NUMERICAL SEMIGROUPS ON $\mathbb{N} \times \mathbb{N}$

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กึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$

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เรานิยามกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ คล้ายกับกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}$ พร้อมทั้งศึกษาเซตย่อยของ $\mathbb{N}\times\mathbb{N}$ ที่ก่อกำเนิดกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ นอกจากนี้เรานิยามและศึกษากึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ แบบลดทอนไม่ได้และแบบสมมาตร เราจำแนกกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ แบบลดทอนไม่ได้และแบบสมมาตร สุดท้ายเราศึกษาเงื่อนไขของกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ ที่สามารถเขียนเป็นอินเตอร์เซกชันของกึ่งกรุปเชิงตัวเลขบน $\mathbb{N}\times\mathbb{N}$ แบบสมมาตร

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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The notion of numerical semigroups on $\mathbb{N}\times\mathbb{N}$ is defined similarly to numerical semigroups on $\mathbb{N}$. We investigate subsets of $\mathbb{N}\times\mathbb{N}$ which generate numerical semigroups on $\mathbb{N}\times\mathbb{N}$. Moreover, we define and investigate irreducible and symmetric numerical semigroups on $\mathbb{N}\times\mathbb{N}$. We provide some characterization of irreducible and symmetric numerical semigroups on $\mathbb{N}\times\mathbb{N}$. Finally, we study conditions of numerical semigroups on $\mathbb{N}\times\mathbb{N}$ that can be expressed as an intersection of symmetric numerical semigroups on $\mathbb{N}\times\mathbb{N}$.

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CONTENTS

LIST OF NOTATIONS

Let $\mathbb{N}$ be the set of all nonnegative integers, $S \subseteq \mathbb{N} \times \mathbb{N}$ a numerical semigroup, a, b nonnegative integers.

$\mathbb{N}^*$	the set $\mathbb{N} \backslash \{0\}$
$\mathbb{N}^2$	the set $\{(m, n) \mid m, n \in \mathbb{N}\}$
$I_{(a,b)}$	the set $\{(m, n) \in \mathbb{N}^2 \mid 0 \leq m \leq a$ and $0 \leq n \leq b\}$
$E_{(a,b)}$	the set $\mathbb{N}^2 \backslash I_{(a,b)}$
S^*	the set $S \backslash \{(0, 0)\}$
$G(S)$	the set $\mathbb{N}^2 \backslash S$
$MG(S)$	the set $\{(m, n) \in G(S) \mid a^2 + b^2 \leq m^2 + n^2$ for all $(a, b) \in G(S)\}$
$PMG(S)$	the set $\{g \in G(S) \mid g + s \in S$ for all $s \in S^*\}$
$SG(S)$	the set $\{g \in G(S) \mid g + g \in S$ and $g + s \in S$ for all $s \in S^*\}$

CHAPTER I
PRELIMINARIES

Let $\mathbb{N}$ be the set of all nonnegative integers. Then $\mathbb{N}$ is a monoid under the usual addition. In this work, we aim to study subsemigroups of $\mathbb{N} \times \mathbb{N}$ (we give the definition in Chapter II). Most of our results are analogous to the results in numerical semigroups. In this chapter, we give some definitions, propositions and examples in numerical semigroups in order to get ideas of our materials.

The study of sets of linear combination of natrual numbers with nonnegative coefficients rose in 1884 by Sylvester : "Let s_1 and s_2 be two relatively prime natural numbers. Determine the largest integer which cannot be written as a linear combination $n_1s_1 + n_2s_2$ where n_1 and n_2 are nonnegative integers". A generalization of Sylvester's problem was proposed by Frobenius.

From Sylvester and Frobenius's problem, we see that the condition that gives the existence of the answer is the greatest common divisor of those given integers which is one. On the other hand, this condition guarantees that there are finitely many positive integers which cannot be written as linear combinations of given positive integers.

In 1987, Froberg, Gottlieb and Haggkvist investigated this structure in [1] and first gave the word numerical semigroup.

Definition 1.1. [1] A *numerical semigroup* is a submonoid S of $\mathbb{N}$ such that $\mathbb{N} \backslash S$ is finite.

Definition 1.2. [6] Let S be a numerical semigroup. An element g of $\mathbb{N} \backslash S$ is called a *gap* of S. The set of all gaps of S is denoted by $G(S)$. The largest interger not in S is called the *Frobenius number* of S, denoted by $F(S)$.

Example 1.3. Let $S = \{0, 7, 8, 9, 10, \ldots\}$. We see that S is a submonoid of $\mathbb{N}$ and $\mathbb{N} \backslash S = \{1, 2, 3, 4, 5, 6\}$ is a finite set. Hence S is a numerical semigroup. We have that $G(S) = \{1, 2, 3, 4, 5, 6\}$ and the Frobenius number of S is 6.

Definition 1.4. [6] Let A be a nonempty subset of $\mathbb{N}$. The *smallest submonoid of* $\mathbb{N}$ *containing* A is the set $\{n_1a_1 + \cdots + n_ka_k \mid k \in \mathbb{N}^*, a_1, \ldots, a_k \in A, n_1, \ldots, n_k \in \mathbb{N}\}$, denoted by $\langle A \rangle$.

Proposition 1.5. *Let A be a nonempty finite subset of $\mathbb{N}$. Then $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$ where $\gcd(A)$ is the greatest common divisor of elements of A.*

The proof of Proposition 1.5 can be seen in [6]. This condition is the same as the condition on Sylvester and Frobenius's problem.

Definition 1.6. [6] A nonempty subset A of $\mathbb{N}$ is a *system of generators* of a submonoid M of $\mathbb{N}$ if $\langle A \rangle = M$. We also say that M is *generated* by A. In this case, we say that A is *minimal* if none of its proper subsets generates M.

Definition 1.7. [6] A submonoid M of $\mathbb{N}$ is *finitely generated* if there exists a system of generators of M with finitely many elements.

Given $A, B \subseteq \mathbb{N}$, we define $A + B = \{a + b \mid a \in A, b \in B\}$. For a subset S of $\mathbb{N}$, we denote $S^* = S \backslash \{0\}$.

Proposition 1.8. [6] *Let S be a submonoid of $\mathbb{N}$. Then $S^* \backslash (S^* + S^*)$ is a system of generators of S. Moreover, every system of generators of S contain $S^* \backslash (S^* + S^*)$.*

Example 1.9. Let $S = \{0, 5, 7, 9, 10, 12, 14, 15, 16, \ldots\}$, we have that $S^* + S^* = \{10, 12, 14, 15, 16, \ldots\}$. Then $S^* \backslash (S^* + S^*) = \{5, 7, 9\}$ and $\gcd(\{5, 7, 9\}) = 1$.

Proposition 1.10. [6] *Every numerical semigroup admits a unique finite minimal system of generators.*

The proof of Proposition 1.8 and Proposition 1.10 can be seen in [6]. In Chapter II, we define an addition operation on $\mathbb{N} \times \mathbb{N}$. We give the definition of numerical semigroups on $\mathbb{N} \times \mathbb{N}$ and system of generators of a submonoid of $\mathbb{N} \times \mathbb{N}$ similarly to numerical semigroups and system of generators of a submonoid of $\mathbb{N}$, respectively. We prove a condition on a subset of $\mathbb{N} \times \mathbb{N}$ which generates a numerical semigroup on $\mathbb{N} \times \mathbb{N}$. Moreover, we provide an algorithm to find the minimal system of generators of numerical semigroups on $\mathbb{N} \times \mathbb{N}$.

Definition 1.11. [3] A numerical semigroup is said to be *irreducible* if it cannot be expressed as the intersection of the numerical semigroups properly containing it.

Example 1.12. Let $S = \{0, 2, 4, 5, 6, \ldots\}$ and $S' = \{0, 4, 5, 6, \ldots\}$. Then S and S' are numerical semigroups. The numerical semigroups containing S are $S_1 = \{0, 2, 3, 4, \ldots\}$ and $S_2 = \mathbb{N}$. Then S is irreducible because $S \neq S_1 \cap S_2$. Let $S_3 = \{0, 3, 4, 5, 6, \ldots\}$. Then S_3 is a numerical semigroup containing S'. Then S' is not irreducible because $S' = S \cap S_3$.

The next proposition shows that irreducible numerical semigroups are maximal in the set of numerical semigroups with fixed Frobenius number.

Proposition 1.13. [6] *Let S be a numerical semigroup. The following statements are equivalent.*

1. *S is irreducible.*
2. *S is maximal among numerical semigroups with Frobenius number $F(S)$.*
3. *S is maximal among numerical semigroups that do not contain $F(S)$.*

Since irreducible numerical semigroups deal with the larger numerical semigroups, a construction of numerical semigroups containing a fixed numerical semigroup is considered.

Definition 1.14. [4] Let S be a numerical semigroup. A gap g of S is a *special gap* if $2g \in S$ and $g + s \in S$ for all $s \in S^*$. The set of all special gaps of S is denoted by $SG(S)$.

Example 1.15. Let $S = \langle 5, 6 \rangle = \{0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20, 21, 22, \ldots\}$ be a numerical semigroup. Then $G(S) = \{1, 2, 3, 4, 7, 8, 9, 13, 14, 19\}$ and $SG(S) = 19$.

Given a numerical semigroup S and $x \in SG(S)$. Following from the definition of special gap, we have that $S \cup \{x\}$ is a numerical semigroup containing S. In [4], the concept of special gap gave a method of finding the set of all numerical semigroups containing a fixed numerical semigroup. We explain this method by the following example.

Example 1.16. Let $S = \{0, 4, 5, 6, \ldots\}$. Then $G(S) = \{1, 2, 3\}$ and $SG(S) = \{2, 3\}$. Then $S_1 = S \cup \{2\} = \{0, 2, 4, 5, 6, \ldots\}$ and $S_2 = S \cup \{3\} = \{0, 3, 4, 5, 6, \ldots\}$. Next, compute $SG(S_1) = \{3\}$ and $SG(S_2) = \{2\}$. Then $S_3 = S_1 \cup \{3\} = S_2 \cup \{2\} = \{0, 2, 3, 4, 5, 6, \ldots\}$. After that compute $SG(S_3) = \{1\}$ so that $S_4 = S_3 \cup \{1\} = \mathbb{N}$. Hence all numerical semigroups containing S are S, S_1, S_2, S_3 and S_4.

The next proposition gives another characterization of irreducible numerical semigroup given by its special gap.

Proposition 1.17. [4] *Let S be a numerical semigroup. Then S is irreducible if and only if $SG(S)$ has at most one element.*

Example 1.18. From Example 1.12, S is irreducible with $SG(S) = \{3\}$ but S' is not irreducible with $SG(S') = \{2, 3\}$.

After we give the definition of numerical semigroup on $\mathbb{N} \times \mathbb{N}$, in Chapter III, we define the special gap of a numerical semigroup on $\mathbb{N} \times \mathbb{N}$ and an irreducible numerical semigroup on $\mathbb{N} \times \mathbb{N}$ similarly to an irreducible numerical semigroup on $\mathbb{N}$. Then we prove that the spacial gap of a numerical semigroup on $\mathbb{N} \times \mathbb{N}$ gives a characterization of irreducible numerical semigroups on $\mathbb{N} \times \mathbb{N}$.

Definition 1.19. [6] A numerical semigroup S is *symmetric* if it is irreducible with odd Frobenius number.

Example 1.20. $S = \{0, 2, 4, 5, 6, \ldots\}$ is a symmetric numerical semigroup because S is irreducible with Frobenius number 3.

The concept of symmetric numerical semigroup can be explained by given two distinct integers a and b having the same distance from $\frac{F(S)}{2}$ on real line. Then only one of a and b belongs to S. The following proposition is a characterization of symmetric numerical semigroups. For the proof see [1] and [6].

Proposition 1.21. [6] *Let S be a numerical semigroup and $\mathbb{Z}$ the set of all integers. Then the following statements are equivalent.*

1. *S is symmetric*

2. *$F(S)$ is odd and for any $x \in \mathbb{Z} \backslash S$, $F(S) - x \in S$.*

3. $|G(S)| = \frac{F(S)+1}{2}$.

When a numerical semigroup is generated by two relatively prime numbers a and b, this is similar to Sylvester's problem. The solution of Sylvester's problem is $ab - a - b$ and the number of positive integers that cannot be written as $n_1a + n_2b$ is $\frac{ab-a-b+1}{2}$. This implies the following proposition.

Proposition 1.22. [6] *Let S be a numerical semigroup generated by $\{a, b\}$. Then S is symmetric.*

Proposition 1.23. [6] *Let S be a numerical semigroup generated by $\{a, b\}$. Then $F(S) = ab - a - b$ and the cardinality of $G(S)$ is $\frac{ab-a-b+1}{2}$.*

In Chapter III, we define symmetric numerical semigroups on $\mathbb{N} \times \mathbb{N}$ and give characterizations of symmetric numerical semigroups on $\mathbb{N} \times \mathbb{N}$. Moreover, we use results from Chapter II to prove some characterization of symmetric numerical semigroups on $\mathbb{N} \times \mathbb{N}$ when system of generators are given.

Given a numerical semigroup S. If S is not irreducible, then we have numerical semigroups $S_1 \supset S$ and $S_2 \supset S$ such that $S = S_1 \cap S_2$. Following from this, one can prove that every numerical semigroup can be expressed as an intersection of irreducible numerical semigroups. But it is not always true that every numerical semigroup can be expressed as an intersection of symmetric numerical semigroups. The condition of those numerical semigroups appeared in [2].

Definition 1.24. [2] Let S be a numerical semigroup. An integer x is called a *pseudo-Frobenius number* if $x \notin S$ and $x + s \in S$ for all $s \in S^*$.

Example 1.25. Let $S = \langle 5, 6 \rangle = \{0, 5, 6, 10, 11, 12, \ldots\}$. Then pseudo-Frobenius number of S are $7, 8$ and 9.

Proposition 1.26. [2] *Let S be a numerical semigroup and $g_1, \ldots, g_i$ be its pseudo-Frobenius numbers. Then S can be written as the intersection of symmetric numerical semigroup if and only if for all even g_i, there exists an odd positive integer y_i such that $g + y_i \notin \langle S \cup \{y_i\} \rangle$.*

In Chapter III, we define the pseudo maximal gap of a numerical semigroup on $\mathbb{N} \times \mathbb{N}$ similarly to the pseudo-Frobenius number. After that we provide conditions of numerical semigroup on $\mathbb{N}\times\mathbb{N}$ that can be expressed as intersection of symmetric numerical semigroups on $\mathbb{N} \times \mathbb{N}$.

In summary, we aim to study numerical semigroups on $\mathbb{N} \times \mathbb{N}$ and investigate their properties analogously to numerical semigroup of $\mathbb{N}$. In Chapter II, we introduce the concept of numerical semigroups on $\mathbb{N} \times \mathbb{N}$ and study conditions of subsets of $\mathbb{N} \times \mathbb{N}$ which generate numerical semigroups on $\mathbb{N} \times \mathbb{N}$. In Chapter III, we investigate basic properties of irreducible and symmetric numerical semigroups on $\mathbb{N} \times \mathbb{N}$ and the most results are analogous to [2], [4], [5] and [6].

CHAPTER II
SYSTEM OF GENERATORS

First, let us introduce an operation on the set $\mathbb{N} \times \mathbb{N}$. For $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, we define an addition operation $+$ on the set $\mathbb{N} \times \mathbb{N}$ by $(a, b) + (c, d) = (a+c, b+d)$. This operation is clearly commutative and associative. We say that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. For a positive integer n, we write $\underbrace{(a, b) + \cdots + (a, b)}_{n \text{ terms}}$ as $n(a, b)$ and denote $0(a, b)$ by $(0, 0)$ so that $n(a, b) = (na, nb)$ for any nonnegative integer n.

Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, we define $(a, b) - (c, d)$ by $(a - c, b - d)$. Note that $(a, b) - (c, d)$ does not need to be an element of $\mathbb{N} \times \mathbb{N}$.

Our main result in this chapter is Proposition 2.24, giving a condition on a nonempty subset of $\mathbb{N}\times\mathbb{N}$ so that it generates a submonoid of $\mathbb{N}\times\mathbb{N}$ under addition $+$ with finite complement. So we define the definition of a system of generators of a submonoid of $\mathbb{N} \times \mathbb{N}$ similarly to the definition of a system of generators of a submonoid of $\mathbb{N}$ in Chapter I.

For the convenience, we denote the set $\mathbb{N} \times \mathbb{N}$ by $\mathbb{N}^2$.

Definition 2.1. Let A be a nonempty subset of $\mathbb{N}^2$. The *smallest submonoid of* $\mathbb{N}^2$ *containing* A is the set $\{n_1a_1 + \cdots + n_ka_k \mid k \in \mathbb{N}^*, a_1, \ldots, a_k \in A, n_1, \ldots, n_k \in \mathbb{N}\}$, denoted by $\langle A \rangle$.

Given $A \subseteq \mathbb{N}^2$ and $(m, n) \in \mathbb{N}^2$, we denote $\langle A \cup \{(m, n)\} \rangle$ by $\langle A, (m, n) \rangle$.

Example 2.2. Let $A = \{(1, 0), (2, 0)\}$. The smallest submonoid of $\mathbb{N}^2$ containing A is $\{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), \ldots\}$.

Definition 2.3. A nonempty subset A of $\mathbb{N}^2$ is a *system of generators* of a submonoid M of $\mathbb{N}^2$ if $\langle A \rangle = M$. We also say that M is *generated* by A. In this case, we say that A is *minimal* if none of its proper subsets generates M.

Example 2.4. From Example 2.2, we have that $\{(1,0),(2,0)\}$ is a system of generators of $\{(0,0),(1,0),(2,0),(3,0),(4,0),\ldots\}$. However, $\{(1,0),(2,0)\}$ is not a minimal system of generators of $\{(0,0),(1,0),(2,0),(3,0),(4,0),\ldots\}$ because $\langle(1,0)\rangle = \{(0,0),(1,0),(2,0),(3,0),(4,0),\ldots\}$.

Definition 2.5. A submonoid M of $\mathbb{N}^2$ is *finitely generated* if there exists a system of generators of M with finitely many elements.

Example 2.6. The submonoid $\{(0,0),(1,0),(2,0),(3,0),(4,0),\ldots\}$ of $\mathbb{N}^2$ is finitely generated because $\langle(1,0)\rangle = \{(0,0),(1,0),(2,0),(3,0),(4,0),\ldots\}$.

Definition 2.7. A subset S of $\mathbb{N}^2$ is a *numerical semigroup* if it contains $(0,0)$, closed under the addition and $\mathbb{N}^2 \backslash S$ is finite.

From now on, the word *numerical semigroup* is referred to a submonoid of $\mathbb{N}^2$ in Definition 2.7. However, the word *numerical semigroup on* $\mathbb{N}$ is referred to a submonoid S of $\mathbb{N}$ under usual addition on $\mathbb{N}$ with $\mathbb{N}\backslash S$ is finite.

Definition 2.8. Let S be a numerical semigroup. An element of $\mathbb{N}^2\backslash S$ is said to be a *gap* of S. We denote the set of all gaps of S by $G(S)$ which is a finite set.

Example 2.9. The set $S = \mathbb{N}^2\backslash\{(1,0),(2,0),(3,0),(4,0)\}$ is a numerical semigroup with $G(S) = \{(1,0),(2,0),(3,0),(4,0)\}$.

To illustrate a submonoid S of $\mathbb{N}^2$ as a diagram in XY-plane, we describe those elements of S and $\mathbb{N}^2\backslash S$ by $\bullet$ and $\circ$, respectively. For any elements that do not appear in the diagram, we assume that they belong to S.

Example 2.10. The numerical semigroup S with $G(S) = \{(0,1),(1,1),(0,2),(1,2)\}$ can be represented by the following diagram.

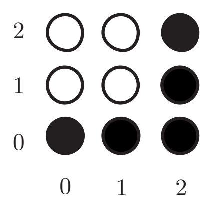

Figure 2.1: The numerical semigroup S

Given subsets $A, B \subseteq \mathbb{N}^2$, we define $A + B = \{a + b \mid a \in A, b \in B\}$. For a subset S of $\mathbb{N}^2$, we denote S^* by $S \backslash \{(0,0)\}$.

Proposition 2.11. *Let S be a submonoid of $\mathbb{N}^2$. Then $S^* \backslash (S^* + S^*)$ is a system of generators of S.*

Proof. Let $(m,n) \in S^*$ be such that $(m,n) \notin S^* \backslash (S^* + S^*)$. Then there exists $(m_1,n_1),(m_2,n_2) \in S^*$ for which $(m_1,n_1) + (m_2,n_2) = (m,n)$. We note that $m_1, m_2 \leq m$ and $n_1, n_2 \leq n$. If $(m_1,n_1) \notin S^* \backslash (S^* + S^*)$, then $(m_3,n_3) + (m_4,n_4) = (m_1,n_1)$ for some $(m_3,n_3),(m_4,n_4) \in S^*$. We apply this process to $(m_2,n_2),(m_3,n_3),(m_4,n_4)$ and those (m_i,n_i)'s occur in this way. This process must stop in finite steps because S is a submonoid of $\mathbb{N}^2$. Hence $(m,n) = (m_{\alpha_1}, n_{\alpha_1}) + \cdots + (m_{\alpha_k}, n_{\alpha_k})$ where $(m_{\alpha_i}, n_{\alpha_i}) \in S^* \backslash (S^* + S^*)$. This shows that $S^* \backslash (S^* + S^*)$ is a system of generators of S. $\square$

Since every numerical semigroup is a submonoid of $\mathbb{N}^2$, the following corollary gives us a system of generators of numerical semigroups.

Corollary 2.12. *Let S be a numerical semigroup. Then $S^* \backslash (S^* + S^*)$ is a system of generators of S.*

Proposition 2.13. *Let S be a numerical semigroup. Then $S^* \backslash (S^* + S^*)$ is a minimal system of generators of S.*

Proof. We show that $S^* \backslash (S^* + S^*)$ is a subset of any system of generators of S. Let A be a system of generators of S. If $s \in S^* \backslash (S^* + S^*)$, then $s = \lambda_1 a_1 + \cdots + \lambda_n a_n$ where $n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathbb{N}^*, a_1, \ldots, a_n \in A$. Since $s \notin (S^* + S^*)$, we have that $s = a_i$ for some $1 \leq i \leq n$. Hence $s \in A$ and $S^* \backslash (S^* + S^*)$ is a minimal system of generators of S. $\square$

For the convenience, given $a, b \in \mathbb{N}$, we define the following sets.

$$I_{(a,b)} = \{(m,n) \in \mathbb{N}^2 \mid 0 \leq m \leq a \text{ and } 0 \leq n \leq b\} \text{ and } E_{(a,b)} = \mathbb{N}^2 \backslash I_{(a,b)}.$$

To show that $E_{(a,b)} \cup \{(0,0)\}$ is a numerical semigroup, let $(x_1,y_1),(x_2,y_2) \in E_{(a,b)}$. If $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq a$, then $y_1 > b$ and $y_2 > b$. Hence $(x_1,y_1) + (x_2,y_2) \in E_{(a,b)}$ because $y_1 + y_2 > b$. If $0 \leq x_1 \leq a$ but $x_2 > a$, then $(x_1,y_1) + (x_2,y_2) \in E_{(a,b)}$

because $x_1+x_2 > a$. For the last case, if $x_1 > a$ and $x_2 > a$, then $(x_1, y_1)+(x_2, y_2) \in E_{(a,b)}$ because $x_1+x_2 > a$. It follows that $E_{(a,b)} \cup \{(0,0)\}$ is a numerical semigroup.

Example 2.14. Let S be a numerical semigroup with $G(S) = \{(1,0),(0,1),(2,1)\}$. The set $S^* + S^*$ can be shown by the following diagram and $S^* \backslash (S^* + S^*) = \{(2,0),(3,0),(1,1),(0,2),(1,2),(0,3)\}$.

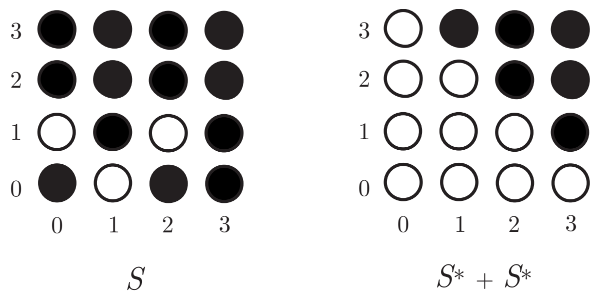

Figure 2.2: The sets S and $S^* + S^*$, respectively

Proposition 2.15. *Every numerical semigroup is finitely generated.*

Proof. Let S be a numerical semigroup. It suffices to show that $S^* \backslash (S^* + S^*)$ is finite. Since $G(S)$ is finite, there exists $(a,b) \in \mathbb{N}^2$ such that $G(S) \subseteq I_{(a,b)}$. Set

$$A = \{(m,n) \in \mathbb{N}^2 \mid 2a+2 \leq m\} \cup \{(m,n) \in \mathbb{N}^2 \mid 2b+2 \leq n\}.$$

Since $A^c = \{(m,n) \in \mathbb{N}^2 \mid 0 \leq m < 2a+2 \text{ and } 0 \leq n < 2b+2\}$ is finite, if we can show that $A \subseteq (S^* + S^*)$, then $S^* \backslash (S^* + S^*)$ is a finite set. Let $(m,n) \in A$. If $2a+2 \leq m$, then $(m,n) = (a+1,0) + (a+k,n)$ for some positive integer k. If $2b+2 \leq n$, then $(m,n) = (0,b+1) + (m,b+k)$ for some positive integer k. Then $(m,n) \in S^* + S^*$ and hence $S^* \backslash (S^* + S^*)$ is a finite set. □

Remark 2.16. It is not always true that any submonoids of $\mathbb{N}^2$ are finitely generated. The example is $S = \{(m,n) \in \mathbb{N}^2 \mid m \neq 0\} \cup \{(0,0)\}$. Then $\{(1,n) \in \mathbb{N}^2 \mid 0 \leq n\} \subseteq S^* \backslash (S^* + S^*)$ becomes an infinite set.

Proppostion 2.15 points that there is an opportunity to find conditions on a finite subset of $\mathbb{N}^2$ so that it generates a numerical semigroup. Next, we give the definitions of the gcd of element of $\mathbb{N}^2$ similarly to the greatest common divisor of positive integers so that we give those conditions via the gcd of element of $\mathbb{N}^2$.

Definition 2.17. Let $(a, b), (c, d) \in \mathbb{N}^2$. We say that (a, b) *divides* (c, d), denoted by $(a, b)|(c, d)$, if there exists a positive integer k such that $k(a, b) = (c, d)$.

Definition 2.18. Let $(a_1, b_1), \ldots, (a_n, b_n) \in \mathbb{N}^2$. An element (x, y) of $\mathbb{N}^2$ is the *gcd* of all of the (a_i, b_i)'s if $(x, y)|(a_i, b_i)$ for all $1 \leq i \leq n$ and any other $(x', y') \in \mathbb{N}^2$ that divides all the (a_i, b_i)'s also divides (x, y). We write $\gcd((a_1, b_1), \ldots, (a_n, b_n)) = \gcd(\{(a_1, b_1), \ldots, (a_n, b_n)\}) = (x, y)$.

Example 2.19. $\gcd((5, 5), (7, 7)) = (1, 1)$ and $\gcd((2, 0), (0, 2))$ does not exist.

Proposition 2.20. *Let $a_1, \ldots, a_n$ be positive integers. Then the following statements hold.*

1. $\gcd(a_1, \ldots, a_n) = d$ *if and only if* $\gcd((a_1, 0), \ldots, (a_n, 0)) = (d, 0)$.
2. $\gcd(a_1, \ldots, a_n) = d$ *if and only if* $\gcd((0, a_1), \ldots, (0, a_n)) = (0, d)$.

Proof. For the first statement, assume that $\gcd(a_1, \ldots, a_n) = d$. Then $d|a_i$ for all i. Hence $(d, 0)|(a_i, 0)$ for all i. Let $\gcd((a_1, 0), \ldots, (a_n, 0)) = (k, 0)$. Then $(d, 0)|(k, 0)$ and $k|a_i$ for all i. This implies that $d|k$ and $k|d$ so that $d = k$. For the converse, assume that $\gcd((a_1, 0), \ldots, (a_n, 0)) = (d, 0)$. Then $d|a_i$ for all i. If $\gcd(a_1, \ldots, a_n) = k$, then $d|k$ and $(k, 0)|(a_i, 0)$ for all i which implies that $d|k$ and $k|d$ so that $d = k$.

Proving the second statement is similar to the proof of the first statement. □

Proposition 2.21. *Let A be a system of generators of a numerical semigroup. The following statements hold.*

1. *There exists a finite subset A_1 of A with* $\gcd(A_1) = (1, 0)$.
2. *There exists a finite subset A_2 of A with* $\gcd(A_2) = (0, 1)$.

Proof. For the first statement, we note that for any $(a, b), (c, d) \in \mathbb{N}^2$, $(a, b) + (c, d) = (a + c, 0)$ if and only if $b = d = 0$. This implies that $(x, 0) \in A$ for some $x \in \mathbb{N}^*$; otherwise, $\mathbb{N}^2 \setminus \langle A \rangle$ has infinitely many elements. Then there exists a finite subset $A' = \{(x, 0)\}$ of A such that $\gcd(A') = (x, 0)$. Let d be the smallest positive integer such that there exists a finite subset $A_1 = \{(a_1, 0), \ldots, (a_n, 0)\}$ of A with $\gcd(A_1) = (d, 0)$. For any $(m, 0) \in A$, we have that $(d, 0)|(m, 0))$. If not, there exist $A_2 = A_1 \cup \{(m, 0)\}$ and $d_1 < d$ such that $\gcd(A_2) = (d_1, 0)$ which yields a contradiction. Hence $(d, 0)|(x, 0)$ for all $(x, 0) \in \langle A\rangle$. Since A generates a numerical semigroup, there exists $(a, 0), (a + 1, 0) \in \langle A\rangle$. Then $d = 1$ because $(d, 0)|(a, 0)$, $(d, 0)|(a + 1, 0)$ and $\gcd((a, 0), (a + 1, 0)) = (1, 0)$.

Proving the second statement is similar to the proof of the first statement. □

Remark 2.22. When a subset A of $\mathbb{N}^2$ holds the condition(1) of the Proposition 2.21, there exists $(x, 0) \in \langle A\rangle$ such that $(x + n, 0) \in \langle A\rangle$ for all $n \in \mathbb{N}$. If A holds the condition(2) of the Proposition 2.21, then there exists $(0, y) \in \langle A\rangle$ such that $(0, y + n) \in \langle A\rangle$ for all $n \in \mathbb{N}$.

Proposition 2.23. *Let $A \subseteq \mathbb{N}^2$ be a system of generators of a numerical semigroup. Then $(1, k), (l, 1) \in A$ for some $k, l \in \mathbb{N}$.*

Proof. Let S be the numerical semigroup generated by A. Since $G(S)$ is a finite set, we have that $(1, y) \in S$ for some $y \in \mathbb{N}$. Then there exists $n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathbb{N}, (a_1, b_1), \ldots, (a_n, b_n) \in A$ such that $\lambda_1(a_1, b_1) + \cdots + \lambda_n(a_n, b_n) = (1, y)$ because A is a system of generators of S. Therefore $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 1$. This shows that $\lambda_i = a_i = 1$ for some $1 \leq i \leq n$. So we have that $(1, k) \in A$. Proving that $(l, 1) \in A$ is similar. □

Proposition 2.24. *A nonempty subset A of $\mathbb{N}^2$ is a system of generators of a numerical semigroup if and only if the following statements hold.*

1. *There exists a finite subset A_1 of A with $\gcd(A_1) = (1, 0)$.*
2. *There exists a finite subset A_2 of A with $\gcd(A_2) = (0, 1)$.*
3. *$(1, k), (l, 1) \in A$ for some $k, l \in \mathbb{N}$.*

Proof. Following Proposition 2.21 and Proposition 2.23, we have that A satisfies those conditions.

For the converse, assume that A satisfies those conditions. By Remark 2.22, let u, v be the smallest positive integers such that $(u + n, 0)$ and $(0, v + n)$ belong to $\langle A\rangle$ for all $n \in \mathbb{N}$. Set $w = \max\{u, v\}$. From the condition (3), we have that $(w, wk), (wl, w) \in \langle A\rangle$ and we may assume that $wl \leq wk$. Observe that

$\{(x,y) \in \mathbb{N}^2 \mid w \leq x \text{ and } w \leq y\} \subseteq \langle A \rangle$. Let $A_1 = \{(x,y) \in \mathbb{N}^2 \mid 0 \leq x < w \text{ and } 2wk \leq y\}$ and $A_2 = \{(x,y) \in \mathbb{N}^2 \mid 2wk \leq x \text{ and } 0 \leq y < w\}$. If we can show that $A_1, A_2 \subseteq \langle A \rangle$, then A is a system of generators of a numerical semigroup.

If $(x,y) \in A_1$, then $y = 2wk + a$ where $a \in \mathbb{N}$. Since $xk < wk$, we have that $(x,y) = (x, 2wk+a) = (x, xk) + (0, wk + a + wk - xk) \in \langle A \rangle$.

If $(x,y) \in A_2$, then $x = 2wk + b$ where $b \in \mathbb{N}$. Since $yl < wl \leq wk$, we have that $(x,y) = (2wk+b, y) = (yl, y) + (wk + b + wk - yl, 0) \in \langle A \rangle$.

Therefore $A_1, A_2 \subseteq \langle A \rangle$ and $\langle A \rangle$ is a numerical semigroup. □

Definition 2.25. Let n be a positive integer. A numerical semigroup S is $n-$ *dimensional* if its minimal system of generators contains exactly n elements.

Proposition 2.26. *A 2-dimensional numerical semigroup is unique.*

Proof. Let A be a minimal system of generators of 2-dimensional numerical semigroup. From Proposition 2.24, we have that $(1,k), (l,1) \in A$ for some $k, l \in \mathbb{N}$. If $k = l = 0$, then $A = \{(1,0), (0,1)\}$ which generates $\mathbb{N}^2$. If $k \neq 0$ and $l \neq 0$, then there exists finite subsets $\{(a_1, 0), \ldots, (a_n, 0)\} = A_1$ and $\{(0, b_1), \ldots, (0, b_m)\} = A_2$ of A such that $\gcd(A_1) = (1,0)$ and $\gcd(A_2) = (0,1)$. This contradicts the minimality of A. If $k \neq 0$ and $l = 0$, then there exists a finite subset $\{(a_1,0), \ldots, (a_n,0)\} = A_1 \subseteq A$ such that $\gcd(A_1) = (1,0)$. This leads to the same contradiction. For the case $k = 0$ and $l \neq 0$, the proof is similar to the case $k \neq 0$ and $l = 0$. Hence a 2-dimensional numerical semigroup is unique. □

Proposition 2.27. *There are no 3-dimensional numerical semigroups.*

Proof. Follow directly from the proof of Proposition 2.26. □

Proposition 2.28. *A minimal system of generators of a 4-dimensional numerical semigroup satisfies one of the following statements.*

1. $\{(0,1), (1,k), (a_1,0), (a_2,0)\}$ *where* $1 < a_1 < a_2$, $\gcd(a_1, a_2) = 1$ *and* $k \in \mathbb{N}^*$.

2. $\{(1,0), (l,1), (0,b_1), (0,b_2)\}$ *where* $1 < b_1 < b_2$, $\gcd(b_1, b_2) = 1$ *and* $l \in \mathbb{N}^*$.

Proof. Let A be a minimal system of generators of a 4-dimensional numerical semigroup. From the proof of Proposition 2.26, we consider the case $k \neq 0$ and

$l = 0$. There exists a finite subset $\{(a_1, 0), \ldots, (a_n, 0)\} = A_1 \subseteq A$ such that $\gcd(A_1) = (1, 0)$. Hence $n = 2$ so that $A = \{(0, 1), (1, k), (a_1, 0), (a_2, 0)\}$.

For the case $k = 0$ and $l \neq 0$, the proof is similar to the first case so that $A = \{(1, 0), (l, 1), (0, b_1), (0, b_2)\}$. □

For $n \geq 4$, n-dimensional numerical semigroups always exist. Consider the set $\{(0, 1), (1, k), (n, 0), \ldots, (2n-3, 0)\}$. Since there are no elements $(n+i, 0)$ such that $(n+i, 0) = a_0(n, 0) + \cdots + a_{i-1}(n+i-1, 0) + a_{i+1}(n+i+1, 0) + \cdots + a_{n-3}(2n-3, 0)$ where $a_i \in \mathbb{N}$ for all $0 \leq i \leq n-3$. Hence this set always be a minimal system of generators of an n-dimensional numerical semigroup.

Proposition 2.29. *Let S be a numerical semigroup and a, b positive integers such that $G(S) = I_{(a-1,b-1)} \backslash \{(0, 0)\}$. Let $A_1 = \{(x, y) \mid a \leq x \leq 2a-1$ and $0 \leq y \leq b-1\}$ and $A_2 = \{(x, y) \mid 0 \leq x \leq a-1$ and $b \leq y \leq 2b-1\}$. Then the minimal system of generators of S is $A_1 \cup A_2$. Moreover, its minimal system of generators has $2ab$ elements.*

Proof. To show that $A_1 \cup A_2$ is a minimal system of generators of S, we show that $A_1 \cup A_2 = S^* \backslash (S^* + S^*)$. Let $(x, y) \in A_1$. Suppose that there exists $(x_1, y_1), (x_2, y_2) \in S^*$ such that $(x_1, y_1) + (x_2, y_2) = (x, y)$. Since $x \leq 2a - 1$, we have that $x_i < a$ for some i. By this i, we have that $b \leq y_i$. This contradits $b \leq y_1 + y_2 = y \leq b - 1$. Then $(x, y) \in S^* \backslash (S^* + S^*)$. When $(x, y) \in A_2$, the proof is similar.

For the reverse inclusion, let $(x, y) \in \mathbb{N}^2$ such that $(x, y) \notin A_1 \cup A_2$. If $x < a$, then $2b \leq y$ and $(x, y) = (x, b) + (0, b + k)$ where $k \in \mathbb{N}$. If $a \leq x < 2a$, then $b \leq y$ and $(x, y) = (x, y - b) + (0, b)$. If $2a \leq x$, then $(x, y) = (a, 0) + (a + k, y)$ where $k \in \mathbb{N}$. Therefore, $(x, y) \notin S^* \backslash (S^* + S^*)$. Since A_1 and A_2 are disjoint, its minimal system of generators has $2ab$ elements. □

Remark 2.30. For a numerical semigroup S with $G(S) \subseteq I_{(a-1,b-1)}$, we set

- $A_1 = \{(x, y) \mid a \leq x \leq 2a-1$ and $0 \leq y \leq b-1\}$;
- $A_2 = \{(x, y) \mid 0 \leq x \leq a-1$ and $b \leq y \leq 2b-1\}$.

From Proposition 2.29, it is not hard to see that $E_{(a-1,b-1)} \subset \langle A_1 \cup A_2 \rangle$. Then the set $(S \cap I_{(a-1,b-1)}) \cup (A_1 \cup A_2)$ is a system of generators of S which contains at most $3ab$ elements.

We end this chapter with an algorithm for finding the minimal system of generators. Given a numerical semigroup S, the following informations are required.

- Set positive integers a and b so that $G(S) \subseteq I_{(a-1,b-1)}$.
- Set $G = \varnothing$, $A_0 = S \cap I_{(a-1,b-1)}$ and A_1, A_2 from the Remark 2.30.
- Set $R_i = \{(x, y) \in A_0 \cup A_1 \cup A_2 \mid y = i\}$ where $0 \leq i \leq 2b - 1$.
- Define an order $\leq$ on $\mathbb{N}^2$ by $(x_1, y_1) \leq (x_2, y_2)$ if $y_1 = y_2$ and $x_1 \leq x_2$.
- The set of all maximal elements of $X \subseteq \mathbb{N}^2$ respect to the relation $\leq$ is denoted by $\text{Max}_{\leq} X$.

The algorithm is the following steps:

1. Set $i = 2b - 1$.
2. If $(r_1, r_2) - (g_1, g_2) \notin S$ for all $(g_1, g_2) \in A_0 \cup A_1 \cup A_2$ and $(r_1, r_2) \in \text{Max}_{\leq} R_i$, then add (r_1, r_2) to G.
3. Remove (r_1, r_2) from R_i and $A_0 \cup A_1 \cup A_2$.
4. If $R_i \neq \varnothing$, then repeat step 2.
5. If $i \neq 0$, then replace R_i in 2., 3. and 4. by R_{i-1} and repeat step 2.
6. G is a system of generators of S.

Example 2.31. From Example 2.14, we set $(a, b) = (3, 2)$ so that $G(S) \subseteq I_{(2,1)}$.

- $A_0 = \{(0, 0), (2, 0), (1, 1)\}$.
- $A_1 = \{(x, y) \mid 3 \leq x \leq 5 \text{ and } 0 \leq y \leq 1\}$.
- $A_2 = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } 2 \leq y \leq 3\}$.
- $R_0 = \{(0, 0), (2, 0), (3, 0), (4, 0), (5, 0)\}$.

- $R_1 = \{(1,1),(3,1),(4,1),(5,1)\}$.
- $R_2 = \{(0,2),(1,2),(2,2)\}$.
- $R_3 = \{(0,3),(1,3),(2,3)\}$.

Set $i = 3$. Then $\{(2,3)\} = \mathrm{Max}_{\leq} R_3$. Since $(2,3) - (0,3) = (2,0) \in S$, we remove $(2,3)$ from R_3. Since $R_3 = \{(0,3),(1,3)\}$ is not the empty set, we repeat the second step with $\{(1,3)\} = \mathrm{Max}_{\leq} R_3$. Since $(1,3) - (0,2) = (1,1) \in S$, we remove $(1,3)$ from R_3. Since $R_3 = \{(0,3)\}$ is not the empty set, we repeat the second step with $\{(0,3)\} = \mathrm{Max}_{\leq} R_3$. Since $(0,3) - (g_1,g_2) \notin S$ for all $(g_1,g_2) \in A_0 \cup A_1 \cup A_2$, we add $(0,3)$ to G. After removing $(0,3)$ from R_3, we have that R_3 is the empty set. Then we consider R_2. We continue this process and finally get $G = \{(2,0),(3,0),(1,1),(0,2),(1,2),(0,3)\}$.

CHAPTER III
IRREDUCIBLE NUMERICAL SEMIGROUPS

In this chapter, we first introduce the special gap of a numerical semigroup and an irreducible numerical semigroup. The concept of special gaps lead to the extension of a numerical semigroup S to a numerical semigroup S' with $|S' \backslash S| = 1$. Then we study the structure of symmetric numerical semigroups and provide some results of symmetric numerical semigroups. After that we give the characterization on a numerical semigroup so that it can be written as an intersection of symmetric numerical semigroups containing it. Finally, we introduce fundamental gaps. This gives another method for finding all numerical semigroups containing a fixed numerical semigroup.

3.1 Irreducible Numerical Semigroups

In this part, we investigate some results of irreducible numerical semigroups analogusly to [4] and [6]. Our main results are to characterize irreducible numerical semigroups by its maximal gap and its special gap.

Definition 3.1. Given a nonempty finite subset A of $\mathbb{N}^2$. An element (a, b) of A is *maximal* if for any $(x, y) \in A$, $x^2 + y^2 \leq a^2 + b^2$. The set of all maximal elements of A is denoted by $M(A)$. For a numerical semigroup S, the set of all maximal gaps of S is denoted by $MG(S)$.

Definition 3.2. Let S be a numerical semigroup. A gap g of S is a *special gap* if for any $s \in S^*, g + s \in S$ and $2g \in S$. The set of all special gaps of S denoted by $SG(S)$.

The next proposition follows directly from the definition of the special gap of a numerical semigroup.

Proposition 3.3. *Let S be a numerical semigroup and g a gap of S. Then $g \in SG(S)$ if and only if $S \cup \{g\}$ is a numerical semigroup.*

To compute all numerical semigroups containing a fixed numerical semigroup, let S be a numerical semigroup. Set $O(S) = \{S\}$. By Proposition 3.3, we have that $S \cup \{g_1\}, \ldots, S \cup \{g_r\}$ are numerical semigroups where $\{g_1, \ldots, g_r\} = SG(S)$. Then add those numerical semigroups $S \cup \{g_i\}$ to $O(S)$ for all $1 \leq i \leq r$. For each $S_i \in O(S)$, we apply Proposition 3.3 and add them to the set $O(S)$ until $\mathbb{N}^2 \in O(S)$.

Proposition 3.4. *Let S be a numerical semigroup. Then $MG(S) \subseteq SG(S)$.*

Proof. Let $(a, b) \in MG(S)$. Since $a^2 + b^2 < (2a)^2 + (2b)^2, (a, b) + (a, b) \in S$. Next, suppose that there exists $(x, y) \in S^*$ such that $(x, y) + (a, b) \in G(S)$. Hence $x^2 + y^2 < (x + a)^2 + (y + b)^2$ and it contradicts the property of $MG(S)$. Therefore $MG(S) \subseteq SG(S)$. □

Proposition 3.5. *Let S and T be numerical semigroups such that $S \subseteq T$. Then $S \cup M(T \backslash S)$ is a numerical semigroup. Moreover, $M(T \backslash S) \subseteq SG(S)$.*

Proof. Let $(a, b) \in M(T \backslash S)$. Suppose that there exists $(x, y) \in S^*$ such that $(a + x, b + y) \in G(S)$. Then $(a + x, b + y) \in T \backslash S$ by $S \subset T$. Since $a^2 + b^2 < (a+x)^2+(b+y)^2$, it contradicts the property of $M(T \backslash S)$. Hence for any $(x, y) \in S^*$, $(a + x, b + y) \in S$. Since $a^2 + b^2 < (2a)^2 + (2b)^2$, we have that $(a, b) + (a, b) \in S$.

For any $(a, b), (c, d) \in M(T \backslash S)$, we have that $a^2 + b^2 < (a + c)^2 + (b + d)^2$. The property of $M(T \backslash S)$ forces that $(a + c, b + d)$ must be an element of S. Now, we can conclude that $S \cup M(T \backslash S)$ is a numerical semigroup. □

Definition 3.6. A numerical semigroup is *irreducible* if it cannot be expressed as ab intersection of two numerical semigroups properly containing it.

Remark 3.7. From Proposition 3.4, any numerical semigroups having more than one maximal gap are not irreducible.

Example 3.8. To check that a numerical semigroup S with $G(S) = \{(0, 1), (1, 1), (0, 2), (1, 2), (2, 2)\}$ is irreducible, we compute $SG(S) = \{(2, 2)\}$ and set $S_1 = S \cup \{(2, 2)\}$. Next, compute $SG(S_1) = \{(1, 1), (1, 2)\}$ then set $S_2 = S_1 \cup \{(1, 2)\}$

and $S_3 = S_1 \cup \{(1,1)\}$. Repeat this process until it return to $\mathbb{N}^2$. The proper numerical semigroups containing S can be shown by the diagram in Figure 3.1. It is easy to check that $S \neq S_i \cap S_j$ for all $i, j \in \{1,2,3,4,5,6\}$.

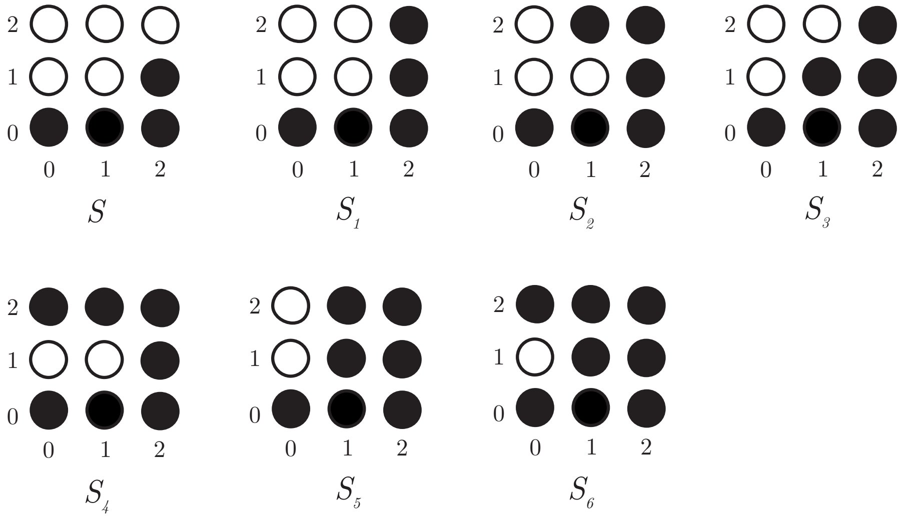

Figure 3.1: Numerical semigroups containing S

When we say that a numerical semigroup is maximal, we mean it is maximal in the sense of subset.

Proposition 3.9. *Let S be a numerical semigroup such that $MG(S)$ has one element. The following statements are equivalent.*

1. *S is irreducible.*

2. *S is maximal among numerical semigroups T with $MG(T) = MG(S)$.*

3. *S is maximal among numerical semigroups T with $T \cap MG(S) = \varnothing$.*

Proof. (1)→(2) Assume that S is irreducible. Let T be a numerical semigroup such that $S \subseteq T$ and $MG(T) = MG(S)$. Note that $S = S \cup \varnothing = (S \cap T) \cup (MG(T) \cap T) = (S \cup MG(T)) \cap T = (S \cup MG(S)) \cap T$. Since $S \cup MG(S)$ is a numerical semigroup properly containing S, it forces that $S = T$ because S is irreducible. Then S is maximal.

(2)→(3) Assume that S is maximal among numerical semigroups T such that $MG(T) = MG(S)$. Let T be a numerical semigroup such that $S \subseteq T$ and $T \cap$

$MG(S) = \varnothing$. Since $MG(S)$ has exactly one element, assume that $MG(S) = \{(a,b)\}$. We have that $T \cup E_{(a,b)}$ is a numerical semigroup and $MG(T \cup E_{(a,b)}) = \{(a,b)\} = MG(S)$. Hence $T \cup E_{(a,b)} \subseteq S$ by the hypothesis. Then $S = T$.

(3)→(1) Assume that S is maximal among numerical semigroup T such that $T \cap MG(S) = \varnothing$. Let S_1 and S_2 be numerical semigroups such that $S \subset S_1$ and $S \subset S_2$. Then $S_1 \cap MG(S)$ and $S_2 \cap MG(S)$ are nonempty sets by the hypothesis. Since $MG(S)$ has one element, we have that $MG(S) \subseteq S_1 \cap S_2$. It follows that S is irreducible. □

Example 3.10. Given $(a,b) \in \mathbb{N}^2$. Irreducible numerical semigroups with the maximal gap (a,b) may not be unique. The following diagram is an example of irreducible numerical semigroups with the maximal gap $(2,2)$.

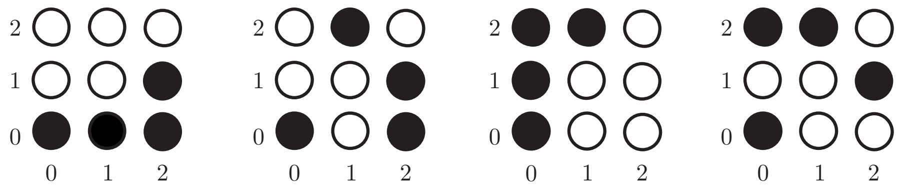

Figure 3.2: Irreducible numerical semigroups with maximal gap $(2,2)$

Proposition 3.11. *Let S be a numerical semigroup such that $\{g_1,\ldots,g_n\} \subseteq G(S)$. Then S is maximal among numerical semigroups T with $T \cap \{g_1,\ldots,g_n\} = \varnothing$ if and only if $SG(S) \subseteq \{g_1,\ldots,g_n\}$.*

Proof. Assume that S is maximal among numerical semigroups T with $T \cap \{g_1,\ldots,g_n\} = \varnothing$. Let $x \in SG(S)$. Then $S \cup \{x\}$ is a numerical semigroup containing S. By the hypothesis, we have that $(S \cup \{x\}) \cap \{g_1,\ldots,g_n\}$ is not empty. This shows that $x \in \{g_1,\ldots,g_n\}$ and hence $SG(S) \subseteq \{g_1,\ldots,g_n\}$.

For the converse, assume that $SG(S) \subseteq \{g_1,\ldots,g_n\}$. Let T be a numerical semigroup such that $S \subset T$. By Proposition 3.5, we have that $M(T \backslash S) \subseteq SG(S) \subseteq \{g_1,\ldots,g_n\}$. Hence $T \cap \{g_1,\ldots,g_n\} \neq \varnothing$ and then S is maximal among numerical semigroups T with $T \cap \{g_1,\ldots,g_n\} = \varnothing$ as desired. □

Proposition 3.12. *A numerical semigroup S is irreducible if and only if $SG(S)$ has at most one element.*

Proof. It is easy to see that $\mathbb{N}^2$ is irreducible. Let $S \neq \mathbb{N}^2$ be a numerical semigroup. Assume that S is irreducible. Suppose that $SG(S)$ has more than one element. Let $g_1, g_2 \in SG(S)$ be distinct elements. Then $(S \cup \{g_1\}) \cap (S \cup \{g_2\}) = S$. This contradicts S is irreducible. Hence $SG(S)$ has exactly one element.

For the converse, assume that $SG(S)$ has one element. Since $MG(S) \subseteq SG(S)$, we have $MG(S) = SG(S)$. By Proposition 3.11, S is maximal among numerical semigroups T with $T \cap MG(S) = \varnothing$. Since $MG(S)$ has exactly one element, S is irreducible by Proposition 3.9. □

The next proposition gives us the process for constructing a numerical semigroup when gaps are fixed. This process can be applied for constructing an irreducible numerical semigroup too.

Proposition 3.13. *Let S be a numerical semigroup such that $\{g_1, \ldots, g_n\} \subseteq G(S)$ and $H = \{g \in G(S) \mid g + g \in S \text{ and } g_i - g \notin S \text{ for all } i\}$. If H is a nonempty set, then $S \cup \{h\}$ is a numerical semigroup not intersecting $\{g_1, \ldots, g_n\}$ for all $h \in M(H)$.*

Proof. We show that $M(H) \subseteq SG(S)$. If $h = (h_1, h_2) \in M(H)$, then $(2h_1, 2h_2) \in S$. Suppose that there exists $(x, y) \in S^*$ such that $(h_1, h_2) + (x, y) \notin S$. Since S is close, we have $(2(h_1 + x, h_2 + y)) = (2h_1, 2h_2) + (2x, 2y) \in S$. Therefore $g_i - (h_1 + x, h_2 + y) \in S$ for some i by $(h_1, h_2) \in M(H)$. It follows that $g_i - (h_1, h_2) \in S$ which yields a contradiction. Hence $M(H) \subseteq SG(S)$ and then $S \cup \{h\}$ is a numerical semigroup. □

Proposition 3.14. *Let S be a numerical semigroup with $\{g_1, \ldots, g_n\} \subseteq G(S)$. Then S is maximal among numerical semigroups T with $T \cap \{g_1, \ldots, g_n\} = \varnothing$ if and only if for any $g \in G(S)$, $g + g \in S$ implies $g_i - g \in S$ for some $1 \leq i \leq n$.*

Proof. Let S be a numerical semigroup be such that S is maximal among numerical semigroups T with $T \cap \{g_1, \ldots, g_n\} = \varnothing$. Let $g \in G(S)$ be such that $g + g \in S$. If $g_i - g \notin S$ for all i, then the set H in Proposition 3.13 is not empty. Hence $S \cup \{h\}$ is a numerical semigroup not intersecting $\{g_1, \ldots, g_n\}$ where $h \in M(H)$. It contradicts the maximality of S. Then $g_i - g \in S$ for some i.

For the converse, assume that for any $g \in G(S)$, $g+g \in S$ implies $g_i - g \in S$ for some $1 \leq i \leq n$. Let T be a numerical semigroup such that $S \subset T$ and $g \in M(T \backslash S)$. Hence $g+g \in S$ by Proposition 3.5. Following from assumption, we have that $g_i - g \in S$ for some i. Since $S \subset T$ and $g \in T$, we have that $g_i \in T$. This shows that S is a maximal numerical semigroup not intersecting $\{g_1, \ldots, g_n\}$. □

Corollary 3.15. *Let S be a numerical semigroup with the maximal gap g'. Then S is irreducible if and only if for any $g \in G(S)$, $g+g \in S$ implies $g' - g \in S$.*

Proof. This follows from Proposition 3.9 and Proposition 3.14. □

To construct a maximal numerical semigroup with gaps $g_1, \ldots, g_m$ containing a fixed numerical semigroup, we can use Proposition 3.13. Firstly, set $S_0 = S \cup \langle E_{g_1} \cap \ldots \cap E_{g_m} \rangle$ where S is a fixed numerical semigroup.

It is not hard to see that for any $(a,b) \in S$ and $(c,d) \in \langle E_{g_1} \cap \ldots \cap E_{g_m} \rangle$, $(a,b) + (c,d) \in \langle E_{g_1} \cap \ldots \cap E_{g_m} \rangle$, $2(a,b) \in S$ and $2(c,d) \in \langle E_{g_1} \cap \ldots \cap E_{g_m} \rangle$. Hence S_0 is a numerical semigroup.

For $n \in \mathbb{N}$, construct $S_{n+1} = S_n \cup \{h(S_n)\}$ where $h(S_n) \in M(H_n)$ and $H_n = \{g \in G(S_n) \mid g+g \in S \text{ and } g_i - g \notin S \text{ for all } 1 \leq i \leq m\}$. Since $G(S_0)$ is a finite set, this process must stop when H_n is the empty set. The last numerical semigroup occured by this process is a maximal numerical semigroup with gaps $g_1, \ldots, g_m$. This process can be illustrated by the following example.

Example 3.16. Let $S = E_{(3,3)} \cup \{(0,0),(1,2),(3,3)\}$ be a numerical semigroup. To construct a maximal numerical semigroup S' with $\{(3,2),(1,3),(2,3)\} \subseteq G(S')$ and $S \subseteq S'$, we set $S_0 = S \cup \left\langle E_{(3,2)} \cap E_{(1,3)} \cap E_{(2,3)} \right\rangle = E_{(3,3)} \cup \{(0,0),(1,2),(3,3)\}$ and compute the set H_n.

- $H_0 = \{(3,0),(2,1),(3,1),(0,2),(2,2),(0,3)\}$; $S_1 = S_0 \cup \{(3,1)\}$
- $H_1 = \{(3,0),(2,1),(0,2),(2,2),(0,3)\}$; $S_2 = S_1 \cup \{(3,0)\}$
- $H_2 = \{(2,1),(2,2),(0,3)\}$; $S_3 = S_2 \cup \{(0,3)\}$
- $H_3 = \{(2,1),(2,2)\}$; $S_4 = S_3 \cup \{(2,2)\}$
- $H_4 = \{(2,1)\}$; $S_5 = S_4 \cup \{(2,1)\}$
- $H_5 = \varnothing$ and S_5 is a maximal numerical semigroup as desired.

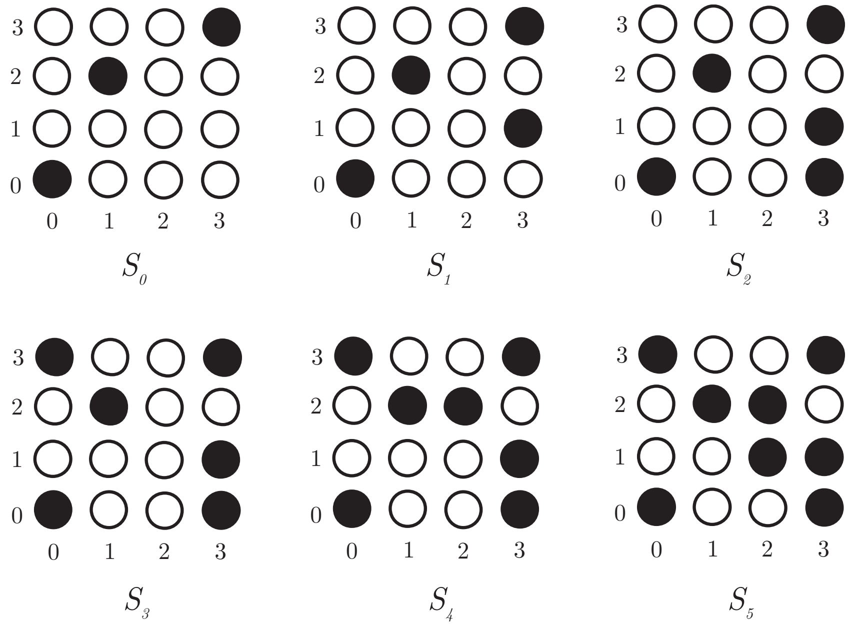

Figure 3.3: A process for constructing a maximal numerical semigroup

3.2 Symmetric Numerical Semigroups

In this part, we investigate a special kind of irreducible numerical semigroups, called symmetric numerical semigroups. We provide some results analogusly to [1], [2] and [6].

Definition 3.17. An element (a, b) of $\mathbb{N}^2$ is *odd(even)* if at least one of a, b is odd(both of a and b are even).

Proposition 3.18. *Let S be an irreducible numerical semigroup. Then there exists a gap (a, b) of S such that $G(S) \subseteq I_{(a,b)}$.*

Proof. Let $(a, b) \in MG(S)$. Suppose that there exists $(x, y) \in G(S)$ but $(x, y) \notin I_{(a,b)}$. If $a < x$, then consider the numerical semigroup $S' = \langle S, (x, y)\rangle$. Hence $(a, b) \in S'$ by Proposition 3.9. There exists $(x_i, y_i) \in S$ and $n \in \mathbb{N}$ such that $(x_1, y_1) + \cdots + (x_n, y_n) + (x, y) = (a, b)$. Since $a < x$, it follows that $(x_1, y_1) + \cdots + (x_n, y_n) = (a, b)$ and it is a contradiction. Hence $G(S) \subseteq I_{(a,b)}$. For the case $x < a$, we have that $b < y$ and the proof is similar. $\square$

Proposition 3.19. *Let S be a numerical semigroup with odd gap g'. If $H = \{g \in G(S) \mid g' - g \notin S\}$ is not the empty set and $h \in M(H)$, then $S \cup \{h\}$ is a numerical* *semigroup not containing g'.*

Proof. Assume that $H\{g \in G(S) \mid g'-g \notin S\}$ is a nonempty set and $h = (h_1, h_2) \in M(H)$. We show that $h \in SG(S)$. Suppose that there exists $s = (s_1, s_2) \in S^*$ such that $h + s \in G(S)$. Since $h_1^2 + h_2^2 < (h_1 + s_1)^2 + (h_2 + s_2)^2$, $g' - (h + s) \in S$. Since $s \in S$, it follows that $g' - h \in S$ and it is a contradiction. Next, suppose that $h + h \in G(S)$. We have that $g' - (h + h) \in S$ by $h_1^2 + h_2^2 < (h_1 + h_1)^2 + (h_2 + h_2)^2$. Following from the fact that g' is odd, $g' - (h + h) = s$ for some $s \in S^*$. Since $g' - h = h + s \in S$, it is a contradiction. So $S \cup \{h\}$ is a numerical semigroup not containing g'. □

Proposition 3.19 may not be true when odd gap is removed. Let S be an irreducible numerical semigroup given by the following diagram. Note that the maximal gap of S is $(4, 2)$ which is even. Compute the set H as in the Proposition 3.19, we have that $H = \{(2, 1)\}$ but $S \cup \{(2, 1)\}$ is not a numerical semigroup because $(4, 2) \notin S$.

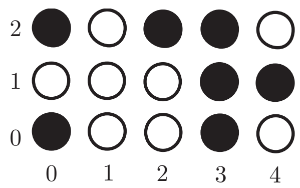

Figure 3.4: Numerical semigroup S

Definition 3.20. A numerical semigroup is *symmetric* if it is irreducible with the odd maximal gap.

Proposition 3.21. *Let S be a numerical semigroup with the maximal gap g'. Then S is symmetric if and only if for any gap g of S, $g' - g \in S$.*

Proof. Assume that S is symmetric. Then g' is odd. If $g \in G(S)$ and $g' - g \notin S$, then the set H of Proposition 3.19 is not the empty set. For $h \in M(H)$, we have that $S \cup \{h\}$ is a numerical semigroup with odd maximal gap g'. This contradicts Proposition 3.9. Hence $g' - g \in S$.

For the converse, assume that $g' - g \in S$ for all $g \in G(S)$. If $g' = (g_1', g_2')$ is even, then $(\frac{g_1'}{2}, \frac{g_2'}{2}) \in G(S)$. This contradicts the hypothesis. Hence g' is odd.

Next, let T be a numerical semigroup such that $S \subset T$. For any $g \in T \backslash S$, $g' - g \in S \subset T$. Hence $g' \in T$. This implies that S is a maximal numerical semigroup with $MG(S) = \{g'\}$. So S is irreducible. $\square$

Proposition 3.22. *Let S be a numerical semigroup with odd maximal gap (a, b). Then S is symmetric if and only if $G(S)$ contains exactly $\frac{(a+1)(b+1)}{2}$ elements.*

Proof. Assume that S is symmetric. Then $G(S) \subseteq I_{(a,b)}$ and hence the cardinality of $G(S)$ is bounded by $(a+1)(b+1)$. We note the fact that for any $(x, y) \in S$, $(a, b)-(x, y) \notin S$. From this fact, we have that $G(S)$ has at least $\frac{(a+1)(b+1)}{2}$ elements. Since S is symmetric, we follow from Proposition 3.21 so that $(a, b)-(g_1, g_2) \in I_{(a,b)}$ for any gaps (g_1, g_2) of S. Hence $G(S)$ has exactly $\frac{(a+1)(b+1)}{2}$ elements.

For the converse, assume that $G(S)$ contains exactly $\frac{(a+1)(b+1)}{2}$ elements. If $(x, y) \in I_{(a,b)} \cap S$, then $(a, b) - (x, y) \in G(S)$ by the above fact. Moreover, $(a, b) - (x, y) \in I_{(a,b)}$. Hence $(x, y) \in I_{(a,b)} \cap S$ implies $(a, b) - (x, y) \in I_{(a,b)} \cap G(S)$. This shows that the set $I_{(a,b)} \cap G(S)$ has at least $\frac{(a+1)(b+1)}{2}$ elements. By the hypothesis, we have that $G(S) \subseteq I_{(a,b)}$. Then for any $(x, y) \in G(S)$, we have that $(a, b) - (x, y) \in S$ otherwise $G(S)$ has more than $\frac{(a+1)(b+1)}{2}$ elements. Hence S is symmetric by the Proposition 3.21. $\square$

We know the form of 4-dimensional numerical semigroups from the Chapter II. To characterize 4-dimensional symmetric numerical semigroups, we need the following proposition.

Proposition 3.23. *Let S be an n-dimensional numerical semigroup with minimal system of generators $\{(0, 1), (1, k), (a_1, 0), \ldots, (a_{n-2}, 0)\}$ with $a_1 < a_2 < \cdots < a_{n-2}$ and $k \geq 1$. Then S is symmetric if and only if $a_1 = n - 2$ and $a_{i+1} = a_i + 1$ for $2 \leq i \leq n - 3$.*

Proof. Assume that S is symmetric. We claim that $(a_1 - 1, (a_1 - 1)k - 1) \in G(S)$. If not, there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ such that $\alpha_1(0, 1) + \alpha_2(1, k) + \alpha_3(a_1, 0) + \cdots + \alpha_n(a_{n-2}, 0) = (a_1 - 1, (a_1 - 1)k - 1)$. Since $a_1 < a_2 < \cdots < a_{n-2}$, we have that $\alpha_2 = a_1 - 1$ and $\alpha_i = 0$ for all $3 \leq i \leq n - 2$. Then $\alpha_1(0, 1) + (a_1 - 1)(1, k) = (a_1 - 1, (a_1 - 1)k - 1)$ forces that $\alpha_1 \in \mathbb{N}$. It is a contradiction. Hence $(a_1 - 1, (a_1 - 1)k - 1) \in G(S)$.

Let l be the largest integer such that $(l, 0) \in G(S)$ and $(l + i, 0) \in S$ for all positive integers $i \geq 1$. By Proposition 3.18, we have that $(l, (a_1 - 1)k - 1) \in G(S)$ and we claim that $MG(S) = \{(l, (a_1 - 1)k - 1)\}$. It is sufficient to show that $(l, (a_1 - 1)k) \in S$. Let α be a nonnegative integer such that $\alpha a_1 < l < (\alpha + 1)a_1$. Hence $0 < l - \alpha a_1 < a_1$. Since $(l, (a_1 - 1)k) = \alpha(a_1, 0) + (l - \alpha a_1)(1, k) + (a_1 k - lk + \alpha a_1 k - k)(0, 1)$, we already prove the claim.

It is not hard to see that $\{(i, (a_1 - 1)k - 1) \mid 0 \leq i \leq a_1 - 2\} \subseteq S$. Then $\{(l - a_1 + 2, 0), (l - a_1 + 3, 0), \ldots, (l - 1, 0), (l, 0)\} \subset G(S)$ by the fact in Proposition 3.22. It follows that $a_1|(l + 1)$ and $a_1|(l - a_1 + 1)$ because $l - a_1 + 2, l - a_1 + 3, \ldots, l$ are consecutive $a_1 - 1$ numbers. By closed property of semigroup, we have that $(l - a_1 + 2 - na_1, 0), (l - a_1 + 3 - na_1, 0), \ldots, (l - na_1, 0) \notin S$ for all $n \in \mathbb{N}$.

To prove that $a_1 = n - 2$, we consider the set $\{(l + 1, 0), \ldots, (l + a_1, 0)\} \subseteq S$. If $a_1 > n - 2$, then the above set has at least $n - 1$ elements. By the system of generators, $(l + i, 0) = \alpha_1(a_1, 0) + \alpha_2(l + 2, 0) + \cdots + \alpha_{i-1}(l + i - 1, 0)$ for some nonnegative integers $\alpha_1, \ldots, \alpha_{i-1}$ and $2 \leq i \leq a_1$. This is impossible. If $a_1 < n-2$, then the above set has at most $n - 3$ elements. Hence $a_{n-2} \notin \{(l + 1, 0), \ldots, (l + a_1, 0)\}$ and then a_{n-2} can be represented by a positive linear combination of $(a_1, 0)$ and $(l + i, 0)$ for some $1 \leq i \leq a_1$. This contradicts the minimality of system of generators. It follows that $a_1 = n-2$ and $a_2 = l+2, a_3 = l+3, \ldots, a_{n-2} = l+n-2$.

For the converse, assume that $a_1 = n - 2$ and $a_{i+1} = a_i + 1$ for $2 \leq i \leq n - 3$. We see that $a_2, \ldots, a_{n-2}$ are consecutive $n - 3$ numbers. Then $ma_1 = a_2 - 1$ for some positive integer m. Hence $((m - 1)a_1 + a_1 - 1, 0) = (ma_1 - 1, 0)$ must be a gap of S.

We claim that $G(S) = \{(pa_1 + r, b) \in \mathbb{N}^2 \mid p, r \in \mathbb{N}, 0 \leq p \leq m - 1,\ 1 \leq r \leq a_1 - 1 \text{ and } 0 \leq b < rk\}$. To prove this, let $(x, y) \in G(S)$. Hence $x \leq ma_1 - 1$. If $a_1|x$, then $(x, y) = \alpha(a_1, 0) + y(0, 1)$ for some $\alpha \in \mathbb{N}$ and it contradicts $(x, y) \in G(S)$. Assume that $x = pa_1 + r$ where $p, r \in \mathbb{N}$, $0 \leq p \leq m - 1$ and $1 \leq r \leq a_1 - 1$. If $rk \leq y$, then $(x, y) = p(a_1, 0) + r(1, k) + (y - rk)(0, 1)$. This contradicts $(x, y) \in G(S)$. Hence $y < rk$.

Next, we show that $(pa_1 + r, rk - 1) \in G(S)$ for all nonnegative integers p, r such that $0 \leq p \leq m - 1$ and $1 \leq r \leq a_1 - 1$. Suppose that there exists $\lambda_i \in \mathbb{N}$ such that $(pa_1 + r, rk - 1) = \lambda_1(a_1, 0) + \lambda_2(1, k) + \lambda_3(0, 1)$. If $\lambda_1 < p$, then $\lambda_2 \geq a_1 + r$ and it is impossible because we have $\lambda_3 < 0$. So $\lambda_1 = p$ and $\lambda_2 = r$. This leads to $\lambda_3 < 0$ again. Therefore $(pa_1+r, rk-1) \in G(S)$ for all $0 \leq p \leq m-1$ and $1 \leq r \leq a_1-1$. It is not hard to see that $\{(1,k),(2,2k),\ldots,(a_1-1,(a_1-1)k)\} \subset S$. Now we have the set $G(S)$ as desired.

From the set $G(S)$, it follows that $MG(S) = \{(ma_1-1,(a_1-1)k-1)\}$. If a_1 is even, then ma_1-1 is odd. If a_1 is odd, then $(a_1-1)k-1$ is odd. Hence the maximal gap of S is odd. By counting the number of gaps of S, we have that the number of its gaps are $m(k+2k+\cdots+(a_1-1)k) = \frac{ma_1(a_1-1)k}{2}$. Hence S is symmetric by Proposition 3.22. □

Corollary 3.24. *Let S be an n-dimensional numerical semigroup with the minimal system of generators $\{(1,0),(l,1),(0,b_1),\ldots,(0,b_{n-2})\}$ with $b_1 < b_2 < \cdots < b_{n-2}$ and $l \geq 1$. Then S is symmetric if and only if $b_1 = n-2$ and $b_{i+1} = b_i + 1$ for $2 \leq i \leq n-3$.*

Proof. For the only if part, assume that S is symmetric. Then we claim that $MG(S) = \{(b_1-1)l-1, k\}$ where k is the largest integer such that $(0,k) \in G(S)$ and $(0,k+i) \in S$ for all positive integer $i \geq 1$. The rest of the proof follows similarly to Proposition 3.23.

For the converse, assume that $b_1 = n-2$ and $b_{i+1} = b_i+1$ for $2 \leq i \leq n-3$. Since $b_2,\ldots,b_{n-2}$ are consecutive $n-3$ numbers. Then $mb_1 = b_2-1$ for some positive integer m. We claim that $G(S) = \{(a, pb_1+r) \in \mathbb{N}^2 \mid p,r \in \mathbb{N}, 0 \leq p \leq m-1, 1 \leq r \leq b_1-1 \text{ and } 0 \leq a < rl\}$. The rest of the proof follows similarly to Proposition 3.23. □

Corollary 3.25. *A 4-dimensional numerical semigroup is symmetric if and only if its minimal system of generators satisfies one of the following sets.*

1. $\{(0,1),(1,c),(2,0),(a,0)\}$ *where a is an odd integer and $c \geq 1$.*
2. $\{(1,0),(c,1),(0,2),(0,a)\}$ *where a is an odd integer and $c \geq 1$.*

Proof. This follows from Proposition 2.28, Proposition 3.23 and Corollary 3.24. □

Proposition 3.26. *Let S be a 5-dimensional numerical semigroup with the minimal system of generators $\{(0,1),(1,k),(a_1,0),(a_2,0),(m,n)\}$ with $0 < k,m,n$ and $1 < a_1 < a_2$. Then S is symmetric if and only if $a_1 = 3$, $m = 2$ and $n = k$.*

Proof. Assume that S is symmetric. Since $0 < k, m, n$, it follows that $(x, 0) \in S$ if and only if $(x, 0) = \alpha_1(a_1, 0) + \alpha_2(a_2, 0)$ when $\alpha_1, \alpha_2 \in \mathbb{N}$. This is equivalent to $(x, 0) \in S$ if and only if $x = \alpha_1 a_1 + \alpha_2 a_2$. By Proposition 1.23, $a_1a_2 - a_1 - a_2$ is the Frobenius number of the numerical semigroup on $\mathbb{N}$ with minimal system of generators $\{a_1, a_2\}$. Hence $(a_1a_2 - a_1 - a_2, 0) \in G(S)$ and $(a_1a_2 - a_1 - a_2 + i, 0) \in S$ for all positive integer $i \geq 1$. Moreover, the set $\{(x, y) \in G(S) \mid y = 0\}$ has exactly $\frac{a_1a_2 - a_1 - a_2 + 1}{2}$ elements by Proposition 1.23. Since S is symmetric, we assume that $MG(S) = \{(a_1a_2 - a_1 - a_2, y)\}$.

To prove that $y = k - 1$, we follow from the results of Proposition 3.18 so that $k - 1 \leq y$. If $k - 1 < y$, then the cardinality of $G(S)$ has at most $(y + 1)(\frac{a_1a_2 - a_1 - a_2 + 1}{2} - 1) + k$. This contradicts Proposition 3.22. Hence $y = k - 1$ and $MG(S) = \{(a_1a_2 - a_1 - a_2, k - 1)\}$.

Next, we show that $a_1 = 3$. If $a_1 = 2$, then $MG(S) = \{(a_2 - 2, k - 1)\}$. Since $\{(0, 1), (1, k), (2, 0), (a_2, 0)\}$ generates an irreducible numerical semigroup S' with $MG(S') = \{(a_2 - 2, k - 1)\}$. Then $S' \subset S$ and $(a_2 - 2, k - 1) \in S$. This is a contradiction. If $a_1 \geq 4$, then we follow from the maximal gap of S so that $(2, k), (3, k) \in S$. Since there are no nonnegative integers α_1, α_2 such that $\alpha_1(0, 1) + \alpha_2(1, k) = (2, k)$, $m = 2$ and $n = k$. For the same reason, we must have $(m, n) = (3, k)$ which yields a contradiction. Hence $a_1 = 3$ and the maximal gap of S forces that $(m, n) = (2, k)$.

For the converse, assume that $a_1 = 3$, $m = 2$ and $n = k$. Then S is generated by $\{(0, 1), (1, k), (2, k), (3, 0), (a_2, 0)\}$. Since $\{(x, y) \in G(S) \mid y = 0\}$ has exactly $a_2 - 1$ elements, let $G_x = \{(g_1, 0), \ldots, (g_{a_2-1}, 0)\} \subseteq G(S)$. We claim that $G(S) = \{(x, y) \in \mathbb{N}^2 \mid (x, 0) \in G_x \text{ and } 0 \leq y \leq k - 1\}$.

Let $(x, y) \in G(S)$. Then $(x, 0) \in G_x$. Suppose that $k \leq y$. If $(x - 1, 0) \in G_x$, then $3|(x - 2)$ so that $(x, y) = (x - 2, 0) + (2, y) = \alpha(3, 0) + (2, k) + (y - k)(0, 1)$ where $\alpha \in \mathbb{N}$. It contradicts $(x, y) \in G(S)$. If $(x - 1, 0) \in S$, then $(x - 1, 0) = \alpha_1(3, 0) + \alpha_2(a_2, 0)$ for some $\alpha_1, \alpha_2 \in \mathbb{N}$. Hence $(x, y) = \alpha_1(3, 0) + \alpha_2(a_2, 0) + (1, k) + (y - k)(0, 1)$. This contradicts $(x, y) \in G(S)$ again. Hence $0 \leq y \leq k - 1$.

Next, let $(x, y) \in \{(x, y) \in \mathbb{N}^2 \mid (x, 0) \in G_x \text{ and } 0 \leq y \leq k - 1\}$. Suppose that $(x, y) = \alpha_1(0, 1) + \alpha_2(1, k) + \alpha_3(2, k) + \alpha_4(3, 0) + \alpha_5(a_2, 0)$ where $\alpha_1, \ldots, \alpha_5 \in \mathbb{N}$. Since $(x, 0) \in G_x$, $\alpha_i \neq 0$ for some $i \in \{2, 3\}$. Hence $y \geq k > k - 1$ which yields a contradiction. Now we have the set $G(S)$ as desired.

From the set $G(S)$, we have that $MG(S) = \{(2a_2-3, k-1)\}$ and the cardinality of S is $(a_2 - 1)k$. Since $2a_2 - 3$ is odd, S is a symmetric numerical semigroup. □

Corollary 3.27. *Let S be a 5-dimensional numerical semigroup with minimal system of generators $\{(1,0),(l,1),(0,b_1),(0,b_2),(m,n)\}$ with $0 < l,m,n$ and $1 < b_1 < b_2$. Then S is symmetric if and only if $b_1 = 3$, $m = l$ and $n = 2$.*

Proof. For the only if part, assume that S is symmetric. We claim that $MG(S) = \{(l-1, b_1b_2 - b_1 - b_2)\}$. The rest of the proof follows similarly to Proposition 3.26.

For the converse, assume that $b_1 = 3$, $m = l$ and $n = 2$. Then S is generated by $\{(1,0),(l,1),(l,2),(0,3),(0,b_2)\}$. Since $\{(x,y) \in G(S) \mid x = 0\}$ has exactly $b_2 - 1$ elements, let $G_y = \{(0,g_1),\ldots,(0,g_{b_2-1})\} \subseteq G(S)$. We claim that $G(S) = \{(x,y) \in \mathbb{N}^2 \mid (0,y) \in G_y \text{ and } 0 \le x \le l-1\}$. The rest of the proof follows similarly to Proposition 3.26. □

Proposition 3.28. *Let a, b be positive integers not simultaneously even. The number of symmetric numerical semigroup S with maximal gap (a,b) is at least $2^{(\lceil \frac{a}{2} \rceil)(\lceil \frac{b}{2} \rceil)}$ where $\lceil x \rceil$ is the least integer which is greater or equal to x.*

Proof. When both of a and b are odd, we construct the following subsets of $\mathbb{N}^2$.

- $A_1 = \{(m,n) \mid 0 \le m \le \frac{a-1}{2} \text{ and } \frac{b+1}{2} \le n \le b\}$.
- $A_2 = \{(m,n) \mid \frac{a+1}{2} \le m \le a \text{ and } \frac{b+1}{2} \le n \le b\} \backslash \{(a,b)\}$.
- $A_3 = \{(m,n) \mid \frac{a+1}{2} \le m \le a \text{ and } 0 \le n \le \frac{b-1}{2}\}$.
- $S = E_{(a,b)} \cup A_2 \cup \{(0,0)\}$.

Note that the cardinalities of A_1 and A_3 are equal which are $\frac{(a+1)(b+1)}{4}$ and the cardinality of $\mathbb{N}^2 \backslash S$ is $\frac{3(a+1)(b+1)}{4}$. For any $(m,n) \in A_1$, we have $(a-m, b-n) \in A_3$ because $\frac{a+1}{2} \le a - m \le a$ and $0 \le b - n \le \frac{b-1}{2}$. Moreover, for any $(m,n) \in A_3$, we have $(a-m, b-n) \in A_1$ because $0 \le m \le \frac{a-1}{2}$ and $\frac{b+1}{2} \le n \le b$. Next, we extend the numerical semigroup S to be the set S' by adding those elements (m,n) from A_1 and A_3 to S and removing $(a-m, b-n)$ from S until the cardinality of $\mathbb{N}^2 \backslash S'$ is equal to $\frac{(a+1)(b+1)}{2}$. The number of the set S' that can be produced in this way is $2^{\frac{(a+1)(b+1)}{4}}$. For any odd integer x, $x = 2k+1$ for some integer k. Then $\lceil \frac{x}{2} \rceil = \lceil \frac{2k+1}{2} \rceil = k+1 = \frac{x+1}{2}$. Hence $2^{\frac{(a+1)(b+1)}{4}}$ can be rewrited as $2^{(\lceil \frac{a}{2} \rceil)(\lceil \frac{b}{2} \rceil)}$.

To prove that S' is closed, it suffices to show that $A_1 + A_3 \subseteq S$. Since $A_1 + A_3 = \{(m,n) \mid \frac{a+1}{2} \leq m \leq \frac{3a-1}{2} \text{ and } \frac{b+1}{2} \leq n \leq \frac{3b-1}{2}\}$, we have that $A_1 + A_3 \subseteq S \subseteq S'$. Now S' is numerical semigroup with $|G(S)| = \frac{(a+1)(b+1)}{2}$ and then S' is a symmetric numerical semigroup.

For the case only one of a and b is odd, we may assume that a is odd and b is even. Then construct the following subsets of $\mathbb{N}^2$.

- $A_1 = \{(m,n) \mid 0 \leq m \leq \frac{a-1}{2} \text{ and } \frac{b}{2} + 1 \leq n \leq b\}$.
- $A_2 = \{(m,n) \mid \frac{a+1}{2} \leq m \leq a \text{ and } \frac{b}{2} + 1 \leq n \leq b\} \backslash \{(a,b)\}$.
- $A_3 = \{(m,n) \mid \frac{a+1}{2} \leq m \leq a \text{ and } 0 \leq n \leq \frac{b}{2} - 1\}$.
- $A_4 = \{(m,n) \mid \frac{a+1}{2} \leq m \leq a \text{ and } n = \frac{b}{2}\}$.
- $S = E_{(a,b)} \cup A_2 \cup A_4 \cup \{(0,0)\}$.

Note that the cardinalities of A_1 and A_3 are equal which are $\frac{(a+1)(b)}{4}$ and the cardinality of $\mathbb{N}^2 \backslash S$ is $\frac{3(a+1)(b)}{4} + \frac{a+1}{2}$. For any $(m,n) \in A_1$, we have $(a-m, b-n) \in A_3$ because $\frac{a+1}{2} \leq a-m \leq a$ and $0 \leq b-n \leq \frac{b}{2}-1$. Moreover, for any $(m,n) \in A_3$, we have $(a-m, b-n) \in A_1$ because $0 \leq m \leq \frac{a-1}{2}$ and $\frac{b}{2}+1 \leq n \leq b$. We extend the numerical semigroup S to be the set S' by adding those elements (m,n) from A_1 and A_3 to S and removing $(a-m, b-n)$ from S. We stop the process when the cardinality of $\mathbb{N}^2 \backslash S'$ is equal to $\frac{(a+1)(b+1)}{2}$. The number of the set S' that can be produced in this way is $2^{\frac{(a+1)(b)}{4}} (= 2^{(\lceil \frac{a}{2} \rceil)(\lceil \frac{b}{2} \rceil)})$. Proving that S' is closed is similar to the first case. □

3.3 Intersections of Symmetric Numerical Semigroups

Given a numerical semigroup S, if S is not irreducible, then there exists numerical semigroups S_1 and S_2 containing S such that $S = S_1 \cap S_2$. If S_1 is not irreducible, then there exist numerical semigroups S_3 and S_4 containing S such that $S_1 = S_3 \cap S_4$. Since $G(S)$ is finite, this process guarantees that S can be written as an intersection of irreducible numerical semigroups.

In this part, we study the condition for numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups. Most of the results in this part are analogus to [2].

Proposition 3.29. *Let S be a numerical semigroup and g an odd gap of S. There exists a symmetric numerical semigroup S' such that $S \subseteq S'$ and $MG(S') = \{g\}$.*

Proof. Let $S_1 = S \cup E_g$. It is not hard to see that S_1 is a numerical semigroup containing S and $MG(S_1) = \{g\}$. By Proposition 3.19, we can construct a symmetric numerical semigroup S' such that $S_1 \subseteq S'$ and $MG(S') = \{g\}$. □

Proposition 3.30. *Let S be a numerical semigroup and g an even gap of S. Then the following statements are equivalent.*

1. *There exists a symmetric numerical semigroup S' with $S \subseteq S'$ and $g \in G(S')$.*

2. *There exists an odd element $x \in \mathbb{N}^2$ such that $g + x \in G(\langle S, x\rangle)$.*

Proof. (1) → (2) Assume that there exists a symmetric numerical semigroup S' with $S \subseteq S'$ and $g \in G(S')$. Since S' is symmetric, we have that $g' - g \in S'$ where $g' \in MG(S')$. Since g' is odd, set $x = g' - g$ which is an odd element. Hence $\langle S, x\rangle \subseteq S'$ which yields $g + x \in G(\langle S, x\rangle)$ by $g + x = g' \in MG(S')$.

(2) → (1) Assume that there exists an odd element $x \in \mathbb{N}^2$ such that $g + x \notin \langle S, x\rangle$. Let $S_0 = \langle S, x\rangle \cup E_{g+x}$ be a numerical semigroup. Hence $MG(S_0) = \{g+x\}$ which is an odd element. Then there exists a symmetric numerical semigroup S' such that $S_0 \subseteq S'$ and $MG(S') = \{g + x\}$. Therefore $S \subseteq S_0 \subseteq S'$. If $g \in S'$, then $g + x \in S'$ which is a contradiction. Hence $g \notin S'$. □

Proposition 3.31. *A numerical semigroup S can be expressed as an intersection of symmetric numerical semigroups if and only if for any even gap g of S, there exists an odd element $x \in \mathbb{N}^2$ such that $g + x \in G(\langle S, x\rangle)$.*

Proof. Assume that S can be expressed as an intersection of symmetric numerical semigroups. Let g be an even gap of S. By the hypothesis, there exists $n \in \mathbb{N}$ and symmetric numerical semigroups $S_1, \ldots, S_n$ such that $S = S_1 \cap S_2 \cap \cdots \cap S_n$. Then $g \in G(S_j)$ for some $1 \leq j \leq n$. We finish the proof by Proposition 3.30.

For the converse, assume that for any even gap g of S, there exists an odd element $x \in \mathbb{N}^2$ such that $g + x \notin \langle S, x \rangle$. Let $g \in G(S)$. If g is odd, then let S_g be a symmetric numerical semigroup such that $S \subsetneq S_g$ and $MG(S_g) = \{g\}$ by Proposition 3.29. If g is even, then let S_g be a symmetric numerical semigroup such that $S \subsetneq S_g$ and $g \in G(S_g)$ by Proposition 3.30. It is not hard to see that $S \subseteq \bigcap_{g \in G(S)} S_g$. Note that $\bigcap_{g \in G(S)} S_g$ is a numerical semigroup because $G(S)$ is finite. Since $g \in G(S)$ implies $g \in G(S_g)$, it follows that $g \in G(\bigcap_{g \in G(S)} S_g)$. This proves that $\bigcap_{g \in G(S)} S_g \subseteq S$. Hence $S = \bigcap_{g \in G(S)} S_g$. □

Proposition 3.32. *Let S be a numerical semigroup such that $S = S_1 \cap \cdots \cap S_n$ for some numerical semigroups $S_1, \ldots, S_n$. Then $MG(S) \subseteq MG(S_1) \cup \cdots \cup MG(S_n)$.*

Proof. Let $(a, b) \in MG(S)$. Since $S = S_1 \cap \ldots \cap S_n$, there exists S_i such that $(a, b) \in G(S_i)$. If $(a, b) \notin MG(S_i)$, then there exists $(c, d) \in G(S_i)$ such that $a^2 + b^2 < c^2 + d^2$. Moreover, $(c, d) \in G(S)$ which contradicts $(a, b) \in MG(S)$. Hence $(a, b) \in MG(S_i)$. □

The maximal gap of every symmetric numerical semigroups is odd. Then the next corollary follows directly from Proposition 3.32.

Corollary 3.33. *Let S be a numerical semigroup which can be expressed as an intersection of symmetric numerical semigroups. Then $MG(S)$ contains only odd gaps.*

Example 3.34. The converse of Corollary 3.33 may not be true. Let S be a numerical semigroup with $MG(S) = \{(0, 3), (3, 0)\}$ described by the diagram in Figure 3.5. Since there is no odd element $x \in \mathbb{N}^2$ such that $(0, 2) + x \in G(\langle S, x \rangle)$, then S cannot be written as an intersection of symmetric numerical semigroups.

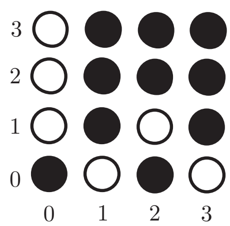

Figure 3.5: Numerical semigroup S

To illustrate all symmetric numerical semigroups containing S, we follow from all odd gaps of S and describe those numerical semigroups by the diagram in figure 3.6.

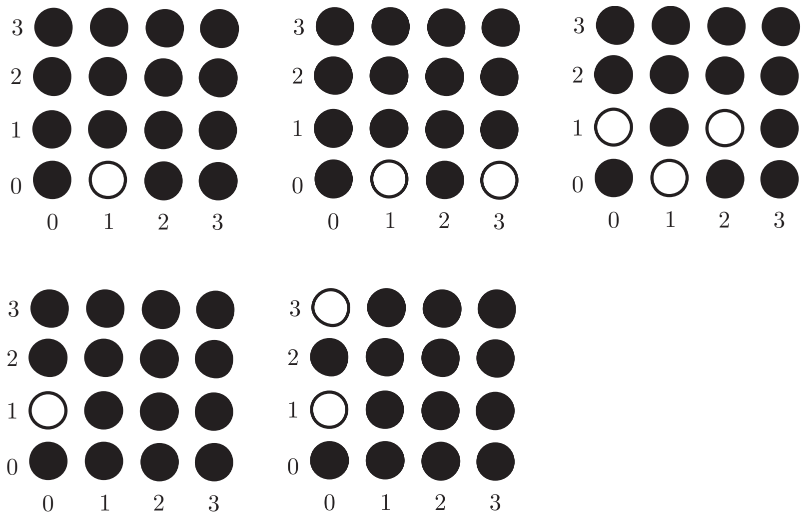

Figure 3.6: All symmetric numerical semigroups containing S

Definition 3.35. Let S be a numerical semigroup. A gap g of S is *pseudo maximal* if $g+s \in S$ for all $s \in S^*$. The set of all pseudo maximal gaps of S is denoted by $PMG(S)$.

Let S be a numerical semigroup. We define a relation $\leq_S$ on $\mathbb{N}^2$ by $(a,b) \leq_S (c,d)$ if and only if $(c-a, d-b) \in S$. It is not hard to see that $\leq_S$ is an order relation. Denote by $\text{Max}_{\leq_S}X$, the set of all maximal elements of X respect to the relation $\leq_S$.

Proposition 3.36. *Let S be a numerical semigroup. Then $PMG(S) = \text{Max}_{\leq_S}G(S)$.*

Proof. Let $g \in PMG(S)$ and $g' \in G(S)$ be such that $g \leq_S g'$ and $g \neq g'$. Then $g' - g = s$ for some $s \in S^*$. Since g is a pseudo maximal gap of S, it follows that $g + (g' - g) \in S$. But it is imposible because $g' \in G(S)$. Hence $g = g'$ so that $g \in \text{Max}_{\leq_S}G(S)$. For the reverse inclusion, suppose that $g \in \text{Max}_{\leq_S}G(S)$ but $g \notin PMG(S)$. Then there exists $s \in S^*$ such that $g + s \in G(S)$. Hence $(g+s) - g = s \in S$ contradicts $g \in \text{Max}_{\leq_S}G(S)$. Therefore $g \in PMG(S)$. $\square$

Proposition 3.37. *Let S be a numerical semigroup and $g_1, \dots, g_n$ all pseudo maximal gaps of S and $g \in \mathbb{N}^2$. Then $g \in G(S)$ if and only if $g_i - g \in S$ for some $1 \leq i \leq n$.*

Proof. Assume that $g \in G(S)$. Following from Proposition 3.36, $g \in G(S)$ implies $g_i - g \in S$ for some $1 \leq i \leq n$. For the converse, assume that $g_i - g \in S$ for some $1 \leq i \leq n$. Suppose that $g \in S$. Then $g_i = g + (g_i - g) \in S$ yields a contradiction. Hence $g \in G(S)$. □

Proposition 3.38. *Any numerical semigroup with all pseudo maximal gaps are odd can be expressed as an intersection of symmetric numerical semigroups.*

Proof. Let S be a numerical semigroup with all pseudo maximal gaps are odd. Let $g_1, \dots, g_n$ be all pseudo maximal gaps of S. For each $1 \leq i \leq n$, let S_{g_i} be a symmetric numerical semigroup such that $S \subseteq S_{g_i}$ and $MG(S_{g_i}) = \{g_i\}$ by Proposition 3.29. It is not hard to see that $S \subseteq S_{g_1} \cap \dots \cap S_{g_n}$. Next, assume that $g \in G(S)$. By Proposition 3.37, we have that $g_k - g \in S$ for some $1 \leq k \leq n$. Since $S \subseteq S_{g_i}$ for all i, it follows that $g_k - g \in S_{g_1} \cap \dots \cap S_{g_n}$ so that $g_k - g \in S_{g_k}$. Since $g_k \in G(S_{g_k})$, it forces that $g \in G(S_{g_k})$. This proves that $S_{g_1} \cap \dots \cap S_{g_n} \subseteq S$. □

Proposition 3.39. *Let S be a numerical semigroup and $g_1, \dots, g_n$ all pseudo maximal gaps of S. Then S can be expressed as an intersection of symmetric numerical semigroups if and only if for each even pseudo maximal gap g_i, there exists an odd element $x \in \mathbb{N}^2$ such that $g_i + x \in G(\langle S, x_i \rangle)$.*

Proof. The only if part follows from Proposition 3.31. For the converse, assume that for each even pseudo maximal gap g_i, there exists an odd element $x \in \mathbb{N}^2$ such that $g_i + x \notin \langle S, x_i \rangle$. If g_i is even, then there exists a numerical semigroup S_{g_i} such that $S \subseteq S_{g_i}$ and $g_i \in G(S_{g_i})$ by Proposition 3.30. If g_i is odd, then we have a numerical semigroup S_{g_i} such that $S \subseteq S_{g_i}$ and $g_i \in MG(S_{g_i})$ by Proposition 3.29. To prove that $S = S_{g_1} \cap \dots \cap S_{g_n}$, we follow the proof similarly to Proposition 3.38. Then S can be expressed as an intersection of symmetric numerical semigroup as desired. □

To express a numerical semigroup as an intersection of symmetric numerical semigroups, we firstly compute the set $PMG(S)$. For each $g_i \in PMG(S)$, we compute a symmetric numerical semigroup as in Proposition 3.39. Then the intersection of those numerical semigroups becomes S. We describe this method by the next example.

Example 3.40. Let S be a numerical semigroup described by the below diagram.

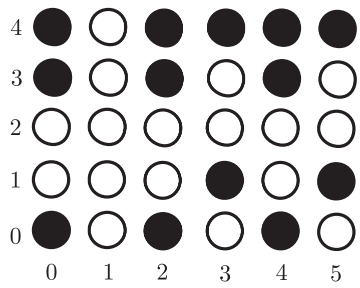

Figure 3.7: Numerical semigroup S

Then $PMG(S) = \{(4,1),(4,2),(5,2),(5,3),(1,4)\}$ and the only even pseudo maximal gap of S is $(4,2)$. Since $(4,2)+(1,0) \in G(\langle S,(1,0)\rangle)$, we have that S can be expressed as an intersection of symmetric numerical semigroups. By the proof of Proposition 3.39, the intersection is $S = S_{(4,1)} \cap S_{(4,2)} \cap S_{(5,2)} \cap S_{(5,3)} \cap S_{(1,4)}$.

To construct a symmetric numerical semigroup $S_{(1,4)}$, we set $S_0 = S \cup E_{(1,4)}$. Then compute the set H in Proposition 3.19.

- $H(S_0) = \{(0,1),(0,2),(1,2),(1,3)\}$ and set $S_1 = S_0 \cup \{(1,3)\}$
- $H(S_1) = \{(0,2),(1,2)\}$ and set $S_2 = S_1 \cup \{(1,2)\}$
- $H(S_2) = \varnothing$ then set $S_{(1,4)} = S_2$

To construct a symmetric numerical semigroup $S_{(4,2)}$, we set $S_0 = \langle S,(1,0)\rangle$. Since $H(S_0) = \varnothing$, set $S_{(4,2)} = S_0$. For constructing numerical semigroups $S_{(5,2)}, S_{(5,3)}$ and $S_{(4,1)}$, we follow the same process as $S_{(1,4)}$. These symmetric numerical semigroups can be seen in Figure 3.8.

The process in Example 3.40 does not guarantee minimal number of the intersection. Then we can remove $S_{(5,2)}$ from the intersection. The next proposition gives a condition for which numerical semigroups can be removed from the intersection.

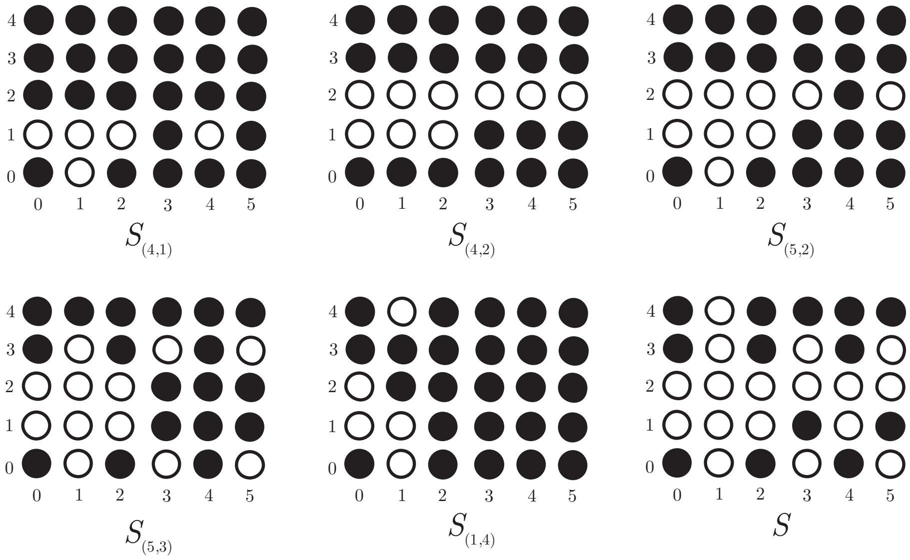

Figure 3.8: Symmetric numerical semigroups containing S

Proposition 3.41. *Let S be a numerical semigroup and $S_1, \ldots, S_n$ numerical semigroups containing S. Then $S = S_1 \cap \cdots \cap S_n$ if and only if for all $g \in SG(S)$, $g \in G(S_i)$ for some $1 \leq i \leq n$.*

Proof. Assume that $S = S_1 \cap \cdots \cap S_n$. Let $g \in SG(S)$. Since $S = S_1 \cap \cdots \cap S_n$, we have that $g \in G(S_i)$ for some $1 \leq i \leq n$. For the converse, assume that for any special gap g of S impies $g \in G(S_i)$ for some $1 \leq i \leq n$. Suppose that $S \subset S_1 \cap \cdots \cap S_n$. Let $g \in M((S_1 \cap \cdots \cap S_n)\backslash S)$ which is a special gap of S by Proposition 3.5. Hence $g \in G(S_i)$ for some i and it contradicts $g \in S_1 \cap \cdots \cap S_n$. Hence $S = S_1 \cap \cdots \cap S_n$. □

From Proposition 3.41, for each S_i we compute the set $CS_i = \{g \in SG(S) \mid g \notin S_i\}$. Then $S = S_{k_1} \cap \cdots \cap S_{k_r}$ if and only if $SG(S) = CS_{k_1} \cup \cdots \cup CS_{k_r}$.

By Example 3.40, we have that $SG(S) = \{(4,1),(4,2),(5,2),(5,3),(1,4)\}$, $CS_{(4,1)} = \{(4,1)\}$, $CS_{(4,2)} = \{(4,2),(5,2)\}$, $CS_{(5,2)} = \{(5,2)\}$, $CS_{(5,3)} = \{(5,3)\}$ and $CS_{(1,4)} = \{(1,4)\}$. Then $S = S_{(4,1)} \cap S_{(4,2)} \cap S_{(5,3)} \cap S_{(1,4)}$.

3.4 Fundamental Gaps

Given a numerical semigroup S, Proposition 3.11 says that $SG(S)$ determines S up to maximality but not uniquely. This shows that we can find a numerical semigroup S' such that $SG(S) = SG(S')$ but $S \neq S'$.

In this part, we study the set that determines a numerical semigroup uniquely, called the fundamental gap. Another application of fundamenatl gaps is a construction of all numerical semigroups containing a fixed numerical semigroup as we presented in the first section. Most of the results in this part are analogus to [5].

Example 3.42. The numerical semigroups described by the diagram below have the same special gaps, $\{(2,1),(1,2)\}$, but they are not the same set.

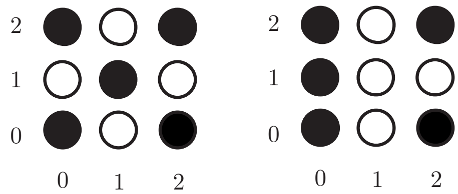

Figure 3.9: Numerical semigroups with the same special gaps

Definition 3.43. Let S be a numerical semigroup. A subset X of $G(S)$ *determines* S if S is the maximum numerical semigroup such that $X \subseteq G(S)$.

From the above example, we see that $\{(2,1),(1,2)\}$ does not determine those numerical semigroups. However, each numerical semigroup determine by the set of all its gap.

For a nonempty subset A of $\mathbb{N}^2$, let $D(A) = \{x \in \mathbb{N}^2 \mid x|a \text{ for some } a \in A\}$.

Example 3.44. Let $X_1 = \{(10,0)\}$ and $X_2 = \{(0,9),(1,2),(4,8)\}$. Then $D(X_1) = \{(1,0),(2,0),(5,0),(10,0)\}$ and $D(X_2) = \{(0,1),(0,3),(0,9),(1,2),(2,4),(4,8)\}$.

Remark 3.45. Given a numerical semigroup S and a nonempty subset $X \subseteq G(S)$. We have that $X \subseteq D(X) \subseteq G(S)$.

Proposition 3.46. *Let X be a nonempty finite subset of $\mathbb{N}^2$. Then X determines a numerical semigroup if and only if $\mathbb{N}^2 \backslash D(X)$ is a numerical semigroup. Moreover, X always determine $\mathbb{N}^2 \backslash D(X)$.*

Proof. Let S be a numerical semigroup determined by X. Assume that $X = \{(x_1,y_1),\ldots,(x_n,y_n)\}$ for some positive integer n. From Remark 3.45, we have that $D(X) \subseteq G(S)$ and then $S \subseteq \mathbb{N}^2 \backslash D(X)$. To complete the proof, we show that $\mathbb{N}^2 \backslash D(X) \subseteq S$. If $(a,b) \in \mathbb{N}^2 \backslash D(X)$, then $k(a,b) \notin X$ for all positive integers k. We set $S' = \left\langle (E_{(x_1,y_1)} \cap \cdots \cap E_{(x_n,y_n)}), (a,b) \right\rangle$ which is a numerical semigroup with $X \subseteq G(S')$. Since X determines S, it implies that $S' \subseteq S$ so that $(a,b) \in S$. Hence $S = \mathbb{N}^2 \backslash D(X)$.

For the converse, assume that $\mathbb{N}^2 \backslash D(X)$ is a numerical semigroup. Let S' be a numerical semigroup such that $X \subseteq G(S')$. Hence $X \subseteq D(X) \subseteq G(S')$ and $S' \subseteq \mathbb{N}^2 \backslash D(X)$. So X determines a numerical semigroup $\mathbb{N}^2 \backslash D(X)$. □

Proposition 3.47. *Let S be a numerical semigroup and X a nonempty subset of $G(S)$. Then X determines S if and only if for any $g \in G(S)$, if $2g, 3g \in S$, then $g \in X$.*

Proof. Assume that X determines S. By Proposition 3.46, we have that $S = \mathbb{N}^2 \backslash D(X)$. Then $G(S) = D(X)$. Let $g \in G(S)$ be such that $2g, 3g \in S$. Since $g \in D(X)$, $kg \in X$ for some positive integer k. But $2g, 3g \in S$ forces that $k = 1$ and hence $g \in X$.

Assume that for any $g \in G(S)$, if $2g, 3g \in S$, then $g \in X$. By Proposition 3.46, it is enough to prove that $S = \mathbb{N}^2 \backslash D(X)$. Since $X \subseteq G(S)$, we have that $D(X) \subseteq G(S)$ and $S \subseteq \mathbb{N}^2 \backslash D(X)$. Let $g \in G(S)$ and k be the largest positive integer such that $kg \in G(S)$. Therefore, $2kg, 3kg \in S$. By the hypothesis, we have that $kg \in X$ and $g \in D(X)$. Then $\mathbb{N}^2 \backslash D(X) \subseteq S$. □

Definition 3.48. A gap g of a numerical semigroup S is *fundamental* if $2g, 3g \in S$. We denote the set of all fundamental gaps of S by $FG(S)$.

By the definition of $SG(S)$, it follows that $SG(S) \subseteq FG(S)$. The next proposition follows from the definition of fundamental gap and Proposition 3.47.

Proposition 3.49. *Let S be a numerical semigroup and X a nonempty subset of $G(S)$. Then X determines S if and only if $FG(S) \subseteq X$. Moreover, $FG(S)$ is the smallest subset of $G(S)$ which determines S.*

Example 3.50. Let $S = \left\langle E_{(2,3)} \cup \{(1,1),(0,2)\} \right\rangle$.

- $G(S) = \{(1,0),(2,0),(0,1),(2,1),(1,2),(0,3),(2,3)\}$

- $SG(S) = \{(2,0),(0,3),(2,3)\}$
- $FG(S) = \{(2,0),(2,1),(1,2),(0,3),(2,3)\}$

Then $FG(S), FG(S) \cup \{(1,0)\}, FG(S) \cup \{(2,0)\}$ and $G(S)$ determine S.

Proposition 3.51. *Let X be a nonempty finite subset of $\mathbb{N}^2$. The following statements are equivalent.*

1. *There exists a numerical semigroup S such that $FG(S) = X$.*
2. *$\mathbb{N}^2 \backslash D(X)$ is a numerical semigroup and for any distinct elements $x_1, x_2 \in X$, there are no positive integers k such that $x_1 = kx_2$.*

Proof. (1)→(2) Assume that there is a numerical semigroup S such that $FG(S) = X$. By Proposition 3.49 and Proposition 3.46, we have that $S = \mathbb{N}^2 \backslash D(X)$ is a numerical semigroup determined by X. Suppose that there are distinct elements $x_1, x_2 \in X$ and a positive integer $k \geq 2$ so that $x_1 = kx_2$. If k is even, then $x_1 = (\frac{k}{2})(2x_2) \in S$. If k is odd, then $x_1 = 3x_2 + (\frac{k-3}{2})(2x_2) \in S$. That is impossible. Hence there are no positive integers k such that $x_1 = kx_2$.

(2)→(1) Assume that $\mathbb{N}^2 \backslash D(X)$ is a numerical semigroup and for any distinct elements $x_1, x_2 \in X$, there are no positive integers k such that $x_1 = kx_2$. Let $S = \mathbb{N}^2 \backslash D(X)$. Then X determines S and $FG(S) \subseteq X$ by Proposition 3.49. We show that $X \subseteq FG(S)$. Let $x \in X$. If $2x \in G(S) = D(X)$, then $k(2x) \in X$ for some positive integer k and it contradicts the hypothesis. For the case $3x \in D(X)$, we have the same contradiction. Hence $2x, 3x \in S$ which yields $X \subseteq FG(S)$. □

Proposition 3.52. *Let S be a numerical semigroup. Then $SG(S) = \mathrm{Max}_{\leq_S} FG(S)$.*

Proof. We note the fact that $SG(S) \subseteq FG(S)$. Let $x \in SG(S)$. Suppose that there exists $y \in FG(S)$ such that $y - x \in S$. This forces that $y = x$ because $x + s \in S$ for all $s \in S^*$. Hence $x \in \mathrm{Max}_{\leq_S} FG(S)$. For the converse, let $x \in \mathrm{Max}_{\leq_S} FG(S)$. Suppose that there exists $s \in S^*$ such that $(x+s) \notin S$. Since $(x+s)+(x+s) = 2x+2s \in S$ and $(x+s)+(x+s)+(x+s) = 3x+3s \in S$, $x+s \in FG(S)$. It is a contradiction because $x \leq_S (x+s)$. Then $x \in SG(S)$. □

From Proposition 3.12, we have that a numerical semigroup S is irreducible if and only if $SG(S)$ has at most one element. Then by Proposition 3.52, we have the following corollary.

Corollary 3.53. *A numerical semigroup is irreducible if and only if* $\mathrm{Max}_{\leq_S} FG(S)$ *has at most one element.*

Proposition 3.54. *Let S be a numerical semigroup and $(a,b) \in G(S)$ such that $S \cup \{(a,b)\}$ is a numerical semigroup. Then* $FG(S \cup \{(a,b)\}) = (FG(S) \backslash \{(a,b)\}) \cup \{(m,n) \in G(S) \mid p(m,n) = (a,b) \text{ when } p \in \{2,3\} \text{ and } (m,n) \notin D(FG(S) \backslash \{(a,b)\})\}$.

Proof. Let $(m,n) \in FG(S \cup \{(a,b)\})$ but $(m,n) \notin FG(S) \backslash \{(a,b)\}$. Then $k(m,n) \in S \cup \{(a,b)\}$ for some positive integer $k \geq 2$. Since $(m,n) \notin FG(S)$, we have that $2(m,n) = (a,b)$ or $3(m,n) = (a,b)$. Next, Suppose that $(m,n) \in D(FG(S) \backslash \{(a,b)\})$. Since $[FG(S) \backslash \{(a,b)\}] \cap [S \cup \{(a,b)\}] = \varnothing$, it follows that $(m,n) \in FG(S) \backslash \{(a,b)\}$ yields a contradiction. Hence $(m,n) \notin D(FG(S) \backslash \{(a,b)\})$.

For the reverse inclusion, it suffices to show that $\{(m,n) \in G(S) \mid p(m,n) = (a,b) \text{ when } p \in \{2,3\} \text{ and } (m,n) \notin D(FG(S) \backslash \{(a,b)\})\} \subseteq FG(S \cup \{(a,b)\})$. Let $(m,n) \in G(S)$ such that $p(m,n) = (a,b)$ with prime $p \leq 3$ and $(m,n) \notin D(FG(S) \backslash \{(a,b)\})$. If $p = 2$, then suppose that $3(m,n) \in G(S \cup \{(a,b)\})$. Since $3(m,n) \in G(S)$, we have that $3(m,n) \in D(FG(S))$. It follows that $(m,n) \in D(FG(S) \backslash \{(a,b)\})$ and it is a contradiction. For the case $p = 3$, the proof is similar to the case $p = 2$. Hence $\{(m,n) \in G(S) \mid p(m,n) = (a,b) \text{ when } p \in \{2,3\} \text{ and } (m,n) \notin D(FG(S) \backslash \{(a,b)\})\} \subseteq FG(S \cup \{(a,b)\})$. □

Example 3.55. Let S be a numerical semigroup with $FG(S) = \{(0,2),(1,2),(2,2)\}$. Since $\mathrm{Max}_{\leq_S} FG(S) = \{(2,2)\}$, set $S_1 = S \cup \{(2,2)\}$. By Proposition 3.54, $FG(S_1) = \{(1,1),(0,2),(1,2)\}$ so that $S_1 = \mathbb{N}^2 \backslash D((1,1),(0,2),(1,2))$. Proceeding in this way, we have the diagram in Figure 3.10.

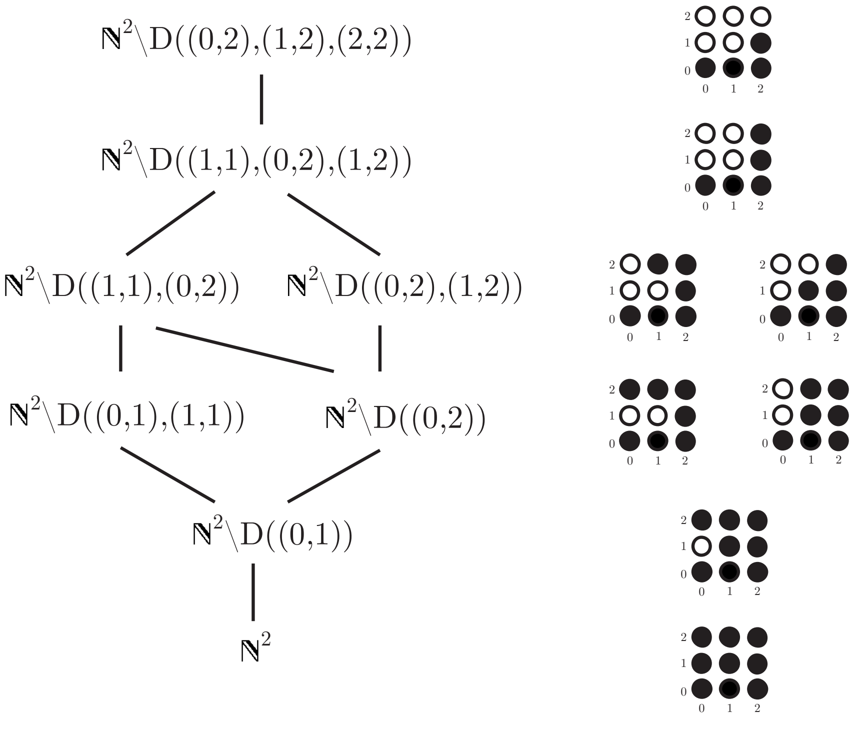

Figure 3.10: All numerical semigroups containing S

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