powers of certain geometric intersection graphs: interval graphs, m-trapezoid graphs and circular-arc graphs.

In this thesis, we investigate values or bounds of the clique covering numbers and the clique partition numbers of the k-power of graphs. We start with finding the values of the clique covering numbers and the clique partition numbers of the k-power of paths and cycles in Chapter 2. Then we further investigate the values of the clique covering numbers and the clique partition numbers of the k-power of pyramids in Chapter 3. And we explore more on ladders and grids in Chapter 4.

Lastly, the conclusion and some open problems for future work are located in Chapter 5.