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THE CONVERGENCE OF DISTRIBUTION FUNCTIONS OF RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES WITH FINITE VARIANCES

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กำหนดให้ (X_{nk}) เป็นลำดับสองชั้นของตัวแปรสุ่มที่มีค่าความแปรปรวนจำกัด ให้ (Z_n) เป็นลำดับของตัวแปรสุ่มที่มีค่าเป็นจำนวนเต็มบวก ซึ่งแต่ละจำนวนนับ n, $Z_n, X_{n1}, X_{n2},...$เป็นอิสระต่อกัน

ในวิทยานิพนธ์นี้ เราให้เงื่อนไขที่จำเป็นและเงื่อนไขเพียงพอที่ทำให้ลำดับของฟังก์ชันการแจกแจงของ

$$X_{n1}+X_{n2}+...+X_{nZ_n}-\sum_{k=1}^{Z_n}\mu_{nk}$$

ลู่เข้าอย่างอ่อน เมื่อ μ_{nk} คือ ค่าคาดคะเนของตัวแปรสุ่ม X_{nk}

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต...
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Let (X_{nk}) be a double sequence of random variables with finite variances. Let (Z_n) be a sequence of positive integral-valued random variables such that for each *n*, $Z_n, X_{n1}, X_{n2},...$ are independent.

In this study, we give necessary conditions and sufficient conditions for the sequence of distribution functions of the sums

$$X_{n1}+X_{n2}+...+X_{nZ_n}-\sum_{k=1}^{Z_n}\mu_{nk}$$

to be weakly convergent, where μ_{nk} is the expectation of X_{nk}.

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Field of Study **Mathematics**	Advisor's signature…………………………
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CHAPTER I

Introduction

In this Chapter, (X_{nk}) is a double sequence of random variables with finite variances σ_{nk}^2 and μ_{nk} the expectation of X_{nk}.

The problem of the convergence of distribution functions of sums of independent random variables has been discussed many times. One of the most important versions of the convergence theorem is the following theorem.

Theorem 1.1(Lindeberg) Let (X_{nk}) be a double sequence of random variables with finite variances σ_{nk}^2, $k=1,2,...,k_n, n=1,2,...$. Assume that for each n, $X_{n1}, X_{n2},..., X_{nk_n}$ are independent and $\mu_{nk}=0$ for all n, k and $\lim_{n\to\infty}\sum_{k=1}^{k_n}\sigma_{nk}^2 = 1$. Then

(i) the sequence of distribution functions of the sums $X_{n1}+X_{n2}+...+X_{nk_n}$ weakly converges to the standard normal distribution function and

(ii) (X_{nk}) is infinitesimal , i.e. $\lim_{n\to\infty}\max_{1\le k\le k_n} P(|X_{nk}|\ge\varepsilon)=0$ for all $\varepsilon>0$,

if and only if (X_{nk}) satisfies Lindeberg condition ,i.e.

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<\varepsilon} x^2 dF_{nk}(x) = 1 \quad \text{for all } \varepsilon>0 .$$

Later , Kolmogorov generalized Theorem 1.1 to the case that the limit distribution function is any infinitely divisible distribution function. There are two important convergence theorems (Theorem 1.2 and Theorem 1.3). In the first theorem (X_{nk}) must satisfy the following conditions:

(α) $(X_{nk}-\mu_{nk})$ is infinitesimal, i.e.,

$$\max_{1\le k\le k_n} P(|X_{nk}-\mu_{nk}|\ge\varepsilon)\to 0 \text{ as } n\to\infty \text{ for all } \varepsilon>0 .$$

(β) There exists a real number C such that $\sum_{k=1}^{k_n}\sigma_{nk}^2 < C$.

In order to prove Theorem 1.2 , Kolmogorov defined the accompanying distribution function of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

to be the distribution function whose logarithm of its characteristic function φ is given by

$$\ln\varphi_n(t) = -iA_n t + it\sum_{k=1}^{k_n}\mu_{nk} + \sum_{k=1}^{k_n}\int_{-\infty}^{\infty}(e^{itx}-1)dF_{nk}(x+\mu_{nk}).$$

This accompanying distribution function is infinitely divisible.

Theorem 1.2 ([6], p.98) Assume that (X_{nk}) satisfies the conditions (α), (β) and for each n, $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that the sequence of distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

converges weakly to a limit distribution function if and only if the sequence of accompanying distribution functions of S_n converges weakly to the same limit distribution function.

Theorem 1.3 ([6], p.100) Assume that (X_{nk}) satisfies the condition (α) and for each n , $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that

(i) the sequence of distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

converges weakly to a limit distribution function F whose variance is σ^2 and

(ii) $\sum_{k=1}^{k_n}\sigma_{nk}^2 \to \sigma^2$

if and only if there exists a function K in $\mathcal{M}$ such that

(i′) $K_{k_n}(u) \to K(u)$ for every continuity point u of K and (ii′) $K_{k_n}(+\infty) \to K(+\infty)$

where

$$K_{k_n}(u) = \sum_{k=1}^{k_n} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk})$$

and m is the set of bounded , non-decreasing , right-continuous functions from $\mathbf{R}$ into $[0,\infty)$ which vanish at $-\infty$.

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \mu_{nk} - \mu$$

where μ is any real number. Logarithm of the characteristic function φ of the limit distribution function is given by

$$\ln\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)\,dK(x)$$

where
$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\dfrac{1}{x^2} & \text{if } x \neq 0 \\ -\dfrac{t^2}{2} & \text{if } x = 0. \end{cases}$$

In [1] , Bethmann generalized Theorem 1.1 to the case that the number of summands is random. The theorem is as follows.

Theorem 1.4 ([1]) Let (X_{nk}) be a double sequence of random variables with finite variances σ_{nk}^2 and (Z_n) a sequence of positive integral-valued random variables. Assume that for each n, $Z_n, X_{n1}, X_{n2}, \ldots$ are independent , $Z_n \xrightarrow{P} \infty$, $\mu_{nk} = 0$ for all n, k and $\lim_{n\to\infty} E(\sum_{k=1}^{Z_n} \sigma_{nk}^2) = 1$. Then

(i) the sequence of distribution functions of $X_{n1} + X_{n2} + \ldots + X_{nZ_n}$ converges weakly to the standard normal distribution function and

(ii) (X_{nk}) is random infinitesimal with respect to (Z_n), i.e.,

$$\max_{1 \le k \le Z_n} P(|X_{nk}| \ge \varepsilon) \xrightarrow{P} 0 \text{ as } n \to \infty \text{ for all } \varepsilon > 0,$$

if and only if

$$\sum_{k=1}^{Z_n} \int_{|x|<\varepsilon} x^2 dF_{nk}(x) \xrightarrow{P} 1 \text{ as } n \to \infty \text{ for all } \varepsilon > 0 .$$

In this work , we generalize Theorem 1.3 to the case that the limit distribution function is any infinitely divisible distribution function.

In the main theorems, we assume (Z_n, X_{nk}) satisfies the following conditions:

(α) $(X_{nk} - \mu_{nk})$ is random infinitesimal with respect to (Z_n), i.e.,

$$\max_{1 \le k \le Z_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \xrightarrow{P} 0 \text{ as } n \to \infty \text{ for all } \varepsilon > 0 .$$

(β) There exists a constant $C>0$ such that $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < C$ for a.e. $q \in (0,1)$.

The main theorems are the followings.

Theorem 1.5 Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the condition $(\tilde{\alpha})$, $(\tilde{\beta})$ and for each n ,Z_n ,X_{n1} ,X_{n2} ,... are independent. If there exists a distribution function F with finite variance σ^2 such that

(i) the sequence of the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - \sum_{k=1}^{Z_n} \mu_{nk}$$

converges weakly to F and

(ii) $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$,

then for a.e. $q \in (0,1)$, there exist a subsequence (n_k) and a function $K^{(q)}$ in $\mathcal{M}$ such that

(i′) $K_{l_{n_k}(q)}(u) \to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$ and

(ii′) $K_{l_{n_k}(q)}(+\infty) \to K^{(q)}(+\infty)$ and $\sigma^2 = \int_0^1 K^{(q)}(+\infty) dq$, where $$K_{l_n(q)}(u) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk}).$$

Theorem 1.6 Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the condition $(\tilde{\alpha}), (\tilde{\beta})$ and for each n, $Z_n, X_{n1}, X_{n2}, \ldots$ are independent. If for a.e. $q \in (0,1)$, there exists a function $K^{(q)}$ in $\mathcal{M}$ such that

(i′) $K_{l_n(q)}(u) \to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$

(ii′) $K_{l_n(q)}(+\infty) \to K^{(q)}(+\infty)$ and $\int_0^1 K^{(q)}(+\infty)dq = \sigma^2$ for some constant σ^2 and

(iii′) for each $x \in \mathbf{R}$, $F^{(q)}(x)$ is measurable in q, where $F^{(q)}$ is the distribution function whose logarithm of its characteristic function $\varphi^{(q)}$ is given by

$$\ln \varphi^{(q)}(t) = \int_{\mathbf{R}} f(t,u)dK^{(q)}(u),$$

then

(i) the sequence of the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - \sum_{k=1}^{Z_n} \mu_{nk}$$

converges weakly to F where $F(x) = \int_0^1 F^{(q)}(x)dq$, $x \in \mathbf{R}$ and

(ii) $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$ and σ^2 is the variance of F.

In chapter II , some important preliminary results and notations , which are necessary for this work , are presented. Chapter III contains our main results.

CHAPTER II

Preliminaries

2.1 Random Variables and Modes of Convergence

A **probability space** is a measure space $(\Omega, \Im, P)$ in which P is a positive measure such that $P(\Omega)=1$. The set Ω will be referred to as a **sample space**. The elements of $\Im$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A.

A function X from a probability space $(\Omega, \Im, P)$ to the set of complex numbers **C** is said to be a **complex-valued random variable** if for every Borel set B in **C**, $X^{-1}(B)$ belongs to $\Im$. If X is real-valued, we say that it is a **real-valued random variable**, or simply a **random variable**. We note that the composition between a Borel function and a complex-valued random variable is also a complex-valued random variable.

We will use the notation $P(X \le x), P(X \ge x)$ and $P(|X| \ge x)$ to denote $P(\{\omega | X(\omega) \le x\}), P(\{\omega | X(\omega) \ge x\})$ and $P(\{\omega | |X(\omega)| \ge x\})$, respectively.

We define the **expectation** of a complex-valued random variable X to be

$$\int_{\Omega} X dP$$

provided that the integral $\int_{\Omega} X dP$ exists. It will be denoted by $E[X]$.

Proposition 2.1.1 ([2], p.174) Let $X_1, X_2, \ldots, X_n$ be random variables. Then

$$E[X_1 + X_2 + \ldots + X_n] = \sum_{j=1}^{n} E[X_j],$$

provided that the sums on the right hand side is meaningful.

Let $(\Omega,\Im,\mu)$ be a measure space and Y a topological space. Let X, $X_1, X_2, \ldots, X_n$ be measurable functions from Ω to Y. We will write

$$X_n \longrightarrow X \quad \text{a.e. } [\mu]$$

if (X_n) converges to X almost everywhere with respect to μ. In the case $\Omega = \mathbf{R}^k$ and μ is the Lebesgue measure on $\mathbf{R}^k$, we simply write

$$X_n \longrightarrow X \quad \text{a.e.}.$$

A sequence (X_n) of measurable complex-valued functions is said to **converge in measure** to a measurable complex-valued function X if

$$\lim_{n\to\infty} \mu(\{x \mid |X_n(x) - X(x)| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

From now on, we shall assume that all our complex-valued random variables, including real-valued random variables, are defined on a common probability space $(\Omega,\Im,P)$.

A sequence (X_n) of complex-valued random variables is said to **converge in probability** to a complex-valued random variable X if (X_n) converges in measure to X with respect to the probability measure. In this case, we write

$$X_n \xrightarrow{P} X.$$

The following theorems are known properties of convergence in probability.

Theorem 2.1.2 ([9], p.201) Let $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots$ be complex-valued random variables. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.

Theorem 2.1.3 ([7], p.46) Let $X, Y, X_1, X_2, \ldots$ be complex-valued random variables.

(i) If $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$, then $X = Y$ a.e.$[P]$.

(ii) If $X_n \xrightarrow{P} X$, then for every subsequence (X_{n_k}) of (X_n),

$X_{n_k} \xrightarrow{P} X$.

2.2 Distribution Functions and Characteristic Functions

A function F from **R** to **R** is said to be a **distribution function** if it is non-decreasing, right-continuous, $F(-\infty)=0$ and $F(+\infty)=1$.

For any random variable X, the function $F:\mathbf{R}\to\mathbf{R}$ defined by

$$F(x)=P(X\le x)$$

is a distribution function. It is the **distribution function of the random variable** X.

Theorem 2.2.1 ([2], p.57) A function F is a distribution function of a random variable if and only if F is non-decreasing, right-continuous, $F(-\infty)=0$ and $F(+\infty)=1$.

Proposition 2.2.2 ([7], p.28) Let X be a random variable with the distribution function F. If $E[X]$ exists, then

$$E[X]=\int_{-\infty}^{\infty} x\,dF(x)\ .$$

The expectation of a random variable X is also known as the **mean** of X. The expectation of $(X-E[X])^2$ is known as the **variance** of X and is denoted by $\sigma^2(X)$. Note that mean and variance of a random variable may be infinite.

Let F be a distribution function. The function $\varphi:\mathbf{R}\to\mathbf{C}$ defined by

$$\varphi(t)=\int_{-\infty}^{\infty} e^{itx}\,dF(x)$$

is called the **characteristic function of the distribution function** F . If F is the distribution function of a random variable X , then φ is also called the **characteristic function of** X .

Proposition 2.2.3 ([6], p.45) For any characteristic function φ, we have

(i) $\varphi(0)=1$.

(ii) $|\varphi(t)|\leq 1$ for every t .

(iii) φ is continuous.

Proposition 2.2.4 ([8], p.45)

(i) The product of two characteristic functions is a characteristic function.

(ii) If φ is a characteristic function, then $|\varphi|^2$ is also a characteristic function.

Proposition 2.2.5 ([3], p.477) Let (F_n) be a sequence of distribution functions and (φ_n) a sequence of corresponding characteristic functions. Let (p_n) be a sequence of non-negative numbers such that $\sum_{k=1}^{\infty} p_k = 1$. Then the function

$$F(x)=\sum_{k=1}^{\infty} p_k F_k(x)$$

is a distribution function and the function

$$\varphi(t)=\sum_{k=1}^{\infty} p_k \varphi_k(t)$$

is the characteristic function of F .

The random variables $X_1, X_2, \ldots, X_n$ are called **independent** if

$$P\left(\bigcap_{j=1}^{n}\{\omega \mid X_j(\omega)\leq x_j\}\right)=\prod_{j=1}^{n} P(X_j \leq x_j)$$

holds for every real numbers $x_1, x_2, \ldots, x_n$.

A sequence of random variables (X_n) is said to be a **sequence of independent random variables** if $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ are independent for all distinct $i_1, i_2, \ldots, i_k$.

Theorem 2.2.6 ([2], p.188,191) Let $X_1, X_2, \ldots, X_n$ be random variables with the characteristic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively. Assume that $X_1, X_2, \ldots, X_n$ are independent. Then the followings hold.

(i) The characteristic function φ of $X_1 + X_2 + \ldots + X_n$ is given by

$$\varphi(t) = \varphi_1(t)\varphi_2(t)\ldots\varphi_n(t) \quad \text{for all } t \in \mathbf{R}.$$

(ii) $\sigma^2(X_1 + X_2 + \ldots + X_n) = \sigma^2(X_1) + \sigma^2(X_2) + \ldots + \sigma^2(X_n)$

if $\sigma^2(X_i) < \infty$ for $i = 1, 2, \ldots, n$.

Let $F, F_1, F_2, \ldots$ be bounded non-decreasing functions. A sequence (F_n) **converges weakly** to F if

(i) for every continuity point x of F, $F_n(x) \to F(x)$ and

(ii) $F_n(+\infty) \to F(+\infty)$ and $F_n(-\infty) \to F(-\infty)$.

We will write

$$F_n \xrightarrow{w} F$$

if (F_n) converges weakly to F. Note that the weak limit of the sequence (F_n), if it exists, is unique. In the following theorem we state some facts of weak convergence which will be used in our work.

Theorem 2.2.7 (Helly's Theorem, [7], p.133) Let (F_n) be a sequence of uniformly bounded, non-decreasing, right-continuous functions. Then (F_n) contains a subsequence which converges weakly to a bounded, non-decreasing, right-continuous function.

Let $\mathcal{M}$ be the set of all bounded, non-decreasing, right-continuous functions M from $\mathbf{R}$ into $[0, \infty)$ which vanish at $-\infty$. The function L defined for any $M_1, M_2 \in \mathcal{M}$ by

$$L(M_1,M_2)=\inf_{h\geq 0}\{h|M_1(x-h)-h\leq M_2(x)\leq M_1(x+h)+h \text{ for every } x\}$$

is a complete metric on $\mathcal{m}$. ([8], p.39)

In the following corollary follows from Theorem 2.2.7 and the fact that the elements in $\mathcal{m}$ vanish at $-\infty$.

Corollary 2.2.8 Let (M_n) be a bounded sequence of elements in $\mathcal{m}$. Then it contains a subsequence which converges weakly to an element in $\mathcal{m}$.

Theorem 2.2.9 ([5], p.39) Let $M, M_1, M_2, \ldots$ be elements of $\mathcal{m}$. Then the following statements are equivalent:

(i) $M_n \xrightarrow{w} M$;

(ii) For every bounded continuous function g on $\mathbf{R}$,

$$\int_{-\infty}^{\infty} g(x)dM_n(x) \to \int_{-\infty}^{\infty} g(x)dM(x);$$

(iii) $L(M_n, M) \to 0$.

In the following, we summarize facts concerning weak convergence of the distribution functions needed for our work.

Theorem 2.2.10 ([11], p.15) Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions. Let F be a distribution function with the characteristic function φ. If $F_n \xrightarrow{w} F$, then (φ_n) converges to φ uniformly in any finite interval.

Theorem 2.2.11 ([11], p.15) Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions. Let φ be a complex-valued function which is continuous at 0. If (φ_n) converges to φ for every t, then there exists a distribution function F such that $F_n \xrightarrow{w} F$ and the characteristic function of F is φ.

Let F_1 and F_2 be distribution functions. The **convolution** of F_1 and F_2 is defined by

$$(F_1 * F_2)(x) = \int_{-\infty}^{\infty} F_2(x-y)dF_1(y) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y) \qquad \text{for all } x \in \mathbf{R}.$$

Theorem 2.2.12 ([4],p.245) Let (a_n) be a sequence of real numbers and E_{a_n} a distribution function defined by

$$E_{a_n}(a) = \begin{cases} 0 & \text{if } a < a_n \\ 1 & \text{if } a \geq a_n \end{cases}.$$

Then the sequence (E_{a_n}) converges weakly if and only if the sequence (a_n) converges in **R**.

Theorem 2.2.13 ([4],p.252) Let $F, G, F_n, G_n, n = 1,2,3,...$ be distribution functions. If $F_n \xrightarrow{w} F$ and $G_n \xrightarrow{w} G$, then $F_n * G_n \xrightarrow{w} F * G$.

2.3 Infinitely Divisible Distribution Functions

A distribution function F with the characteristic function φ is said to be **infinitely divisible** if for every natural number n, there exists a characteristic function φ_n such that for every t,

$$\varphi(t) = \{\varphi_n(t)\}^n.$$

The characteristic function of any **infinitely divisible** function is also said to be infinitely divisible. A random variable is said to be **infinitely divisible** if its distribution function is infinitely divisible.

Theorem 2.3.1 ([8], p.81)

(i) If φ is an infinitely divisible characteristic function, then for every t, $\varphi(t) \neq 0$.

(ii) If φ is an infinitely divisible characteristic function, then $|\varphi|^2$ is also infinitely divisible.

(iii) The product of a finite number of infinitely divisible characteristic functions is infinitely divisible.

(iv) A characteristic function which is the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible.

Theorem 2.3.2 ([5], p.307) In order that a distribution function F with finite variance is infinitely divisible it is necessary and sufficient that there exist a unique constant μ and a non-decreasing , right-continuous function of bounded variation K such that $K(-\infty)=0$ and the logarithm of its characteristic function φ is given by

$$\ln\varphi(t)=i\mu t+\int_{-\infty}^{\infty} f(t,x)dK(x) \tag{1}$$

where

$$f(t,x)=\begin{cases}(e^{itx}-1-itx)\dfrac{1}{x^2} & \text{if } x\neq 0\\ -\dfrac{t^2}{2} & \text{if } x=0.\end{cases}$$

In the sequel, $f(t,x)$ always denotes this function. The formula (1) is known as **Kolmogorov formula.**

Remark 2.3.3 ([12],p.618) For each $t\in\mathbf{R}$, $|f(t,x)|\leq\frac{1}{2}t^2$ for all $x\in\mathbf{R}$.

Theorem 2.3.4 ([6], p.85) Let X be an infinitely divisible random variable with finite variance. Let the constant μ and the function K be given as in the Kolmogorov formula of the characteristic function of X. Then

(i) $E[X] = \mu$

(ii) $\sigma^2(X) = K(+\infty)$.

2.4 Kolmogorov Theorems

In this section, we let (X_{nk}), $k=1,2,3,\ldots,k_n$, $n=1,2,3,\ldots$ be a double sequence of random variables with finite variances.

For each n and k , we let μ_{nk}, σ_{nk}^2 and F_{nk} be the expectation, variance and distribution function of X_{nk} , respectively.

In [6], Kolmogorov gave necessary and sufficient conditions for weak convergence of the sequence of distribution functions of sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

where (A_n) is a sequence of real numbers. There are two important convergence theorems (Theorem 2.4.2 and Theorem 2.4.3). In the first theorem, (X_{nk}) must satisfy the following conditions.

(α) $(X_{nk} - \mu_{nk})$ is infinitesimal, i.e., for every $\varepsilon > 0$

$$\max_{1 \le k \le k_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \to 0 .$$

(β) There exists a real number C such that

$$\sum_{k=1}^{k_n} \sigma_{nk}^2 < C .$$

In order to prove the first theorem, Kolmogorov defined **the accompanying distribution function** of sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

to be the distribution function whose logarithm of its characteristic function is given by

$$\ln \Psi_n(t) = -iA_n t + it \sum_{k=1}^{k_n} \mu_{nk} + \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} (e^{itx} - 1) dF_{nk}(x + \mu_{nk}) .$$

Remark 2.4.1 Let $K_{k_n}:\mathbf{R}\to\mathbf{R}$ be defined by

$$K_{k_n}(u)=\sum_{k=1}^{k_n}\int_{-\infty}^{u}x^2dF_{nk}(x+\mu_{nk}) \quad \text{for all } u\in\mathbf{R}.$$

If (X_{nk}) satisfies the condition (β), then $K_{k_n}\in\mathcal{M}$ and the logarithm of the characteristic function Ψ_n of the accompanying distribution function of S_n is given by

$$\ln\Psi_n(t)=-iA_nt+it\left(\sum_{k=1}^{k_n}\mu_{nk}\right)+\int_{\mathbf{R}}f(t,u)dK_{k_n}(u)\ .$$

Proof. Clearly , K_{k_n} is non-decreasing and $K_{k_n}(-\infty)=0$ for all $n\in\mathbf{N}$. Since (X_{nk}) satisfies the condition (β) , $K_{k_n}(+\infty)$ is bounded . Next , we will show that the function K_{k_n} is right - continuous. Let $u\in\mathbf{R}$ and (u_m) be a decreasing sequence such that $\lim_{m\to\infty}u_m=u$ .We must show that $\lim_{m\to\infty}K_{k_n}(u_m)=K_{k_n}(u)$. Since (u_m) is decreasing and $\lim_{m\to\infty}u_m=u$, so $(-\infty,u_1]\supseteq(-\infty,u_2]\supseteq...$ and $\bigcap_{m=1}^{\infty}(-\infty,u_m]=(-\infty,u]$.

Then

$$\begin{aligned}\lim_{m\to\infty}K_{k_n}(u_m)&=\lim_{m\to\infty}\sum_{k=1}^{k_n}\int_{-\infty}^{u_m}x^2dF_{nk}(x+\mu_{nk})\\&=\sum_{k=1}^{k_n}\lim_{m\to\infty}\int_{-\infty}^{u_m}x^2dF_{nk}(x+\mu_{nk})\\&=\sum_{k=1}^{k_n}\int_{-\infty}^{u}x^2dF_{nk}(x+\mu_{nk})\\&=K_{k_n}(u).\end{aligned}$$

So K_{k_n} is right - continuous. Hence $K_{k_n}\in\mathcal{M}$. It's easy to see that the logarithm of the characteristic function Ψ_n of the accompanying distribution function of S_n can be rewritten in the following form :

$$\ln\Psi_n(t)=-iA_nt+it\left(\sum_{k=1}^{k_n}\mu_{nk}\right)+\int_{\mathbf{R}}f(t,u)dK_{k_n}(u)\ . \qquad \#$$

Theorem 2.4.2 ([6], p.98) Assume that (X_{nk}) satisfies the conditions (α), (β) and for each $n, X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that the sequence of distribution functions of sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

converges weakly to a limit distribution function if and only if the sequence of accompanying distribution functions of S_n converges weakly to the same limit distribution function.

Theorem 2.4.3 ([6], p.100) Assume that (X_{nk}) satisfies the condition (α) and for each n, $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that

(i) the sequence of distribution functions of sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$$

converges weakly to a limit distribution function F whose variance is σ^2 and

(ii) $\sum_{k=1}^{k_n} \sigma_{nk}^2 \to \sigma^2$

if and only if there exists a function K in $\mathcal{M}$ such that

(i′) $K_{k_n}(u) \to K(u)$ for every continuity point u of K and

(ii′) $K_{k_n}(+\infty) \to K(+\infty)$

where

$$K_{k_n}(u) = \sum_{k=1}^{k_n} \int_{-\infty}^{u} x^2 \, dF_{nk}(x + \mu_{nk}) .$$

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \mu_{nk} - \mu$$

where μ is any real number. Logarithm of the characteristic function of the limit distribution function is given by

$$\ln \varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t, x) \, dK(x) .$$

2.5 Random Sums of Random Variables

Let (Z_n) be a sequence of positive integral-valued random variable. Let (X_{nk}) be a double sequence of complex-valued random variables. Here our double sequence is infinite in both directions, i.e, $n = 1,2,3,\ldots$ and $k = 1,2,3,\ldots$. For each n, a value $Z_n(\omega)$ of Z_n determines a finite sequence of values

$$X_{n1}(\omega), X_{n2}(\omega), \ldots, X_{nZ_n(\omega)}(\omega)$$

of $X_{n1}, X_{n2}, \ldots, X_{nZ_n(\omega)}$. It can be seen that for each n, Z_n and (X_{nk}) together define a random experiment in which each outcome gives rise to a finite sequence of complex numbers .

However, the length of this finite sequence is random.We shall call the system (Z_n, X_{nk}), a **random double sequence of complex-valued random variables.**

Let (Z_n, X_{nk}) be a random double sequence of complex-valued random variables. For each n, we define

$$\sum_{k=1}^{Z_n} X_{nk} \text{ , } \prod_{k=1}^{Z_n} X_{nk} \text{ and } X_{nZ_n}$$

to be the functions from Ω to **C** given by the following formulas

$$(\sum_{k=1}^{Z_n} X_{nk})(\omega) = (\sum_{k=1}^{Z_n(\omega)} X_{nk})(\omega)$$

$$(\prod_{k=1}^{Z_n} X_{nk})(\omega) = (\prod_{k=1}^{Z_n(\omega)} X_{nk})(\omega)$$

and

$$(X_{nZ_n})(\omega) = (X_{nZ_n(\omega)})(\omega) \text{ ,}$$

respectively.

In case X_{nk} 's are real-valued random variables, we define

$$\sup_{1 \le k \le Z_n} X_{nk}$$

to be the function from Ω to **R** given by

$$(\sup_{1 \le k \le Z_n} X_{nk})(\omega) = (\sup_{1 \le k \le Z_n(\omega)} X_{nk})(\omega) .$$

It will be shown that $\sum_{k=1}^{Z_n} X_{nk}$, $\prod_{k=1}^{Z_n} X_{nk}$ and X_{nZ_n} are complex-valued random variables and $\sup_{1\le k\le Z_n} X_{nk}$ is a real-valued random variable. These facts are special cases of a more general result that follows.

Proposition 2.5.1 ([10],p.17) Let (Y_k) be a sequence of complex-valued random variables. Let Z be a positive integral-valued random variable. Let Y_Z denote a function from Ω to **C** defined by

$$Y_Z(\omega) = (Y_{Z(\omega)})(\omega)$$

for all $\omega \in \Omega$. Then Y_Z is a complex-valued random variable.

Proposition 2.5.2 ([10],p.17) Let (Z_n, X_{nk}) be a random double sequence of complex-valued random variables. For each n, $\sum_{k=1}^{Z_n} X_{nk}$, $\prod_{k=1}^{Z_n} X_{nk}$ and X_{nZ_n} are complex-valued random variables. Furthermore, in case where the X_{nk}'s are real-valued random variables, $\sup_{1\le k\le Z_n} X_{nk}$ is a real-valued random variable.

We will consider sums of the form

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

where (Z_n, X_{nk}) is a random double sequence of random variables and (A_{nk}) is a double sequence of real numbers. We will refer to them as **random sums**.

We say that (X_{nk}) is **random infinitesimal** with respect to (Z_n) if for every $\varepsilon > 0$,

$$\max_{1\le k\le Z_n} P(|X_{nk}| \ge \varepsilon) \xrightarrow{P} 0 \text{ as } n \to \infty.$$

CHAPTER III

Convergence of Distribution Functions of Random Sums

The purpose of this chapter is to find necessary and sufficient conditions for the weak convergence of the sequence of distribution functions of random sums to a limit distribution function.

In [1], Bethmann gave necessary and sufficient conditions for the weak convergence of the sequence of distribution functions of random sums to the standard normal distribution function.One of the important tools used by Bethmann is what is known as the "*q*-quantiles". We will also make use of this tool.

3.1 Definition and properties of q-quantiles

Let Z be a positive integral-valued random variable. Let $l:(0,1)\to \mathbf{N}$ be defined by

$$l(q)=\max\{k\in \mathbf{N} \,|\, P(Z<k)\le q\}.$$

The function l is called the **q-quantiles** of $\mathbf{Z}$.

Remark 3.1.1 For a positive integral-valued random variable Z , the function q -quantiles of Z is non-decreasing .

Lemma 3.1.2 Let Z be a positive integral-valued random variable and $g:\mathbf{N}\to \mathbf{C}$. Then

(i) l is a Borel function .

(ii) $E(goZ)=\int_0^1 g(l(q))dq$.

Proof.

Let $\operatorname{Im} Z=\{k_j \,|\, j\in \mathbf{N}\}$ where $k_j<k_{j+1}$ for $j=1,2,\ldots$, $q_j=\sum_{k=1}^{k_j} P(Z=k)$ and $q_0=0$.

For each k_j and $q \in [q_{j-1}, q_j)$, we have $l(q) = k_j$.

Then $l^{-1}(\{k_j\}) = [q_{j-1}, q_j)$ for $j = 2,3,4,...$ and $l^{-1}(\{k_1\}) = (0, q_1)$.

(i) For any open set O in $\mathbf{N}$,

$$l^{-1}(O) = \bigcup_{k_j \in O} [q_{j-1}, q_j) \cap (0,1)$$

This implies that $l^{-1}(O)$ is a Borel set in $(0,1)$. Therefore l is a Borel function .

(ii) Since l is a Borel function and domain of g is $\mathbf{N}$, $g \circ l$ is a Borel function.

Then
$$\int_{[q_{j-1}, q_j)} g(l(q))dq = g(k_j)(q_j - q_{j-1})$$

for all $k_j \in \operatorname{Im} Z$.

If $\operatorname{Im} Z = \{k_1, k_2, ..., k_t\}$, then $\bigcup_{j=1}^{t} [q_{j-1}, q_j) = [0,1)$.

If $\operatorname{Im} Z = \{k_1, k_2, ...\}$, then $\lim_{n\to\infty} q_n = \lim_{n\to\infty} \sum_{k=1}^{k_n} P(Z = k) = \sum_{k=1}^{\infty} P(Z = k) = 1$.

Thus $\bigcup_{j=1}^{\infty} [q_{j-1}, q_j) = [0,1)$.

Therefore
$$\begin{aligned} E(g \circ Z) &= \sum_{k_j \in \operatorname{Im} Z} g(k_j) P(Z = k_j) \\ &= \sum_{k_j \in \operatorname{Im} Z} g(k_j)(q_j - q_{j-1}) \\ &= \sum_{k_j \in \operatorname{Im} Z} \int_{[q_{j-1}, q_j)} g(l(q))dq \\ &= \int_0^1 g(l(q))dq \ . \end{aligned}$$
#

Proposition 3.1.3 [1] For every n , let (a_{nk}), $k = 1,2,...$ be a nondecreasing sequence of non-negative real numbers and Z_n an integral - valued random variable.

Furthermore , let $a \geq 0$ be fixed. Then we have $a_{nZ_n} \xrightarrow{P} a$ if and only if $a_{nl_n(q)} \to a$ for all $q \in (0,1)$.

Proposition 3.1.4 Let (Z_n) be a sequence of positive integral-valued random variables.

(i) If $Z_n \xrightarrow{P} \infty$, then $l_n(q) \to \infty$ for every $q \in (0,1)$.

(ii) If (Z_n) is increasing, then $(l_n(q))$ is increasing for all $q \in (0,1)$.

Proof.

(i) Fix $q \in (0,1)$ and let M be any positive integer. Since $\lim_{n\to\infty} P(Z_n \geq M) = 1$, there exists $n_0 \in \mathbf{N}$ such that $P(Z_n \geq M) > 1-q$ for all $n \geq n_0$. That is $P(Z_n < M) < q$ for all $n \geq n_0$. This show that for $n \geq n_0$, $l_n(q) \geq M$. Hence $l_n(q) \to \infty$ as $n \to \infty$.

(ii) Let $q \in (0,1)$ and $n_1 < n_2$ .Since $Z_{n_1} \leq Z_{n_2}$, $P(Z_{n_2} < l_{n_1}(q)) \leq P(Z_{n_1} < l_{n_1}(q)) \leq q$. By definition of q -quantiles, $l_{n_1}(q) \leq l_{n_2}(q)$. #

Theorem 3.1.5 Let (Z_n, X_{nk}) be a random double sequence of random variables such that for each $n, Z_n, X_{n1}, X_{n2}, \ldots$ are independent. Let φ_{nk} be the characteristic function of X_{nk} and (A_{nk}) a double sequence of real numbers.Then the characteristic functions φ_n of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - A_{nZ_n}$$

are given by

$$\varphi_n(t) = E\left[\exp(-itA_{nZ_n})\prod_{k=1}^{Z_n}\varphi_{nk}(t)\right]$$

for $t \in \mathbf{R}$.

Proof. For each $n, j \in \mathbf{N}$, let $S_n^j = X_{n1} + X_{n2} + ... + X_{nj} - A_{nj}$ and F_n^j and φ_n^j be the distribution function and characteristic function of S_n^j, respectively . Then

$\varphi_n^j(t) = \exp(-itA_{nj})\prod_{k=1}^{j}\varphi_{nk}(t)$ for $t \in \mathbf{R}$.

Therefore

$$\begin{aligned} F_n(x) &= P(S_{Z_n} \le x) \\ &= \sum_{j=1}^{\infty} P(Z_n = j \wedge S_n^j \le x) \\ &= \sum_{j=1}^{\infty} P(Z_n = j)P(S_n^j \le x) \\ &= \sum_{j=1}^{\infty} P(Z_n = j)F_n^j(x) \ . \end{aligned}$$

By Proposition 2.2.5 , for $t \in \mathbf{R}$,

$$\begin{aligned} \varphi_n(t) &= \sum_{j=1}^{\infty} P(Z_n = j)\varphi_n^j(t) \\ &= \sum_{j=1}^{\infty} P(Z_n = j)\exp(-itA_{nj})\prod_{k=1}^{j}\varphi_{nk}(t) \\ &= E\left[\exp(-itA_{nZ_n})\prod_{k=1}^{Z_n}\varphi_{nk}(t)\right] . \end{aligned}$$

#

3.2 Convergence of distribution functions of random sums

Let (X_{nk}) be a double sequence of random variables with finite variances σ_{nk}^2 and (Z_n) a sequence of positive integral-valued random variables. Assume that for each n , $Z_n, X_{n1}, X_{n2}, \ldots$ are independent.

Put

$$S_n^{(q)} = X_{n1} + X_{n2} + \ldots + X_{nl_n(q)} - \sum_{k=1}^{l_n(q)} \mu_{nk} \ ,$$

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - \sum_{k=1}^{Z_n} \mu_{nk} \ ,$$

and let $F_n^{(q)}$ and F_n be the distribution functions of $S_n^{(q)}$ and S_{Z_n}, respectively.

To prove the main theorems (Theorem 3.2.6 and Theorem 3.2.7), we need the following results.

Theorem 3.2.1 ([8],p.199) Let $\{G \mid G : \mathbf{R} \times \mathbf{R} \to \mathbf{R}\}$ be a family of functions which has the following properties :

(i) For each value of y, the function $G(x, y)$ is a distribution function in the variable x.

(ii) For each value of x, $G(x, y)$ is a measurable function in the variable y.

Then for any arbitrary distribution function $H(y)$,

$$F(x) = \int_{-\infty}^{\infty} G(x, y) dH(y)$$

is a distribution function and the corresponding characteristic function is given by

$$\varphi(t) = \int_{-\infty}^{\infty} g(t, y) dH(y)$$

where $g(\cdot, y)$ is the characteristic function of the distribution function $G(\cdot, y)$.

Proposition 3.2.2 For a.e. $q \in (0,1)$, let $F^{(q)} : \mathbf{R} \to [0,1]$ be a distribution function. If for each $x \in \mathbf{R}$, $F^{(q)}(x)$ is a measurable function in q, then

$F(x) = \int_0^1 F^{(q)}(x) dq$, $x \in \mathbf{R}$, is a distribution function and the corresponding

characteristic function is given by $\varphi(t) = \int_0^1 \varphi^{(q)}(t) dq$ for $t \in \mathbf{R}$, where $\varphi^{(q)}$ is the

characteristic function of $F^{(q)}$.

Proof. Let A be a measurable subset of $(0,1)$ such that for every $q \in A$, $F^{(q)}$ is a distribution function and $l(A)=1$ where l is the Lebesgue measure on $(0,1)$. Let N be an arbitrary distribution function and $G:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ defined by

$$G(x,q)=\begin{cases} F^{(q)}(x) & \text{if } q\in A \\ N(x) & \text{if } q\notin A \,. \end{cases}$$

For each $q \in \mathbf{R}$, the function $G(x,q)$ is a distribution function in x. That is the condition (i) of Theorem 3.2.1 holds. Next , we will show that the condition (ii) of Theorem 3.2.1 holds. That is we must show that for each $x \in \mathbf{R}$, $G(x,q)$ is measurable in q. Let $x \in \mathbf{R}$ and O be any open set in $\mathbf{R}$.

If $N(x) \in O$, then $G^{-1}(x,O)=\left\{t\in A \,\middle|\, F^{(t)}(x)\in O\right\}\cup A^c$.

If $N(x) \notin O$, then $G^{-1}(x,O)=\left\{t\in A \,\middle|\, F^{(t)}(x)\in O\right\}$.

Since $F^{(q)}(x)$ is measurable in q, $\left\{t \in A \,\middle|\, F^{(t)}(x) \in O\right\}$ is a measurable set.

Since A is a measurable set , A^c is a measurable set.

Hence for each $x \in \mathbf{R}$, $G(x,q)$ is a measurable function in q.

Let $H:\mathbf{R}\to[0,1]$ be a distribution function defined by

$$H(q)=\begin{cases} 0 & \text{if } q\le 0 \\ q & \text{if } 0<q<1 \\ 1 & \text{if } q\ge 1 \,. \end{cases}$$

By Theorem 3.2.1 , we have

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} G(x,q)\,dH(q) \\ &= \int_0^1 G(x,q)\,dq \\ &= \int_0^1 F^{(q)}(x)\,dq \quad \text{for } x\in\mathbf{R}\,, \end{aligned}$$

is a distribution function and the corresponding characteristic function is given by

$$\varphi(t)=\int_{-\infty}^{\infty} g(t,q)\,dH(q)$$

$$=\int_0^1 g(t,q)dq$$

$$=\int_0^1 \varphi^{(q)}(t)dq$$

for $t\in\mathbf{R}$, where $g(\cdot,y)$ and $\varphi^{(q)}$ are the characteristic functions of the distribution functions $G(\cdot,y)$ and $F^{(q)}$, respectively. #

Proposition 3.2.3 If for a.e. $q\in(0,1)$, there exists a distribution function $F^{(q)}$ such that $F_n^{(q)}\xrightarrow{w}F^{(q)}$ and for each $x\in\mathbf{R}$, $F^{(q)}(x)$ is a measurable function in q, then we also have $F_n\xrightarrow{w}F$ where F is a distribution function defined by

$$F(x)=\int_0^1 F^{(q)}(x)dq,\quad x\in\mathbf{R}.$$

Proof. Assume that for a.e. $q\in(0,1)$, there exists a distribution function $F^{(q)}$ such that $F_n^{(q)}\xrightarrow{w}F^{(q)}$ and for each $x\in\mathbf{R}$, $F^{(q)}(x)$ is a measurable function in q.

Let $\varphi_n^{(q)}$ and $\varphi^{(q)}$ be the characteristic functions of $F_n^{(q)}$ and $F^{(q)}$, respectively.

Observe that $$\varphi_n^{(q)}(t)=\exp(-it\sum_{k=1}^{l_n(q)}\mu_{nk})\prod_{k=1}^{l_n(q)}\varphi_{nk}(t)\quad \text{for all } t\in\mathbf{R}.$$

By Proposition 3.2.2, $F(x)=\int_0^1 F^{(q)}(x)dq$, $x\in\mathbf{R}$, is a distribution function and the corresponding characteristic function is $\varphi(t)=\int_0^1\varphi^{(q)}(t)dq$, $t\in\mathbf{R}$.

Since $F_n^{(q)}\xrightarrow{w}F^{(q)}$, by Theorem 2.2.10, $\varphi_n^{(q)}(t)\to\varphi^{(q)}(t)$ for all $t\in\mathbf{R}$.

Since $\left|\varphi_n^{(q)}\right|\le 1$, by Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_0^1\varphi_n^{(q)}(t)dq=\int_0^1\varphi^{(q)}(t)dq$$

for all $t\in\mathbf{R}$. By Theorem 3.1.5, the characteristic function φ_n of S_{Z_n} is defined by

$\varphi_n(t)=E[\exp(-it\sum_{k=1}^{Z_n}\mu_{nk})\prod_{k=1}^{Z_n}\varphi_{nk}(t)]$ for all $t\in\mathbf{R}$. Let $g_{n,t}:\mathrm{N}\to\mathrm{C}$ be defined by

$g_{n,t}(m)=\exp(-it\sum_{k=1}^{m}\mu_{nk})\prod_{k=1}^{m}\varphi_{nk}(t)$. By Lemma 3.1.2(ii) ,

$$\varphi_n(t)=\int_0^1\exp(-it\sum_{k=1}^{l_n(q)}\mu_{nk})\prod_{k=1}^{l_n(q)}\varphi_{nk}(t)dq=\int_0^1\varphi_n^{(q)}(t)dq\,.$$

Hence $\varphi_n(t)\to\varphi(t)$ for all $t\in\mathbf{R}$. By Theorem 2.2.11, $F_n\xrightarrow{w}F$. #

Proposition 3.2.4 ([13]) Let (X_{nk}) be random infinitesimal with respect to (Z_n) . If $F_n\xrightarrow{w}F$ for some distribution function F , then there exist distribution functions $\bar{F}^{(q)}$ and bounded sequences of real numbers $(a_n^{(q)})$, $n\in\mathbf{N}$ such that for some subsequence (n') ,

$$F_{n'}^{(q)}*E_{a_{n'}^{(q)}}\xrightarrow{w}\bar{F}^{(q)}$$

for a.e. $q\in(0,1)$, where E_a stands for the degenerated distribution function with unit-jump at $a\in\mathbf{R}$.

Corollary 3.2.5 Let (X_{nk}) be random infinitesimal with respect to (Z_n). If $F_n\xrightarrow{w}F$ for some distribution function F , then for a.e. $q\in(0,1)$, there exists distribution function $F^{(q)}$ such that for some subsequence (n'), $F_{n'}^{(q)}\xrightarrow{w}F^{(q)}$.

Proof. By Proposition 3.2.4, for a.e. $q\in(0,1)$, there exist a distribution function $\bar{F}^{(q)}$ and a bounded sequence $(a_n^{(q)})$ such that for some subsequence (n') ,

$$F_{n'}^{(q)}*E_{a_{n'}^{(q)}}\xrightarrow{w}\bar{F}^{(q)}.$$

Since $(a_n^{(q)})$ is bounded, there is a subsequence (n'') of (n') such that $(a_{n''}^{(q)})$ converges. Let $\lim_{n''\to\infty}a_{n''}^{(q)}=a^{(q)}$. Then

$$F_{n''}^{(q)}*E_{a_{n''}^{(q)}}\xrightarrow{w}\bar{F}^{(q)}$$

and by Theorem 2.2.12, we have

$$E_{-a_{n''}^{(q)}} \xrightarrow{w} E_{-a^{(q)}} .$$

By Theorem 2.2.13 , $F_{n''}^{(q)} \xrightarrow{w} \overline{F}^{(q)} * E_{-a^{(q)}}$. Let $F^{(q)} = \overline{F}^{(q)} * E_{-a^{(q)}}$. Then $F^{(q)}$ is a distribution function, so the corollary is proved. #

In the main theorems , we assume (Z_n, X_{nk}) satisfies the following conditions :

$(\widetilde{\alpha})$ $(X_{nk} - \mu_{nk})$ is random infinitesimal with respect to (Z_n).

$(\widetilde{\beta})$ There exists a constant $C > 0$ such that $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < C$ for a.e. $q \in (0,1)$.

The following theorems are the main theorems of this chapter.

Theorem 3.2.6 Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the condition $(\widetilde{\alpha})$, $(\widetilde{\beta})$ and for each n ,Z_n ,X_{n1} ,X_{n2} ,... are independent. If there exists a distribution function F with finite variance σ^2 such that

(i) the sequence of the distribution functions of random sums

$$S_{Z_n} = X_{n1} + X_{n2} + ... + X_{nZ_n} - \sum_{k=1}^{Z_n} \mu_{nk}$$

converges weakly to F and

(ii) $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$,

then for a.e. $q \in (0,1)$, there exist a subsequence (n_k) and a function $K^{(q)}$ in $\mathcal{M}$ such that

(i′) $K_{l_{n_k}(q)}(u) \to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$ and (ii′) $K_{l_{n_k}(q)}(+\infty) \to K^{(q)}(+\infty)$ and $\sigma^2 = \int_0^1 K^{(q)}(+\infty)dq$,

where
$$K_{l_n(q)}(u) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk}).$$

Proof. We divide the proof into 3 steps as follows.

Step 1 We will show that for a.e. $q \in (0,1)$, there exist a subsequence (n_k) and a function $K^{(q)}$ in $\mathcal{M}$
such that
$$\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) \to \int_{\mathbf{R}} f(t,u)dK^{(q)}(u) .$$

Since $F_n \xrightarrow{w} F$, by Corollary 3.2.5 , for a.e. $q \in (0,1)$, there exists a distribution function $F^{(q)}$ such that for some subsequence (n_k), $F_{n_k}^{(q)} \xrightarrow{w} F^{(q)}$. By Theorem 2.4.2, the sequence of accompanying distribution functions of $S_{n_k}^{(q)}$ converges weakly to $F^{(q)}$. Let $\psi_{n_k}^{(q)}$ be a characteristic function of the accompanying distribution function of $S_{n_k}^{(q)}$ and $\varphi^{(q)}$ a characteristic function of $F^{(q)}$. By Theorem 2.2.10 , $\psi_{n_k}^{(q)}(t) \to \varphi^{(q)}(t)$ for all $t \in \mathbf{R}$. It follows that $\ln\psi_{n_k}^{(q)}(t) \to \ln\varphi^{(q)}(t)$ for all $t \in \mathbf{R}$.
By Remark 2.4.1 , we have
$$\ln\psi_{n_k}^{(q)}(t) = -i\left(\sum_{j=1}^{l_{n_k}(q)} \mu_{n_k j}\right)t + it\left(\sum_{j=1}^{l_{n_k}(q)} \mu_{n_k j}\right) + \int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u)$$
$$= \int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u).$$

Since $\psi_{n_k}^{(q)}$ is infinitely divisible , by Theorem 2.3.1(iv) , $\varphi^{(q)}$ is infinitely divisible. By Theorem 2.3.2 , $\ln\varphi^{(q)}(t) = i\mu_q t + \int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$ for some constant μ_q and function $K^{(q)}$ in $\mathcal{M}$.

Therefore $\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) \to i\mu_q t + \int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$.

For $t \neq 0$, $\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) \to i\mu_q + \frac{1}{t}\int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$.

To show that $\mu_q = 0$, it suffices to show that $\lim_{t\to 0}\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) = 0$ and $\lim_{t\to 0}\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK^{(q)}(u) = 0$. By Remark 2.3.3 , $|f(t,u)| \le \frac{1}{2}t^2$ for all $t \in \mathbf{R}$ .Thus

$$\left|\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u)\right| \le \frac{1}{|t|}\int_{\mathbf{R}} |f(t,u)|dK_{l_{n_k}(q)}(u) \le \frac{|t|}{2}K_{l_{n_k}(q)}(+\infty) = \frac{|t|}{2}\sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2 .$$

By the condition $(\tilde{\beta})$, $\sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2$ is bounded , so $\lim_{t\to 0}\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) = 0$.

Similarly, we can prove that $\lim_{t\to 0}\frac{1}{t}\int_{\mathbf{R}} f(t,u)dK^{(q)}(u)=0$. Hence $\mu_q = 0$. It follows that $\int_{\mathbf{R}} f(t,u)dK_{l_{n_k}(q)}(u) \to \int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$.

Step 2 We will show that for a.e. $q \in (0,1)$, $K_{l_{n_k}(q)} \xrightarrow{w} K^{(q)}$ where (n_k) is a sequence in step 1.

First , we will show that for a.e. $q \in (0,1)$, there exists a subsequence of $(K_{l_{n_k}(q)})$ which converges weakly to $K^{(q)}$.

Since for each $k \in \mathbf{N}$, $K_{l_{n_k}(q)}$ is non-decreasing and by condition $(\tilde{\beta})$, $\sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2 < C$, that is $(K_{l_{n_k}(q)}(+\infty))$ is bounded, so $(K_{l_{n_k}(q)})$ is bounded. By Corollary 2.2.8 , there exists a subsequence $(K_{l_{n_{k_r}}(q)})$ of $(K_{l_{n_k}(q)})$ and a function $\overline{K}^{(q)}$ in m such that $K_{l_{n_{k_r}}(q)} \xrightarrow{w} \overline{K}^{(q)}$.By Remark 2.3.3, $f(t,u)$ is continuous and bounded for all $t \in \mathbf{R}$, thus by Theorem 2.2.9 ,

$$\int_{\mathbf{R}} f(t,u)dK_{l_{n_{k_r}}(q)}(u) \to \int_{\mathbf{R}} f(t,u)d\overline{K}^{(q)}(u) \qquad .....(*)$$

for all $t \in \mathbf{R}$.

By (*) and step 1 , we have $\int_{\mathbf{R}} f(t,u)d\overline{K}^{(q)}(u) = \int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$.

By Theorem 2.3.2 , $\overline{K}^{(q)} = K^{(q)}$. So $K_{l_{n_{k_r}}(q)} \xrightarrow{w} K^{(q)}$. By the same argument , we can prove that for every subsequence of $(K_{l_{n_k}(q)})$, it contains a subsequence which converges weakly to $K^{(q)}$. By Theorem 2.2.9 , for every subsequence of $(K_{l_{n_k}(q)})$, it contains a subsequence which converges to $K^{(q)}$ with respect to the metric L . This implies that $(K_{l_{n_k}(q)})$ converges to $K^{(q)}$ with respect to the metric L . By Theorem 2.2.9 , $K_{l_{n_k}(q)} \xrightarrow{w} K^{(q)}$.

Step 3 **We will show that** $\sigma^2 = \int_0^1 K^{(q)}(+\infty)dq$ **.**

Since $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$, by Lemma 3.1.2(ii) , we have $\lim_{n\to\infty} \int_0^1 \sum_{j=1}^{l_n(q)} \sigma_{nj}^2 dq = \sigma^2$.

Therefore $\lim_{k\to\infty} \int_0^1 \sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2 dq = \sigma^2$. By condition $(\tilde{\beta})$ and it follows from the Lebesgue Dominated Convergence Theorem that $\int_0^1 \lim_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2 dq = \sigma^2$. By step 2 , we have $K_{l_{n_k}(q)}(+\infty) \to K^{(q)}(+\infty)$. That is $\lim_{k\to\infty} \sum_{j=1}^{l_{n_k}(q)} \sigma_{n_k j}^2 = K^{(q)}(+\infty)$. Therefore $\sigma^2 = \int_0^1 K^{(q)}(+\infty)dq$.

By step 1- step 3, the theorem is proved. #

Theorem 3.2.7 Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the condition $(\tilde{\alpha}),(\tilde{\beta})$ and for each $n, Z_n, X_{n1}, X_{n2},\ldots$ are independent. If for a.e. $q\in(0,1)$, there exists a function $K^{(q)}$ in $\mathcal{M}$ such that

(i′) $K_{l_n(q)}(u)\to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$

(ii′) $K_{l_n(q)}(+\infty)\to K^{(q)}(+\infty)$ and $\int_0^1 K^{(q)}(+\infty)dq=\sigma^2$ for some constant σ^2 and

(iii′) for each $x\in\mathbf{R}$, $F^{(q)}(x)$ is measurable in q, where $F^{(q)}$ is the distribution function whose logarithm of its characteristic function $\varphi^{(q)}$ is given by

$$\ln\varphi^{(q)}(t)=\int_{\mathbf{R}} f(t,u)dK^{(q)}(u),$$

then

(iii) the sequence of the distribution functions of sums

$$S_{Z_n}=X_{n1}+X_{n2}+\ldots+X_{nZ_n}-\sum_{k=1}^{Z_n}\mu_{nk}$$

converges weakly to F where $F(x)=\int_0^1 F^{(q)}(x)dq$, $x\in\mathbf{R}$ and

(iv) $E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]\to\sigma^2$ and σ^2 is the variance of F.

Proof. For a.e. $q\in(0,1)$, let $\psi_n^{(q)}$ be a characteristic function of the accompanying distribution function of $X_{n1}+X_{n2}+\ldots+X_{nl_n(q)}-\sum_{k=1}^{l_n(q)}\mu_{nk}$. By Remark 2.4.1,

$$\ln\psi_n^{(q)}(t)=-i\left(\sum_{k=1}^{l_n(q)}\mu_{nk}\right)t+it\left(\sum_{k=1}^{l_n(q)}\mu_{nk}\right)+\int_{\mathbf{R}} f(t,u)dK_{l_n(q)}(u)$$

$$=\int_{\mathbf{R}} f(t,u)dK_{l_n(q)}(u).$$

By (i′), (ii′) and Theorem 2.2.9, $\int_{\mathbf{R}} f(t,u)dK_{l_n(q)}(u) \to \int_{\mathbf{R}} f(t,u)dK^{(q)}(u)$. Let $F^{(q)}$ be the distribution function whose logarithm of its characteristic function $\varphi^{(q)}$ is given by

$$\ln\varphi^{(q)}(t) = \int_{\mathbf{R}} f(t,u)dK^{(q)}(u).$$

Then $\ln\psi_n^{(q)}(t) \to \ln\varphi^{(q)}(t)$ for all $t \in \mathbf{R}$, so $\psi_n^{(q)}(t) \to \varphi^{(q)}(t)$ for all $t \in \mathbf{R}$. By Theorem 2.2.11 and Theorem 2.4.2, $F_n^{(q)} \xrightarrow{w} F^{(q)}$. By Theorem 2.3.4, $F^{(q)}$ has zero mean and variance of $F^{(q)}$ is $K^{(q)}(+\infty)$. By Proposition 3.2.3, $F_n \xrightarrow{w} F$ where F is a distribution function defined by $F(x) = \int_0^1 F^{(q)}(x)dq$, $x \in \mathbf{R}$. Since

$\lim_{n\to\infty} \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 = K^{(q)}(+\infty)$ and $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < C$ for a.e. $q \in (0,1)$, by Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \int_0^1 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 dq = \int_0^1 K^{(q)}(+\infty)dq = \sigma^2$. Hence, by Lemma 3.1.2(ii), $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$.

It remains to show that σ^2 is the variance of F. Observe that

$$\int_{\mathbf{R}} xdF(x) = \int_{\mathbf{R}} xd\left(\int_0^1 F^{(q)}(x)dq\right) = \lim_{m\to\infty} \int_{-m}^{m} xd\left(\int_0^1 F^{(q)}(x)dq\right).$$

For each $m \in \mathbf{N}$, let $P_{n(m)}$ be the partition of the interval $[-m, m]$,

$$-m = x_0 < x_1 < x_2 < \ldots < x_{n(m)-1} < x_{n(m)} = m$$

such that $x_i = x_{i-1} + \frac{2m}{n}$ for $i = 1,2,3,\ldots,n(m)$. Then for any $m \in \mathbf{N}$ and u_i in $[x_{i-1}, x_i]$, $i = 1,2,3,\ldots,n(m)$, we have

$$\left|\sum_{i=1}^{n(m)} u_i\left(F^{(q)}(x_i) - F^{(q)}(x_{i-1})\right)\right| \le m\left|\sum_{i=1}^{n(m)} \left(F^{(q)}(x_i) - F^{(q)}(x_{i-1})\right)\right| \le m$$

…..(1)

and for all $m \in \mathbf{N}$,

$$\left|\int_{-m}^{m} x dF^{(q)}(x)\right| \le \int_{-m}^{m} |x| dF^{(q)}(x)$$

$$\le \int_{-\infty}^{\infty} |x| dF^{(q)}(x)$$

$$= \int_{|x|\le 1} |x| dF^{(q)}(x) + \int_{|x|>1} |x| dF^{(q)}(x)$$

$$\le 1 + \int_{|x|>1} |x|^2 dF^{(q)}(x)$$

$$\le 1 + \int_{-\infty}^{\infty} |x|^2 dF^{(q)}(x)$$

$$= 1 + K^{(q)}(+\infty)$$

and $\int_0^1 (1+K^{(q)}(+\infty))dq = 1+\sigma^2$. …..(2)

For each $m \in \mathbf{N}$,

$$\int_{-m}^{m} xd(\int_0^1 F^{(q)}(x)dq) = \lim_{n\to\infty} \sum_{i=1}^{n(m)} u_i[\int_0^1 F^{(q)}(x_i)dq - \int_0^1 F^{(q)}(x_{i-1})dq]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n(m)} u_i[\int_0^1 (F^{(q)}(x_i) - F^{(q)}(x_{i-1}))dq]$$

$$= \lim_{n\to\infty} \int_0^1 \sum_{i=1}^{n(m)} u_i(F^{(q)}(x_i) - F^{(q)}(x_{i-1}))dq$$

$$= \int_0^1 \lim_{n\to\infty} \sum_{i=1}^{n(m)} u_i(F^{(q)}(x_i) - F^{(q)}(x_{i-1}))dq \qquad \text{(by (1))}$$

$$= \int_0^1 \int_{-m}^{m} xdF^{(q)}(x)dq \qquad \text{…..(3)}$$

Therefore ,

$$\int_{\mathbf{R}} xdF(x) = \lim_{m\to\infty} \int_{-m}^{m} xd(\int_0^1 F^{(q)}(x)dq)$$

$$= \lim_{m\to\infty} \int_0^1 \int_{-m}^{m} x dF^{(q)}(x) dq \qquad \text{(by(3))}$$

$$= \int_0^1 \lim_{m\to\infty} \int_{-m}^{m} x dF^{(q)}(x) dq \qquad \text{(by(2))}$$

$$= \int_0^1 \int_{\mathbf{R}} x dF^{(q)}(x) dq$$

$$= \int_0^1 0 dq \qquad \text{(since } \int_{\mathbf{R}} x dF^{(q)}(x) = 0 \text{)}$$

$$= 0 .$$

Similarly , we can show that $\int_{\mathbf{R}} x^2 dF(x) = \int_0^1 \int_{\mathbf{R}} x^2 dF^{(q)}(x) dq$.

Hence the variance of F is equal to $\int_{\mathbf{R}} x^2 dF(x) = \int_0^1 K^{(q)}(+\infty) dq = \sigma^2$. #

Corollary 3.2.8 Let (X_{nk}), k = 1, 2, …, k_n , n = 1, 2, … be a double sequence of random variables which satisfies the condition (α) and for each n, $X_{n1}, X_{n2}, ...$ are independent. Then there exists a distribution function F with finite variance σ^2 such that

(i) the sequence of the distribution functions of sums

$$S_n = X_{n1} + X_{n2} + ... + X_{nk_n} - \sum_{k=1}^{k_n} \mu_{nk}$$

converges weakly to F and

(ii) $\sum_{k=1}^{k_n} \sigma_{nk}^2 \to \sigma^2$

if and only if there exists a function K in $\mathcal{M}$ such that

(i′) $K_{k_n}(u) \to K(u)$ for all continuity point u of K and

(ii′) $K_{k_n}(+\infty) \to K(+\infty)$,

where
$$K_{k_n}(u) = \sum_{k=1}^{k_n} \int_{-\infty}^{u} x^2 dF_{nk}(x + \mu_{nk}) .$$

Proof. In order that S_n can be viewed as a random sums, we define Z_n and $\widetilde{X}_{nk}$ as follows. For any positive integer *n*, we define

$$Z_n(\omega) = k_n$$

for all $\omega \in \Omega$.

For $k = 1, 2, \dots k_n$, define $\widetilde{X}_{nk}(\omega) = X_{nk}(\omega)$ for all $\omega \in \Omega$

and for $k > k_n$, define $\widetilde{X}_{nk}(\omega) = 0$ for all $\omega \in \Omega$.

It follows that $(Z_n, \widetilde{X}_{nk})$ is a random double sequence of random variables which are independent in each row. We denote the distribution function, characteristic function, mean and variance of $\widetilde{X}_{nk}$ by $\widetilde{F}_{nk}, \widetilde{\varphi}_{nk}, \widetilde{\mu}_{nk}$ and $\widetilde{\sigma}_{nk}$, respectively. Since $Z_n(\omega) = k_n$ for all $\omega \in \Omega$, $l_n(q) = k_n$ for all $q \in (0,1)$. First, we will show that $(Z_n, \widetilde{X}_{nk})$ satisfies the condition $(\widetilde{\alpha})$. Let $\varepsilon > 0$ be given. Since (X_{nk}) satisfies the condition (α), we have $\max\limits_{1 \le k \le k_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \to 0$. Let $a_{nk_n} = \max\limits_{1 \le k \le k_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon)$. so $a_{nk_n} \to 0$.

By Proposition 3.1.3, $a_{nZ_n} \xrightarrow{P} 0$. That is $\max\limits_{1 \le k \le Z_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \xrightarrow{P} 0$.

Therefore $(Z_n, \widetilde{X}_{nk})$ satisfies condition $(\widetilde{\alpha})$. We note that (ii) and (ii′) are equivalent and $l_n(q) = k_n$ for $q \in (0,1)$, so the condition $(\widetilde{\beta})$ holds.

Next, we will prove that the sequence of the distribution functions of sums

$$\widetilde{S}_{Z_n} = \widetilde{X}_{n1} + \widetilde{X}_{n2} + \dots + \widetilde{X}_{nZ_n} - \sum_{k=1}^{Z_n} \widetilde{\mu}_{nk}$$

converges weakly to F if and only if

the sequence of the distribution functions of sums

$$S_n = X_{n1} + X_{n2} + \dots + X_{nk_n} - \sum_{k=1}^{k_n} \mu_{nk}$$

converges weakly to F.

Let φ be the characteristic function of F. According to the fact that $P(Z_n = k_n) = 1$, we have the characteristic function $\widetilde{\varphi}_n$ of $\widetilde{S}_{Z_n}$ is given by

$$\widetilde{\varphi}_n(t) = E[\exp(-it\sum_{j=1}^{Z_n} \widetilde{\mu}_{nj}) \prod_{j=1}^{Z_n} \widetilde{\varphi}_{nj}(t)]$$

$$=\sum_{k=1}^{\infty}P(Z_n=k)\exp(-it\sum_{j=1}^{k}\widetilde{\mu}_{nj})\prod_{j=1}^{k}\widetilde{\varphi}_{nj}(t)$$

$$=P(Z_n=k_n)\exp(-it\sum_{k=1}^{k_n}\widetilde{\mu}_{nk})\prod_{k=1}^{k_n}\widetilde{\varphi}_{nk}$$

$$=\exp(-it\sum_{k=1}^{k_n}\mu_{nk})\prod_{k=1}^{k_n}\varphi_{nk}(t)$$

which is the characteristic function φ_n of S_n.Then $\varphi_n(t)\to\varphi(t)$ for all $t\in\mathbf{R}$ if and only if $\widetilde{\varphi}_n(t)\to\varphi(t)$ for all $t\in\mathbf{R}$. Hence the sequence of the distribution functions of $\widetilde{S}_{Z_n}$ converges weakly to F if and only if the sequence of the distribution functions of S_n converges weakly to F.

($\to$) Since $Z_n(\omega)=k_n$ for all $\omega\in\Omega$, $E[\sum_{k=1}^{Z_n}\sigma_{nk}^2]=\sum_{k=1}^{k_n}\sigma_{nk}^2$ .By the same proof of step 1 of Theorem 3.2.6, there exists a function $K\in\mathcal{M}$ such that

$$\int_{\mathbf{R}}f(t,u)dK_{k_n}(u)\to\int_{\mathbf{R}}f(t,u)dK(u)\ .$$

Let (n') be any subsequence of (n). By Theorem 3.2.6, for a.e. $q\in(0,1)$, there exist a subsequence (n'') of (n') and a function $\overline{K}^{(q)}$ in $\mathcal{M}$ such that $K_{l_{n''}(q)}\xrightarrow{w}\overline{K}^{(q)}$. Since $l_{n''}(q)=k_{n''}$ for all $q\in(0,1)$, so $K_{k_{n''}}\xrightarrow{w}\overline{K}^{(q)}$ and $\overline{K}^{(q)}$ are equal for all $q\in(0,1)$. Let $\overline{K}=\overline{K}^{(q)}$. Thus $K_{k_{n''}}\xrightarrow{w}\overline{K}$. By Theorem 2.2.9,

$$\int_{\mathbf{R}}f(t,u)dK_{k_{n''}}(u)\to\int_{\mathbf{R}}f(t,u)d\overline{K}(u)\ .$$

By the uniqueness of the representation by Kolmogorov's formula, we conclude that $\overline{K}=K$. Hence $K_{k_n}\xrightarrow{w}K$.

($\leftarrow$) By (i′) and (ii′), we have $K_{l_n(q)}\xrightarrow{w}K$ for all $q\in(0,1)$ and

$\int_0^1K^{(q)}(+\infty)dq=\int_0^1K(+\infty)dq=K(+\infty)$. Let $\sigma^2=K(+\infty)$. Since $K^{(q)}=K$ for all $q\in(0,1)$, clearly, for each $x\in\mathbf{R}$, $F^{(q)}(x)$ is measurable in q, where $F^{(q)}$ is the distribution function whose logarithm of its characteristic function $\varphi^{(q)}$ is given by

$$\ln\varphi^{(q)}(t)=\int_{\mathbf{R}} f(t,u)dK^{(q)}(u).$$

Therefore the sufficiency follows from Theorem 3.2.7. #

Theorem 3.2.9 Let (Z_n, X_{nk}) be a random double sequence of random variables which satisfies the condition $(\tilde{\alpha})$, for a.e. $q \in (0,1)$, $(K_{l_n(q)})$ is monotone and for each n, $Z_n, X_{n1}, X_{n2}, \ldots$ are independent. If there exists a distribution function F with finite variance σ^2 such that

(i) the sequence of the distribution functions of sums

$$S_{Z_n} = X_{n1} + X_{n2} + \ldots + X_{nZ_n} - \sum_{k=1}^{Z_n} \mu_{nk}$$

converges weakly to F and

(ii) $E\left[\sum_{k=1}^{Z_n} \sigma_{nk}^2\right] \to \sigma^2$,

then for a.e. $q \in (0,1)$, there exists a function $K^{(q)}$ in $\mathcal{M}$ such that

(i′) $K_{l_n(q)}(u) \to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$ and

(ii′) $K_{l_n(q)}(+\infty) \to K^{(q)}(+\infty)$ and $\sigma^2 = \int_0^1 K^{(q)}(+\infty)dq$,

where
$$K_{l_n(q)}(u) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk}).$$

The converse is true if for each $x \in \mathbf{R}$, $F^{(q)}(x)$ is measurable in q, where $F^{(q)}$ is the distribution function whose logarithm of its characteristic function $\varphi^{(q)}$ is given by

$$\ln\varphi^{(q)}(t)=\int_{\mathbf{R}} f(t,u)dK^{(q)}(u).$$

Proof. First, we will show that for each $q \in (0,1)$, $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2$ is bounded.

Since $E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]\to\sigma^2$, $E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]$ is bounded. For each $n\in\mathbb{N}$ and $q\in(0,1)$,

$$E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]=\sum_{y\in \mathrm{Im}Z_n}(\sum_{k=1}^{y}\sigma_{nk}^2)P(Z_n=y)$$

$$=\sum_{l=1}^{\infty}\sum_{k=1}^{l}\sigma_{nk}^2 P(Z_n=l)$$

$$\geq\sum_{l=l_n(q)}^{\infty}\sum_{k=1}^{l}\sigma_{nk}^2 P(Z_n=l)$$

$$=\sigma_{n1}^2 P(Z_n=l_n(q))+\sigma_{n2}^2 P(Z_n=l_n(q))+...+\sigma_{nl_n(q)}^2 P(Z_n=l_n(q))$$
$$+\sigma_{n1}^2 P(Z_n=l_n(q)+1)+\sigma_{n2}^2 P(Z_n=l_n(q)+1)+...+\sigma_{nl_n(q)}^2 P(Z_n=l_n(q)+1)$$
$$+\sigma_{n(l_n(q)+1)}^2 P(Z_n=l_n(q)+1)+...$$

$$\geq\sum_{k=1}^{l_n(q)}\sigma_{nk}^2 P(Z_n\geq l_n(q))$$

$$\geq\sum_{k=1}^{l_n(q)}\sigma_{nk}^2(1-q). \qquad (\text{ since } P(Z_n<l_n(q))\leq q)$$

Since $E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]$ is bounded , $\sum_{k=1}^{l_n(q)}\sigma_{nk}^2$ is bounded for each $q\in(0,1)$ .From the fact that for each $q\in(0,1)$, $\sum_{k=1}^{l_n(q)}\sigma_{nk}^2$ is bounded, we can use the same argument of step1-step3 in Theorem 3.2.6 to show that for a.e. $q\in(0,1)$, there exist a subsequence (n_k) and a function $K^{(q)}$ in $\mathcal{M}$ such that $K_{l_{n_k}(q)}(u)\to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$ and $K_{l_{n_k}(q)}(+\infty)\to K^{(q)}(+\infty)$. But $(K_{l_n(q)})$ is monotone,we have $K_{l_n(q)}(u)\to K^{(q)}(u)$ for all continuity point u of $K^{(q)}$ and $K_{l_n(q)}(+\infty)\to K^{(q)}(+\infty)$.

To prove the necessity , it remains to show that $\sigma^2=\int_0^1 K^{(q)}(+\infty)dq$.

Since $\lim_{n\to\infty} E\left[\sum_{k=1}^{Z_n}\sigma_{nk}^2\right]=\sigma^2$, by Lemma 3.1.2(ii) , we have $\lim_{n\to\infty}\int_0^1\sum_{k=1}^{l_n(q)}\sigma_{nk}^2 dq=\sigma^2$.

Since $(K_{l_n(q)})$ is monotone, $\int_0^1 \lim_{n\to\infty}\sum_{k=1}^{l_n(q)}\sigma_{nk}^2 dq=\sigma^2$. That is $\sigma^2=\int_0^1 K^{(q)}(+\infty)dq$. The converse can be proved in the similar way as the proof of Theorem 3.2.7. #

Example 3.2.10 Let Z_n be a random variable defined by

$$P(Z_1=1)=0 \text{ and } P(Z_1=2)=1 ,$$

for $n\geq 2$, $$P(Z_n=n)=\frac{1}{n} \text{ and } P(Z_n=n+1)=1-\frac{1}{n} .$$

For each n and k, defined X_{nk} by $P(X_{nk}=-\frac{1}{\sqrt{n}})=P(X_{nk}=\frac{1}{\sqrt{n}})=\frac{1}{2}$.

Then $$F_{nk}(x)=\begin{cases} 0 & \text{if } x<-\frac{1}{\sqrt{n}} \\ \frac{1}{2} & \text{if } -\frac{1}{\sqrt{n}}\leq x<\frac{1}{\sqrt{n}} \\ 1 & \text{if } x\geq\frac{1}{\sqrt{n}} \end{cases}$$

for all n , k .

Assume that for each $n, Z_n, X_{n1}, X_{n2},\ldots$ are independent.

It's easy to see that $\mu_{nk}=0$ and $\sigma_{nk}^2=\frac{1}{n}$ for all n , k .

Then

1. $l_1(q)=2$ for all $q\in(0,1)$ and $l_n(q)=\begin{cases} n & \text{if } 0<q<\frac{1}{n} \\ n+1 & \text{if } \frac{1}{n}\leq q<1 \end{cases}$ for $n\geq 2$.

2. (X_{nk}, Z_n) satisfies the conditon $(\tilde{\alpha})$,i.e.,

$$\max_{1\leq k\leq Z_n} P(|X_{nk}|\geq\varepsilon)\xrightarrow{P}0 \quad \text{for all } \varepsilon>0 .$$

3. $\sum_{k=1}^{l_n(q)}\sigma_{nk}^2<2$ for all $q\in(0,1)$,i.e., (X_{nk}) satisfies the conditon $(\tilde{\beta})$.

4. For all $q\in(0,1)$, $K_{l_n(q)}\xrightarrow{w}K$ where $K:\mathbf{R}\to\mathbf{R}$ defined by

$$K(u)=\begin{cases} 0 & \text{if } u<0 \\ 1 & \text{if } u\geq 0 \end{cases}.$$

5. The sequence of the distribution functions of random sums

$$S_{Z_n}=X_{n1}+X_{n2}+...+X_{nZ_n}$$

converges to a limit distribution function.

Next, we will show that 1-5 hold.

1. For $q\in(0,1)$, $l_1(q)=\max\{k\in\mathrm{N}|P(Z_1<k)\leq q\}=2$.

   For $n\geq 2$, let $q\in(0,1)$.

   Case 1 $0<q<\frac{1}{n}$.

   $P(Z_n<n)=0<q$ and $P(Z_n<n+1)=P(Z_n=n)=\frac{1}{n}>q$.

   Then $l_n(q)=n$.

   Case 2 $\frac{1}{n}\leq q<1$.

   $P(Z_n<n+1)=P(Z_n=n)=\frac{1}{n}\leq q$ and

   $P(Z_n<n+2)=P(Z_n=n)+P(Z_n=n+1)=\frac{1}{n}+(1-\frac{1}{n})=1>q$.

   Then $l_n(q)=n+1$.

2. To prove (X_{nk},Z_n) satisfies the conditon $(\tilde{\alpha})$, by Proposition 3.1.3 , it suffice to show that for each $q\in(0,1)$, $\lim_{n\to\infty}\max_{1\leq k\leq l_n(q)} P(|X_{nk}|\geq\varepsilon)=0$ for all $\varepsilon>0$.

Let $q\in(0,1)$ and $\varepsilon>0$. Let $N\in\mathrm{N}$ be such that $\frac{1}{N}<q$ and $\frac{1}{\sqrt{N}}<\varepsilon$. For $n\geq N$,

$$\max_{1\leq k\leq l_n(q)} P(|X_{nk}|\geq\varepsilon)=\max_{1\leq k\leq n+1} P(|X_{nk}|\geq\varepsilon)=\max_{1\leq k\leq n+1}(P(X_{nk}\geq\varepsilon)+P(X_{nk}\leq-\varepsilon))=0.$$

Thus $\lim_{n\to\infty}\max_{1\leq k\leq l_n(q)} P(|X_{nk}|\geq\varepsilon)=0$.

3. $\sum_{k=1}^{l_n(q)}\sigma_{nk}^2\leq\sum_{k=1}^{n+1}\sigma_{nk}^2=\sum_{k=1}^{n+1}\frac{1}{n}=\frac{n+1}{n}\leq 2$ for all $q\in(0,1)$.

4. Let $q \in (0,1)$ and $u \in \mathbf{R}-\{0\}$. Let $N_1 \in \mathbf{N}$ be such that $\frac{1}{N_1^2} < q$.

Case 1 $u<0$.

Let $N_2 \in \mathbf{N}$ be such that $u < -\frac{1}{\sqrt{N_2}}$. Let $N = \max\{N_1, N_2\}$.

For $n \geq N$, $K_{l_n(q)}(u) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk})$

$$= \sum_{k=1}^{n+1} \int_{-\infty}^{u} x^2 dF_{nk}(x)$$

$$= 0.$$

Then $K_{l_n(q)}(u) \to 0$.

Case 2 $u>0$.

Let $N_2 \in \mathbf{N}$ be such that $u > \frac{1}{\sqrt{N_2}}$. Let $N = \max\{N_1, N_2\}$.

For $n \geq N$, $K_{l_n(q)}(u) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{u} x^2 dF_{nk}(x+\mu_{nk})$

$$= \sum_{k=1}^{n+1} \int_{-\infty}^{u} x^2 dF_{nk}(x)$$

$$= \sum_{k=1}^{n+1} \left(\int_{\{-\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) + \int_{\{\frac{1}{\sqrt{n}}\}} x^2 dF_{nk}(x) \right)$$

$$= \sum_{k=1}^{n+1} \left(\left(\frac{1}{2}-0\right)\left(-\frac{1}{\sqrt{n}}\right)^2 + \left(1-\frac{1}{2}\right)\left(\frac{1}{\sqrt{n}}\right)^2 \right)$$

$$= \sum_{k=1}^{n+1} \frac{1}{n}$$

$$= \frac{n+1}{n}.$$

Then $K_{l_n(q)}(u) \to 1$.

5. Follows from Theorem 3.2.7. #

REFERENCES

1. Bethmann,J. The Lindeberg-Feller theorem for sums of random number of independent random variables in a triangular array. Theory of Prob. Applications. Vol.33, No.2 (1988) :334-339.
2. Cramer,H. Mathematical method of statistics. Uppsala:Almqvist&Wilesells, 1945.
3. Feller,W. An introduction to probability theory and its applications. New York: John Wiley&Sons, 1966.
4. Fristedt, B. and Gray, L. A Modern Approach to Probability Theory. Boston: Birkhauser, 1997.
5. Gnedenko, B.V. Theory of probability. New York:Chelsca Publishing Company, 1962.
6. Gnedenko, B.V. and Kolmogorov, A.N. Limit distributions for sums of independent random variables. Cambridge:Addison-Wesley, 1954.
7. Laha, R.G. and Rohatgi, V.K. Probability Theory. New York:John Wiley&Sons, 1979.
8. Lukacs,E. Characteristic functions. New York:Hafner Publishing Company, 1960.
9. Munroe,M.E. Measure and integration. California:Addison-Wesley, 1971.
10. Neammanee, K. A limit theorem for random sums of independent random variables with finite variances. Doctoral dissertation, Department of Mathematics, Graduate School, Chulalongkorn University, 1993.
11. Petrov, V.V. Sums of independent random variables. Berlin:Springer-Verlag, 1975.
12. Shapiro,J.M. Error estimates for certain probability limit theorems. Ann. Math. Statist.Vol.26(1955):617-630.
13. Szasz, D. Limit theorems for the distributions of the sums of random number of random variables. Ann. Math. Statist. Vol.43 (1972):1902-1913.

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