CHAPTER IV

THE APPROXIMATE DENSITY OF STATES

THE APPROXIMATE PROPAGATOR

As discussed in previous chapter, in order to calculate the average propagator (2.20), we shall approximate the propagator with the help of the constructed trial action (3.12). The approximation which we will use is the first order cumulant expansion (Kubo, 1962). Let us first introduce a path integral average with respect to the trial action $S_0[\mathbf{x}(\tau)]$ by

$$\langle O \rangle = \frac{\int D(\mathbf{x}(\tau))\, O \, \exp\left[\frac{i}{\hbar} S_0[\mathbf{x}(\tau)]\right]}{\int D(\mathbf{x}(\tau))\, \exp\left[\frac{i}{\hbar} S_0[\mathbf{x}(\tau)]\right]} \, , \tag{4.1}$$

where O denotes a function to be averaged function. Accordingly, the average propagator can be rewritten as

$$\begin{aligned} \bar{K}(\mathbf{x}_2, \mathbf{x}_1; t, 0) &= \int D(\mathbf{x}(\tau))\, \exp\left[\frac{i}{\hbar}(\bar{S} - S_0) + \frac{i}{\hbar} S_0\right] \\ &= K_0(\mathbf{x}_2, \mathbf{x}_1; t, 0) \left\langle \exp\frac{i}{\hbar}(\bar{S} - S_0) \right\rangle_0 , \end{aligned} \tag{4.2}$$

where K_0 is the trial propagator, (3.23), The equation (4.2) for the average propagator still is and exact expression but it still cannot be solved. The cumulant expansion,

$$\langle \exp[a] \rangle = \exp\left[\langle a \rangle + \frac{1}{2}\left(\langle a^2 \rangle - \langle a \rangle^2 \right) - \frac{1}{3!}\left(\langle a^3 \rangle - 3\langle a^2 \rangle \langle a \rangle + 2\langle a \rangle^3 \right) + \cdots \right], \quad (4.3)$$

to first order allows us the approximate propagator,

$$\overline{K}_1(\mathbf{x}_2, \mathbf{x}_1; t, 0) = K_0(\mathbf{x}_2, \mathbf{x}_1; t, 0) \exp\left[\frac{i}{\hbar} \langle \overline{S} - S_0 \rangle_0 \right], \quad (4.4)$$

where the index 1 denotes the first order approximation. Since the kinetic energy terms of both actions are identical, then the exponent $\langle \overline{S} - S_0 \rangle_0$ can be replaced by $\langle \overline{S}' - S_0' \rangle_0$, where the prime symbol means excluding the kinetic energy term. That is,

$$\overline{K}_1(\mathbf{x}_2, \mathbf{x}_1; t, 0) = K_0(\mathbf{x}_2, \mathbf{x}_1; t, 0) \exp\left[\frac{i}{\hbar} \langle \overline{S}' - S_0' \rangle_0 \right]. \quad (4.5)$$

Since the trial propagator K_0 has been already carried out, the remaining work is to evaluate $\langle \overline{S}' \rangle_0$ and $\langle S_0' \rangle_0$.

By substituting $\overline{S}$ from (2.21), the path integral average of $\overline{S}'$ is expressed as

$$\langle \overline{S}' \rangle_0 = -n\overline{V}t + \frac{i}{2\hbar} \int_0^t d\tau \int_0^t d\sigma \langle W(\mathbf{x}(\tau) - \mathbf{x}(\sigma)) \rangle_0, \quad (4.6)$$

where $W(\mathbf{x}(\tau) - \mathbf{x}(\sigma))$ is given by (2.22) and (2.23) for Gaussian and screened Coulomb potentials, respectively. To find the average of the autocorrelation function, we shall use its Fourier integral. The Fourier transforms of the Gaussian and screened Coulomb autocorrelation functions are given below respectively;

$$W(\mathbf{x}(\tau)-\mathbf{x}(\sigma))=\int \frac{d\mathbf{k}}{(2\pi)^3}W(\mathbf{k})\ \exp[i\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}(\sigma))], \tag{4.7}$$

where $W(\mathbf{k})$ is the Fourier transform of the autocorrelation function.

$$W(\mathbf{k})=u^2\exp\left[-\frac{1}{4}L^2k^2\right], \tag{4.8}$$

and

$$W(\mathbf{k})=\frac{Z^2e^4}{\varepsilon_0^2}\frac{(4\pi)^2}{\left(k^2+Q^2\right)^2}. \tag{4.9}$$

Inserting (4.7) into (4.6),

$$\langle\bar{S}'\rangle_0=-n\bar{V}t+\frac{i}{2\hbar}n\int_0^t d\tau\int_0^t d\sigma\int\frac{d\mathbf{k}}{(2\pi)^3}W(\mathbf{k})\langle\exp[i\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}(\sigma))]\rangle_0, \tag{4.10}$$

$$=-n\bar{V}t+\frac{i}{2\hbar}\int_0^t d\tau\int_0^t d\sigma\int\frac{d\mathbf{k}}{(2\pi)^3}W(\mathbf{k})\ \exp\left[i\mathbf{k}\cdot\mathcal{A}-k^2\mathcal{B}^{\,2}\right], \tag{4.11}$$

where

$$\mathcal{A}=\langle\mathbf{x}(\tau)-\mathbf{x}(\sigma)\rangle_0, \tag{4.12}$$

and

$$\mathcal{B}=\frac{1}{2}\left(\frac{1}{3}\left\langle(\mathbf{x}(\tau)-\mathbf{x}(\sigma))^2\right\rangle_0-\langle x(\tau)-x(\sigma)\rangle_0^2\right). \tag{4.13}$$

The last equation (4.11) is the cumulant expansion of (4.10) by using the reason that only the first two cumulant functions are non-zero because S_0 is quadratic. For a Gaussian case, putting (4.8) into (4.11) and applying the Gaussian integration formula (3.8) for each Cartesian component of **k**-integration, we get

$$\langle \bar{S}' \rangle_0 = -n\bar{V}t + \frac{i}{2\hbar}(4\pi)^{-3/2} nu^2 \int_0^t d\tau \int_0^t d\sigma \left(B + \frac{L^2}{4} \right)^{-3/2} \exp\left[-\frac{A^2}{4\left(B + \frac{L^2}{4}\right)} \right]. \qquad (4.14)$$

In the case of the screened Coulomb potential, the extra transformation,

$$\frac{1}{a^2} = \int_0^\infty dx\, x\, \exp[-ax], \qquad (4.15)$$

is employed. The expression like (4.14) is obtained, except the extra integration,

$$\langle \bar{S}' \rangle = -n\bar{V}t + \frac{i}{2\hbar} 2\sqrt{\pi} \frac{nZ^2 e^4}{\varepsilon_0^2} \int_0^t d\tau \int_0^t d\sigma \int_0^\infty dy\, y\, (B + y)^{-3/2} \exp\left[-\frac{A^2}{4(B + y)} - Q^2 y \right]. \qquad (4.16)$$

Note that we have interchanged the order of integrations in (4.16) and carried out in the manner as the Gaussian case.

Now, consider the average of the trial action, we have

$$\langle S_0' \rangle_0 = -\frac{1}{8}\kappa\omega \int_0^t d\tau \int_0^t d\sigma \left\langle |\mathbf{x}(\tau) - \mathbf{x}(\sigma)|^2 \right\rangle_0 \frac{\cos\omega\left(\frac{1}{2}t - |\tau - \sigma|\right)}{\sin\frac{1}{2}\omega t}. \qquad (4.17)$$

Because of the symmetry under interchanging between τ and σ of S_0', above equation can be written as

$$\langle S_0' \rangle_0 = -\frac{1}{4}\kappa\omega \int_0^t d\tau \int_0^t d\sigma \left\langle (\mathbf{x}(\tau) - \mathbf{x}(\sigma))^2 \right\rangle_0 \frac{\cos\omega(\frac{1}{2}t - |\tau - \sigma|)}{\sin\frac{1}{2}\omega t}. \tag{4.18}$$

Here, the approximate propagator can be obtained by (4.5), (4.14) or (4.16), and (4.18), however, the terms such as A , B , and $\left\langle (\mathbf{x}(\tau) - \mathbf{x}(\sigma))^2 \right\rangle_0$ are not solved explicitly yet. These terms will be determined in next section.

DETAILED CALCULATIONS

Let us begin with calculating A , B , and $\left\langle (\mathbf{x}(\tau) - \mathbf{x}(\sigma))^2 \right\rangle_0$. We can see that all these terms consists of at most two average types, namely $\langle \mathbf{x}(\tau) \rangle_0$ and $\langle \mathbf{x}(\tau) \cdot \mathbf{x}(\sigma) \rangle_0$. The method used to obtain these two average govern with the so-called generating functional (Feynman and Hibbs, 1965). If the action S is Gaussian then the action $S^f = S + \int f(\tau) x(\tau)\, d\tau$ is also Gaussian. As discussed in chapter III, the path integrals of these actions can be written in the form (3.10) so that

$$\left\langle \exp\left[\int_0^t d\tau f(\tau) x(\tau)\right] \right\rangle = \exp\left[\frac{i}{\hbar}\left(S_{cl}^f - S_{cl}\right)\right], \tag{4.19}$$

where $\langle\ \rangle$ denotes the path integral average with respect to S. Differentiating both sides of (4.19) with respect to f and set $f = 0$, we have

$$\langle x(\tau) \rangle = \left. \frac{\delta S_{cl}^f}{\delta f(\tau)} \right|_{f=0} \tag{4.20}$$

If we continue differentiating (4.19) and set $f=0$, the successive moments will come out, for example,

$$\langle x(\tau)x(\sigma)\rangle = \left[\frac{\hbar}{i}\frac{\delta^2 S_{cl}^f}{\delta f(\tau)\delta f(\sigma)} + \frac{\delta S_{cl}^f}{\delta f(\tau)}\cdot\frac{\delta S_{0,cl}^f}{\delta f(\sigma)}\right]_{f=0}, \tag{4.21}$$

In our problem, the action S should be the trial action S_0, so that the action S^f is the forced trial action S_0^f.

Using the forced classical trial action $S_{0,cl}^f$ from (3.23), the first and second functional derivatives can be obtained:

$$\left.\frac{\delta S_{0,cl}^f}{\delta f(\tau)}\right|_{f=0} = x_2\left(\frac{\mu}{m}\frac{\cos\frac{1}{2}\nu(t-\tau)\sin\frac{1}{2}\nu\tau}{\sin\frac{1}{2}\nu t} + \frac{\mu\tau}{Mt}\right)$$
$$+x_1\left(\frac{\mu}{m}\frac{\sin\frac{1}{2}\nu(t-\tau)\cos\frac{1}{2}\nu\tau}{\sin\frac{1}{2}\nu t} + \frac{\mu(t-\tau)}{Mt}\right) \tag{4.22}$$

and

$$\left.\frac{\delta^2 S_{0,cl}^f}{\delta f(\tau)\delta f(\sigma)}\right|_{f=0} = -3\left[H(\tau-\sigma)\left(\frac{2\mu}{m^2\nu}\frac{\sin\frac{1}{2}\nu(t-\tau)\sin\frac{1}{2}\nu\sigma\cos\frac{1}{2}\nu(\tau-\sigma)}{\sin\frac{1}{2}\nu t} + \frac{\mu(t-\tau)\sigma}{mMt}\right)\right.$$
$$\left.+H(\tau-\sigma)\left(\frac{2\mu}{m^2\nu}\frac{\sin\frac{1}{2}\nu(t-\sigma)\sin\frac{1}{2}\nu\tau\cos\frac{1}{2}\nu(\tau-\sigma)}{\sin\frac{1}{2}\nu t} + \frac{\mu(t-\sigma)\tau}{mMt}\right)\right], \tag{4.25}$$

where H is the Heaviside step function. Putting (4.22) and (4.23) into (4.20) and (4.21), we get

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$$\langle \mathbf{x}(\tau)\rangle_0 = \mathbf{x}_2\left(\frac{\mu}{m}\frac{\cos\frac{1}{2}\nu(t-\tau)\sin\frac{1}{2}\nu\tau}{\sin\frac{1}{2}\nu t}+\frac{\mu\tau}{Mt}\right)$$

$$+\mathbf{x}_1\left(\frac{\mu}{m}\frac{\sin\frac{1}{2}\nu(t-\tau)\cos\frac{1}{2}\nu\tau}{\sin\frac{1}{2}\nu t}+\frac{\mu(t-\tau)}{Mt}\right), \qquad (4.24)$$

and, for $\tau > \sigma$,

$$\langle \mathbf{x}(\tau)-\mathbf{x}(\sigma)\rangle_0 = 3i\hbar\left(\frac{2\mu}{m^2\nu}\frac{\sin\frac{1}{2}\nu(t-\tau)\sin\frac{1}{2}\nu\sigma\cos\frac{1}{2}\nu(\tau-\sigma)}{\sin\frac{1}{2}\nu t}+\frac{\mu(t-\tau)\sigma}{mMt}\right)$$

$$+\langle \mathbf{x}(\tau)\rangle_0\cdot\langle \mathbf{x}(\sigma)\rangle_0 \qquad (4.25)$$

and, for $\tau < \sigma$,

$$\langle \mathbf{x}(\tau)-\mathbf{x}(\sigma)\rangle_0 = 3i\hbar\left(\frac{2\mu}{m^2\nu}\frac{\sin\frac{1}{2}\nu(t-\sigma)\sin\frac{1}{2}\nu\tau\cos\frac{1}{2}\nu(\tau-\sigma)}{\sin\frac{1}{2}\nu t}+\frac{\mu(t-\sigma)\tau}{mMt}\right)$$

$$+\langle \mathbf{x}(\tau)\rangle_0\cdot\langle \mathbf{x}(\sigma)\rangle_0. \qquad (4.26)$$

It can be easily obtained by inserting (4.24) into (4.12) that

$$\mathcal{A} = \left(\frac{\mu}{m}\frac{\sin\frac{1}{2}\nu(\tau-\sigma)\cos\frac{1}{2}\nu(t-(\tau+\sigma))}{\sin\frac{1}{2}\nu t}+\frac{\mu(\tau-\sigma)}{Mt}\right)(\mathbf{x}_2-\mathbf{x}_1). \qquad (4.27)$$

For $\mathcal{B}$ and $\left\langle(\mathbf{x}(\tau)-\mathbf{x}(\sigma))^2\right\rangle_0$, we must separately substitute for $\tau > \sigma$ and $\tau < \sigma$ cases, (4.25) and (4.26), into the new forms of them:

$$\mathcal{B} = \frac{1}{6}\left(\langle \mathbf{x}^2(\tau)\rangle_0 - 2\langle \mathbf{x}(\tau)\cdot\mathbf{x}(\sigma)\rangle_0 + \langle \mathbf{x}^2(\sigma)\rangle_0 - \langle \mathbf{x}(\tau)\rangle_0^2 + 2\langle \mathbf{x}(\tau)\rangle_0\cdot\langle \mathbf{x}(\sigma)\rangle_0 - \langle \mathbf{x}(\sigma)\rangle_0^2\right),$$

$$(4.28)$$

and

$$\left\langle (\mathbf{x}(\tau)\cdot\mathbf{x}(\sigma))^2 \right\rangle_0 = \left\langle \mathbf{x}^2(\tau)\right\rangle_0 - 2\left\langle \mathbf{x}(\tau)\cdot\mathbf{x}(\sigma)\right\rangle_0 + \left\langle \mathbf{x}^2(\sigma)\right\rangle_0 . \tag{4.29}$$

Note that, in (4.28), we have set

$$\langle x(\tau) - x(\sigma)\rangle_0^2 = \frac{1}{3}\langle \mathbf{x}(\tau) - \mathbf{x}(\sigma)\rangle_0^2 \tag{4.30}$$

because of the directional symmetry of the system. Now, we are able to write down $\mathcal{B}$ and $\left\langle (\mathbf{x}(\tau) - \mathbf{x}(\sigma))^2 \right\rangle_0$ as

$$\mathcal{B} = i\hbar\frac{\mu}{m}\left(\frac{\sin\frac{1}{2}\nu|\tau-\sigma|\sin\frac{1}{2}\nu(t-|\tau-\sigma|)}{m\nu\sin\frac{1}{2}\nu t} + \frac{(t-|\tau-\sigma|)|\tau-\sigma|}{2Mt}\right), \tag{4.31}$$

and

$$\left\langle (\mathbf{x}(\tau) - \mathbf{x}(\sigma))^2 \right\rangle_0 = 6i\hbar\frac{\mu}{m}\left(\frac{\sin\frac{1}{2}\nu|\tau-\sigma|\sin\frac{1}{2}\nu(t-|\tau-\sigma|)}{m\nu\sin\frac{1}{2}\nu t} + \frac{(t-|\tau-\sigma|)|\tau-\sigma|}{2Mt}\right)$$

$$+\left(\frac{\mu}{m}\frac{\sin\frac{1}{2}\nu(\tau-\sigma)\cos\frac{1}{2}\nu(t-(\tau+\sigma))}{\sin\frac{1}{2}\nu t} + \frac{\mu(\tau-\sigma)}{Mt}\right)^2 (\mathbf{x}_2 - \mathbf{x}_1)^2 . \tag{4.32}$$

It is worth to note that $\mathcal{B}$ has the following property

$$\mathcal{B}\,(|\tau-\sigma|) = \mathcal{B}\,(t-|\tau-\sigma|). \tag{4.33}$$

<u>EVALUATING THE APPROXIMATE DENSITY OF STATES</u>

According to the last two sections, it can be concluded the approximate density of states defined by (4.5) has a translational symmetry. This means that the formula

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for calculating it is given by (2.33) in which only the diagonal element of the propagator is governed with. Combining (4.5), (4.14) or (4.16), (4.18), (4.27), (4.31), and (4.33) with the condition $\mathbf{x}_2 = \mathbf{x}_1$, the approximate density of states can be written as

$$\rho_1(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}} \left(\frac{\nu \sin\frac{1}{2}\omega t}{\omega \sin\frac{1}{2}\nu t}\right)^{3}$$
$$\exp\left[\frac{i}{\hbar}(E - n\bar{V})t + \frac{3}{2}\frac{\mu}{m}\left(\frac{1}{2}\nu t \cot \nu\frac{t}{2} - 1\right) - \frac{1}{2\hbar^2}[\cdots]\right], \qquad (4.34)$$

where

$$[\cdots] = \xi_L\left(\frac{L^2}{4}\right)^{3/2}\int_0^t d\tau\int_0^t d\sigma\left(\frac{L^2}{4} + B\right)^{-\frac{3}{2}} \text{ ; for a Gaussian case,}$$

$$[\cdots] = \xi_Q\frac{Q}{\pi}\int_0^{\infty} dy\exp[-Q^2 y]\int_0^t d\tau\int_0^t d\sigma\,(y + B)^{-\frac{3}{2}} \text{ ; for a screened Coulomb case,}$$

and $\xi_L = \dfrac{nu^2}{(\pi L^2)^{\frac{3}{2}}}$, $\xi_Q = \dfrac{2\pi nZ^2e^4}{Q\epsilon_o^2}$ and B is given by (4.31). When the second component is derived from directly integrating the trial action term. Based on the property (4.33), the double-time integration can be reduced to a single integration as following,

$$\int_0^t d\tau\int_0^t d\sigma\left(B\left(|\tau - \sigma|\right) + C\right)^{-\frac{3}{2}} = t\int_0^t dx\left(B(x) + C\right)^{-\frac{3}{2}}. \qquad (4.35)$$

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In a low-lying energy tail, suppose that ν is large and $\gamma = \left(\frac{m}{m+M}\right)^{\frac{1}{2}}$ is very small which is called the "full-ground-state" approximation. Hence, we are able to approximate

$$\frac{\sin\frac{1}{2}\nu x \sin\frac{1}{2}\nu(t-x)}{\sin\frac{1}{2}\nu t} \cong \frac{1}{2i}, \tag{4.36}$$

$$\sin\frac{1}{2}\omega t \cong \frac{1}{2}\nu\gamma\, t, \tag{4.37}$$

$$\left(\sin\frac{1}{2}\nu t\right)^{-3} \cong -8i \exp\left[-i\frac{3}{2}\nu t\right], \tag{4.38}$$

$$\frac{1}{2}\nu t \cot\frac{1}{2}\nu t - 1 \cong \frac{i}{2}\nu t. \tag{4.39}$$

By using (4.36) and keeping the terms up to t^2 only, (4.35) can be integrated (Jeffrey, 1995) giving

$$\int_0^t d\tau \int_0^t d\sigma \left(B\left(|\tau-\sigma|\right)+C\right)^{-\frac{3}{2}} = \left(\frac{\hbar(1-\gamma^2)}{2m\nu}+C\right)^{-\frac{3}{2}} t^2, \tag{4.40}$$

Applying (4.37), (4.38), (4.39) and (4.40), the expression for the density of states becomes

$$\rho_g(E) = \frac{1}{2\pi\hbar}\left(\frac{m}{2\pi\hbar}\right)^{\frac{3}{2}}\left(\frac{E_\nu}{\hbar}\right)^3 \int_{-\infty}^{\infty} dt\,(it)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar}\left(E - n\bar{V} - \frac{3}{2}E_\nu + \frac{3}{4}E_\nu\left(1-\gamma^2\right)\right)t - \beta^2 t^2\right], \tag{4.41}$$

where

$$\beta^2 = \frac{1}{2\hbar^2}\xi_L\left(1+\frac{4E_L(1-\gamma^2)}{E_v}\right)^{-\frac{1}{2}} \text{;for the Gaussian case,}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\hbar^2}\xi_Q \exp\left[\frac{E_Q(1-\gamma^2)}{2E_v}\right]D_{-3}\left(\sqrt{\frac{2E_Q(1-\gamma^2)}{E_v}}\right) \text{;for the screened Coulomb case.}$$

In (4.40), the following variables $E_v = \hbar v$, $E_L = \frac{\hbar^2}{2mL^2}$ and $E_Q = \frac{\hbar^2 Q^2}{2m}$ are used and $D_p(z)$ is the parabolic cylinder function. In addition, the formula (Gradshteyn and Ryzhik, 1965),

$$\int_0^\infty dx\, x^{\upsilon-1}(x+\beta)^{-\upsilon+\frac{1}{2}} \exp[\mu x] = 2^{\upsilon-\frac{1}{2}}\Gamma(\upsilon)\mu^{-\frac{1}{2}}\exp\left[\frac{\beta\mu}{2}\right]\cdot D_{1-2\upsilon}\left(\sqrt{2\beta\mu}\right), \qquad (4.42)$$

has been used for calculating β^2 in a screened Coulomb case. With the aid of the identity,

$$\int_{-\infty}^{\infty} dx\,(ix)^\upsilon \exp[-\beta^2x^2 - iqx] = 2^{\frac{-\upsilon}{2}}\sqrt{\pi}\beta^{-\upsilon-1}\exp\left[-\frac{q^2}{8\beta^2}\right]D_\upsilon\left(\frac{q}{\sqrt{2}\beta}\right), \qquad (4.43)$$

we have

$$\rho_g(E) = \frac{1}{2\sqrt{\pi}\hbar}2^{\frac{-3}{4}}\left(\frac{m}{2\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{E_v}{\hbar}\right)^3\beta^{\frac{-3}{2}}\exp\left[-\frac{\left(E-n\bar{V}-\frac{3}{2}E_v+\frac{3}{4}E_v(1-\gamma^2)\right)}{8\hbar^2\beta^2}\right].$$

$$\times D_{\frac{1}{2}}\left(-\frac{E-n\bar{V}-\frac{3}{2}E_v+\frac{3}{4}E_v(1-\gamma^2)}{\sqrt{2}\hbar\beta}\right). \qquad (4.44)$$

Here, the density of states can be written down in the form which is obtained by Sa-yakanit et al. (1982). For the Gaussian case,

$$\rho_g(\eta\,;x,x')=\frac{L^{-3}}{E_L\xi_L'^{3/4}}\cdot\frac{a(\eta\,;x,x')}{b^{3/4}(\eta\,;x,x')}\exp\left[\frac{-b(\eta\,;x,x')}{4\xi_L'}\right]D_{\frac{3}{2}}\left(\sqrt{\frac{b(\eta\,;x,x')}{\xi_L'}}\right), \tag{4.45}$$

where $\xi_L'=\dfrac{\xi_L}{E_L^2}, \eta=\dfrac{n\bar{V}-E}{E_L}$, $x=\dfrac{E_v}{E_L}$ and $x'=\dfrac{E_v}{E_L}\left(1-\gamma^2\right)$.

When the two dimensionless functions are defined by

$$a(\eta\,;x,x')=\left(\frac{3}{2}x-\frac{3}{4}x'+\eta\right)^{\frac{3}{2}}\frac{x^3}{8\sqrt{2}\pi^2}\left(1+\frac{4}{x'}\right)^3 \tag{4.46}$$

and

$$b(\eta\,;x,x')=\left(\frac{3}{2}x-\frac{3}{4}x'+\eta\right)^2\left(1+\frac{4}{x'}\right)^{\frac{3}{2}}. \tag{4.47}$$

For the screened Coulomb case,

$$\rho_g(\eta\,;z,z')=\frac{Q^3}{E_Q\xi_Q'^{3/4}}\cdot\frac{a(\eta\,;z,z')}{b^{3/4}(\eta\,;z,z')}\exp\left[\frac{-b(\eta\,;z,z')}{4\xi_Q'}\right]D_{\frac{3}{2}}\left(\sqrt{\frac{b(\eta\,;z,z')}{\xi_Q'}}\right), \tag{4.48}$$

where $\xi_Q'=\dfrac{\xi_Q}{E_Q^2}, \eta=\dfrac{n\bar{V}-E}{E_Q}$, $z=\sqrt{\dfrac{2E_Q}{E_v}}$ and $z'=\sqrt{\dfrac{2E_Q}{E_v}\left(1-\gamma^2\right)}$.

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The two dimensionless functions are given by

$$a(\eta\ ;z,z')=\frac{\sqrt{\pi}}{8\sqrt{2}}z^{-6}\left(3z^{-2}-\frac{3}{2}z^{-4}z'^{2}+\eta\right)^{\frac{1}{2}}\exp\left[\frac{-z'^{2}}{2}\right]D_{-3}^{-2}(z') \tag{4.49}$$

and

$$b(\eta\ ;z,z')=\frac{\sqrt{\pi}}{2\sqrt{2}}\left(3z^{-2}-\frac{3}{2}z^{-4}z'^{2}+\eta\right)^{2}\exp\left[\frac{-z'^{2}}{4}\right]D_{-3}^{-1}(z'). \tag{4.50}$$

In the deep-tail approximation, we can consider this limit by two methods: one is obtained by fixing the fluctuation strength ξ' and limiting $E\rightarrow\infty$ and another is by keeping E constant and limiting $\xi'\rightarrow 0$. We shall use the first method relying on the asymptotic behaviour of the parabolic-cylinder function,

$$D_p(z)=z^{p}\exp\left[\frac{-z^{2}}{4}\right]\qquad \text{as}\quad z\rightarrow\infty. \tag{4.51}$$

Our two density of states have the same forms with those derived by Halperin and Lax (1966,1967) and Sa-yakanit (1979). The density of states of both cases are

$$\rho_d(\eta\ ;x,x')=\frac{L^{-3}}{E_L\xi_L'^{2}}a(\eta\ ;x,x')\exp\left[\frac{-b(\eta\ ;x,x')}{2\xi_L'}\right] \tag{4.52}$$

and

$$\rho_d(\eta\,;z,z') = \frac{Q^3}{E_Q \xi_Q'^2} a(\eta\,;z,z') \exp\left[\frac{-b(\eta\,;z,z')}{2\xi_Q'}\right]. \tag{4.53}$$

Note that g and d denote the "full-ground-state" and the "deep-tail" approximations, respectively.

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