CHAPTER II

BASIC PRINCIPLES

Equations of Motion in the Time Domain

In dynamic analyses, the equations of motion can be derived by means of Hamilton's variational principle. This principle has been widely used and referred to in most standard text books and related papers. Hamilton's principle can be written as (see, e.g. Clough and Penzien (1975)) :

$$\int_{t_1}^{t_2} \delta(T-V)dt + \int_{t_1}^{t_2} \delta W_{nc}dt = 0 \tag{2.1}$$

in which T and V are the kinetic energy and the potential energy of the system, respectively, and W_{nc} is the work done resulting from nonconservative forces acting on the system, including damping and any other external forces. The symbol $\delta(\cdot)$ denotes the first variational operator. The time domain from t_1 to t_2 is any interval of time under consideration.

Let $\{q_1, q_2, ..., q_N\}$ be a set of generalized coordinates that describes the displacement field of the system. The kinetic energy, the potential energy and the work done by nonconservative forces can be expressed in the forms

$$T = T(q_1, q_2, ..., q_N, \dot{q}_1, \dot{q}_2, ..., \dot{q}_N, t) \tag{2.2a}$$

$$V = V(q_1, q_2, ..., q_N, t) \tag{2.2b}$$

$$\delta W_{nc} = Q_1\delta q_1 + Q_2\delta q_2 + ... + Q_N\delta q_N \tag{2.2c}$$

where $\dot{q}_1, \dot{q}_2, ..., \dot{q}_N$ are the time derivatives of the generalized coordinates and

Q_1, Q_2, ..., Q_N are the generalized nonconservative forces associated with the generalized coordinates q_1, q_2, ..., q_N, respectively.

Substitution of Eqs. (2.2) into Eg. (2.1) yields

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \dots + \frac{\partial T}{\partial q_N}\delta q_N \right.$$
$$+ \frac{\partial T}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\delta \dot{q}_2 + \dots + \frac{\partial T}{\partial \dot{q}_N}\delta \dot{q}_N$$
$$- \frac{\partial V}{\partial q_1}\delta q_1 - \frac{\partial V}{\partial q_2}\delta q_2 - \dots - \frac{\partial V}{\partial q_N}\delta q_N$$
$$\left. + Q_1\delta q_1 + Q_2\delta q_2 + \dots + Q_N\delta q_N\right) dt = 0 \tag{2.3}$$

Integrating the velocity dependent terms by parts leads to

$$\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{q}_i}\delta \dot{q}_i dt = \left[\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i dt \tag{2.4}$$

According to the condition for the validity of the Hamilton principle $\delta q_1(t_1)$ and $\delta q_1(t_2)$ are equal to zero. Consequently, the first bracket on the right-hand side of Eq. (2.4) is always zero. Eq. (2.3), after introducing Eq. (2.4), becomes

$$\int_{t_1}^{t_2}\left\{\sum_{i=1}^{N}\left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} + Q_i\right]\delta q_i\right\} dt = 0 \tag{2.5}$$

Due to the fact that δq_i ($i = 1, 2, \dots, N$) are arbitrary, the bracketed expression of Eq. (2.5) must be equal to zero, viz.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad , \quad i = 1, 2, \dots, N \tag{2.6}$$

which are the well-known Lagrange's equations of motion. In general, the kinetic energy of the linear systems with small displacement is velocity dependent only (ie. not depending on q_i). Thus the terms $\frac{\partial T}{\partial q_i}$ in the above equations vanish, and Eqs. (2.6) are simplified to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = Q_i \quad , \quad i = 1, 2, \ldots, N \tag{2.7}$$

The kinetic energy and the potential energy of a system in a volume domain Ω bounded by a closed surface Γ can be written in matrix forms,

$$T = \frac{1}{2}\int_{\Omega} \rho \{\dot{u}\}^T \{\dot{u}\} d\Omega \tag{2.8}$$

and

$$V = \frac{1}{2}\int_{\Omega} \{\sigma\}^T \{\varepsilon\} d\Omega - \int_{\Omega} \{u\}^T \{\bar{X}\} d\Omega - \int_{\Gamma} \{u\}^T \{\bar{T}\} d\Gamma \tag{2.9}$$

respectively, where $\{u\}$ is the displacement vector of any point in the system; $\{\dot{u}\}$ is the velocity vector (ie. time derivative of the displacement); $\{\sigma\}$ is the stress vector; $\{\varepsilon\}$ is the strain vector; $\{\bar{X}\}$ is the body force vector; $\{\bar{T}\}$ is the surface traction vector; ρ is the mass density. $\{\cdot\}^T$ stands for the transpose of a vector.

Finite Element Formulation

In the finite element formulation we can consider the nodal displacements of the discretized system as a set of generalized coordinates, ie.

$$\{u\} = [\bar{N}]\{q\} \tag{2.10}$$

where $[\bar{N}]$ is the global interpolation function matrix, and $\{q\}$ is the global nodal displacement vector.

It is of convenience at this step to recall the basic equations in the linear theory of elasticity. The constitutive relations for an elastic body undergoing small displacements can be written as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{2.11}$$

and the strain-displacement relationship is

$$\{\varepsilon\} = [L]\{u\} \tag{2.12}$$

where [D] is the elasticity matrix and [L] is the strain operator matrix.

Introducing Eq. (2.10) into Eq. (2.12) we have

$$\{\varepsilon\} = [L][\bar{N}]\{q\} = [B]\{q\} \tag{2.13}$$

where [B] represents the strain interpolation function. Substitution of $\{\varepsilon\}$ from Eq. (2.13) into Eq. (2.11) yields

$$\{\sigma\} = [D][B]\{q\} \tag{2.14}$$

By virtue of Eq. (2.10), Eq. (2.13) and Eq. (2.14) we can express the kinetic energy and the potential energy in terms of nodal quantities as

$$T = \frac{1}{2}\int_{\Omega} \rho\{\dot{q}\}^T[\bar{N}]^T[\bar{N}]\{\dot{q}\}d\Omega$$

$$= \frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\} \tag{2.15}$$

where

$$[M] = \int_{\Omega} \rho[\bar{N}]^T[\bar{N}]\,d\Omega \tag{2.16}$$

and

$$V = \frac{1}{2}\int_{\Omega} \{q\}^T[B]^T[D][B]\{q\}\, d\Omega$$

$$- \int_{\Omega} \{q\}^T[\bar{N}]^T\{\bar{X}\}\, d\Omega - \int_{\Gamma} \{q\}^T[\bar{N}]^T\{\bar{T}\}\, d\Gamma$$

$$= \frac{1}{2}\{q\}^T[K]\{q\} - \{q\}^T\{f_\Omega\} - \{q\}^T\{f_\Gamma\} \quad (2.17)$$

in which

$$[K] = \int_{\Omega} [B]^T[D][B]\, d\Omega \quad (2.18)$$

$$\{f_\Omega\} = \int_{\Omega} [\bar{N}]^T\{\bar{X}\}\, d\Omega \quad (2.19)$$

$$\{f_\Gamma\} = \int_{\Gamma} [\bar{N}]^T\{\bar{T}\}\, d\Gamma \quad (2.20)$$

In the above equations, [M] and [K] are the mass matrix and the stiffness matrix of the global system respectively, and $\{f_\Omega\}$ and $\{f_\Gamma\}$ are respectively vectors of the body forces and surface tractions.

As it is well-known in the finite element analysis method, the global interpolating functions $[\bar{N}]$ are defined to be piecewise continuous functions which are non-zero within the individual element only. Therefore, any integration taken over the whole domain of the system can be evaluated by the total summation of the integrations over each element. Thus, the global matrices [M] and [K] can be synthesized from the element matrices as follows:

$$[M] = \sum_{e=1}^{n} [M^e] = \sum_{e=1}^{n} \int_{\Omega^e} \rho [N^e]^T[N^e]\, d\Omega^e \quad (2.21)$$

and

$$[K] = \sum_{e=1}^{n} [K^e] = \sum_{e=1}^{n} \int_{\Omega^e} [B^e]^T [D][B^e]\, d\Omega^e \tag{2.22}$$

where n is the total number of elements in the structure.

In fact, $[N^e]$ and $[B^e]$ are of the same size as their global counterparts but they are equal to zeroes except in the element domain considered, thus integration outside the domain is always zero. However, in practice in which the well-known method called 'the direct stiffness method' is used, we can reduce the size of $[N^e]$ and $[B^e]$ to the minimum (size) and after performing the element integration, we can assemble them into the proper places of the global matrices.

The element mass matrix computed from the integral in Eq. (2.21) is called the 'consistent mass matrix' and contains non-zero off diagonal terms. Rather than using the consistent mass matrix, the lumped mass procedure as widely used in dynamic analysis is followed in this study. The lumped mass matrix, which is diagonal, is easily obtained, and yet yields acceptable results for dynamic analyses of structures.

The substitution of Eq. (2.15) and Eq. (2.17) into Eq. (2.7) yields

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} + \{f_\Omega\} + \{f_\Gamma\} \tag{2.23}$$

The generalized nonconservative force vector, $\{Q\}$, may be conveniently separated into two components, viz. the system internal damping forces, $\{Q_c\}$, and the external nonconservative forces, $\{Q_f\}$. Thus,

$$\{Q\} = \{Q_c\} + \{Q_f\} \tag{2.24}$$

One of the most difficult job in dynamic analyses is the evaluation of the internal damping effect. For viscous damping which is widely assumed in dynamic analyses of structures, the damping force is proportional to the velocity. Thus,

$$\{Q_c\} = -[C]\{\dot{q}\} \tag{2.25}$$

The minus sign appearing in Eq. (2.25) denotes the fact that the damping force acts in the opposite direction of $\{\dot{q}\}$. Substitution of Eq. (2.24) into Eq. (2.23) incorporating Eq. (2.25) leads to

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q_f\} + \{f_\Omega\} + \{f_\Gamma\} \tag{2.26}$$

The three force vectors on the right hand side of Eq. (2.26) represent any generalized force acting on the system. These forces can be conservative or nonconservative forces and will collectively be denoted by a single force vector $\{F\}$. Eq. (2.26) can then be written as

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F\} \tag{2.27}$$

<u>Solution of Equations of Motion in Frequency Domain</u>

The equations of motion can be solved either in the time domain or in the frequency domain. In contrast with the solution in the time domain, the characteristics of the system can be readily viewed by means of solution in the frequency domain, for example, the system predominant frequency, and the relative contribution of various frequency contents to the total response.

Consider the load vector, $\{F\}$, in the form :

$$\{F\} = \{P\}e^{i\omega t} \tag{2.28}$$

The steady state response, after the initial transient motion, can be assumed to be

$$\{q\} = \{u\}e^{i\omega t} \tag{2.29}$$

at each excitation circular frequency ω, where $\{P\}$ and $\{u\}$ are the complex amplitudes of load and displacement respectively; $i = \sqrt{-1}$ and t is the time in seconds. One obtains, upon substitution of Eq. (2.28) and Eq. (2.29) and the corresponding time derivatives into Eq. (2.27),

$$[S]\{u\} = \{P\} \tag{2.30}$$

in which $[S]$ is the dynamic-stiffness or impedance matrix defined as

$$[S] = [K] + i\omega[C] - \omega^2[M] \tag{2.31}$$

At each forcing frequency, Eq. (2.30) can be directly solved by means of Gauss elimination method in complex form :

$$\{u\} = [S]^{-1}[P]$$

$$= \{F_1 + iF_2\} \tag{2.32}$$

in which F_1 and F_2 are respectively the real and imaginary parts of the response. This solution is normally called 'the displacement function' or equivalently 'the compliance'.

Linear Hysteretic Damping

The material damping in the soil is usually assumed to be caused by friction in the grains, resulting in linear hysteretic damping which is frequency independent. According to the correspondance principle (see, e.g. Wolf (1985)) which states that the damped solution can be obtained from the elastic result by replacing the elastic constants by the corresponding complex values. Thus the static-stiffness matrix can be written as

$$[K^*] = [K](1+2\xi i) \tag{2.33}$$

where ξ is the damping ratio.

In this study, only the linear hysteretic damping is considered, i.e. the viscous damping mechanism is neglected as is normally assumed. Thus, the dynamic-stiffness, $[S]$, in Eq. (2.30) can be expressed as

$$[S] = [K](1+2\xi i) - \omega^2[M] \tag{2.34}$$

Substructure Formulation of Equations of Motion for Soil-structure Interaction

When the super-structure is considered together with its foundation and support soil medium, it is convenient to solve the whole system by substructuring.

Fig. 2.1a depicts a general model of soil-structure interaction problem. For simplicity it is assumed that end boundaries are sufficiently far away from the existing structure so that no special boundary treatment is required. The equations of motion, in terms of absolute displacements of the complete structure, are given by Eq. (2.30) i.e. :

$$\begin{bmatrix} [S_{ss}] & [S_{sb}] & [0] & [0] \\ [S_{bs}] & [S_{bb}] & [S_{ba}] & [0] \\ [0] & [S_{ab}] & [S_{aa}] & [S_{ac}] \\ [0] & [0] & [S_{ca}] & [S_{cc}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_b\} \\ \{u_a\} \\ \{u_c\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{P_f\} \end{Bmatrix} \tag{2.35}$$

In Eq. (2.35), partitioning are performed corresponding to the various parts of the model illustrated in Fig. 2.1. The subscript notations denote different nodal positions: s for super-structure nodes, b for boundary connection nodes of structures at the ground surface, a for soil interior nodes, and c for outer boundary nodes.

In linear elastic analyses, the total displacements at any point, $\{u\}$, can be expressed as the superposition of independent displacements. In this problem, the displacement vector, $\{u\}$, is decomposed into

$$\{u\} = \{u^g\} + \{u^i\} \tag{2.36}$$

where $\{u^g\}$ denotes the free field displacement vector which is the site response for the soil medium alone in the absence of super-structure (Fig. 2.1b), and $\{u^i\}$ represents the interaction displacement vector affected by the presence of the super-structure as shown in Fig. 2.1c. Models in Fig. 2.1a and 2.1 b are prescribed by the free-body reaction force, $\{P_f\}$. As far as the soil medium is concerned the interaction displacements can be interpreted as the change in the response from the free field solution as influenced by the interaction of the super-structure with the soil mass.

The equations of motion for the free-field problem corresponding to Fig. 2.1b can be written as

$$\begin{bmatrix} [0] & [0] & [0] & [0] \\ [0] & [S^g_{bb}] & [S^g_{ba}] & [0] \\ [0] & [S^g_{ab}] & [S^g_{aa}] & [S^g_{ac}] \\ [0] & [0] & [S^g_{ca}] & [S^g_{cc}] \end{bmatrix} \begin{Bmatrix} \{u^g_s\} \\ \{u^g_b\} \\ \{u^g_a\} \\ \{u^g_c\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{P_f\} \end{Bmatrix} \tag{2.37}$$

where superscript g stands solely for the ground model. Now observing that the contribution to the impedance terms $[S_{ba}]$, $[S_{ab}]$, $[S_{aa}]$, $[S_{ac}]$, $[S_{ca}]$ and $[S_{cc}]$ are derived from the soil mass only, we have

$$[S_{aa}] = [S^g_{aa}], \qquad [S_{ab}] = [S^g_{ab}], \qquad [S_{ac}] = [S^g_{ac}],$$

$$[S_{ba}] = [S^g_{ba}], \qquad [S_{ca}] = [S^g_{ca}] \quad \text{and} \quad [S_{cc}] = [S^g_{cc}].$$

Substituting Eq. (2.36) and (2.37) into Eq. (2.35), incorporating the above equalities yields

$$\begin{bmatrix} [S_{ss}] & [S_{sb}] & [\,0\,] & [\,0\,] \\ [S_{bs}] & [S_{bb}] & [S_{ba}] & [\,0\,] \\ [\,0\,] & [S_{ab}] & [S_{aa}] & [S_{ac}] \\ [\,0\,] & [\,0\,] & [S_{ca}] & [S_{cc}] \end{bmatrix} \begin{Bmatrix} \{u_s^i\} \\ \{u_b^i\} \\ \{u_a^i\} \\ \{u_c^i\} \end{Bmatrix} = \begin{bmatrix} -[S_{ss}] & -[S_{sb}] & [\,0\,] & [\,0\,] \\ -[S_{bs}] & [S_{bb}^g]-[S_{bb}] & [\,0\,] & [\,0\,] \\ [\,0\,] & [\,0\,] & [\,0\,] & [\,0\,] \\ [\,0\,] & [\,0\,] & [\,0\,] & [\,0\,] \end{bmatrix} \begin{Bmatrix} \{u_s^g\} \\ \{u_b^g\} \\ \{u_a^g\} \\ \{u_c^g\} \end{Bmatrix} \tag{2.38}$$

Performing condensation on Eq. (2.38), we obtain

$$\begin{bmatrix} [S_{ss}] & [S_{sb}] \\ [S_{bs}] & [S_{bb}]+[S_{bb}^c] \end{bmatrix} \begin{Bmatrix} \{u_s^i\} \\ \{u_b^i\} \end{Bmatrix} = \begin{bmatrix} -[S_{ss}] & -[S_{sb}] \\ -[S_{bs}] & [S_{bb}^g]-[S_{bb}] \end{bmatrix} \begin{Bmatrix} \{u_s^g\} \\ \{u_b^g\} \end{Bmatrix} \tag{2.39}$$

where the submatrix $[S_{bb}^c]$ results from the condensation manipulations that

$$-\begin{bmatrix} [\,0\,] & [\,0\,] \\ [S_{ba}] & [\,0\,] \end{bmatrix} \begin{bmatrix} [S_{aa}] & [S_{ac}] \\ [S_{ca}] & [S_{cc}] \end{bmatrix}^{-1} \begin{bmatrix} [\,0\,] & [S_{ab}] \\ [\,0\,] & [\,0\,] \end{bmatrix} = \begin{bmatrix} [\,0\,] & [\,0\,] \\ [\,0\,] & [S_{bb}^c] \end{bmatrix} \tag{2.40}$$

Substituting $\{u^i\} = \{u\}-\{u^g\}$ into Eq. (2.39), after some manipulations, Eq. (2.39) becomes

$$\begin{bmatrix} [S_{ss}] & [S_{sb}] \\ [S_{bs}] & [S_{bb}]+[S_{bb}^c] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_b\} \end{Bmatrix} = \begin{bmatrix} [\,0\,] & [\,0\,] \\ [\,0\,] & [S_{bb}^g]+[S_{bb}^c] \end{bmatrix} \begin{Bmatrix} \{u_s^g\} \\ \{u_b^g\} \end{Bmatrix}$$

$$= \begin{Bmatrix} [\,0\,] \\ ([S_{bb}^g]+[S_{bb}^c])\{u_b^g\} \end{Bmatrix} \tag{2.41}$$

The matrix $[S_{bb}]$ can be separately written as

$$[S_{bb}] = [S_{bb}^s] + [S_{bb}^a] \tag{2.42}$$

where $[S^s_{bb}]$ is the stiffness contributed from the super-structure and $[S^a_{bb}]$ is the stiffness contributed from all of the underlying soil medium including embedded piles, if present. It is obvious that $[S^a_{bb}]$, by definition, is exactly the same as $[S^g_{bb}]$. In view of Eq. (2.42), one can rewrite Eq. (2.41) in the form

$$\begin{bmatrix} [S_{ss}] & [S_{sb}] \\ [S_{bs}] & [S^s_{bb}]+[\bar{S}_{bb}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_b\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ [\bar{S}_{bb}]\{u^g_b\} \end{Bmatrix} \tag{2.43}$$

where

$$[\bar{S}_{bb}] = ([S^g_{bb}]+[S^c_{bb}]) \tag{2.44}$$

The physical meaning of $[\bar{S}_{bb}]$ is that it represents the condensed dynamic stiffness or impedance of the soil foundation associated with the boundary connection nodes b. One may note that the reduced system governed by Eq. (2.43) can be graphically represented by the simple model shown in Fig. 2.2a. Moreover, the system can be further simplified as exemplified by the model in Fig. 2.2b provided that the following conditions are satisfied

a) the type of seismic wave that affects the motion is the vertically propagating shear wave.

b) the vertical response of structure is negligibly small by virtue of sufficiently designed vertical members to take vertical loadings due to gravity.

c) at the ground level, a rigid mat foundation exists.

<u>Impedance and Transfer Functions</u>

To solve Eq. (2.43) the force vector on the right hand side must be pre-evaluated. Determination of $[\bar{S}_{bb}]$ is known as 'impedance problem', which is equivalent to the determination of dynamic-stiffness of the foundation. For some special foundations, for example, strip foundation and circular foundation resting on elastic half space, impedance functions can be found in some well-known publications, for instance, Karasudhi et al. (1968), Luco and Westmann (1971) and (1972), and Veletsos and Wei (1974). However, in more complicated conditions such as an embedded foundation or a foundation on piles, resort to numerical methods like the finite element or boundary integral methods are inevitable.

The displacements of the foundation without super-structure, $\{u_b^g\}$, can be determined directly from Eq. (2.37) under the conditions that $\{P_f\} = \{0\}$ and $\{u_c^g\}$ is specified, noting that the displacements solution $\{u_a\}$ and $\{u_g\}$ are the absolute displacement vectors.

Interpretation of $\{u_b^g\}$ is useful since one can conceive that how much the soil layer amplify the seismic wave. Furthermore, signal of the predominant frequency of the soil layer can be seen in the presentations of $\{u_b^g\}$.

Alternatively, one may solve for $\{u_b^g\}$ by using relative displacement formulation. The equations of motion in the time domain, Eq. (2.27), can be applied to solve the problem shown in Fig. 2.1b where the fixed base ground acceleration is specified (see, e.g. Clough and Penzien (1975)). The resulting equations of motion can be written in the form

$$[M]\{\ddot{q}_r\} + [C]\{\dot{q}_r\} + [K]\{q_r\} = -\{m\}\ddot{q}_f \qquad (2.45)$$

where $\{q_r\}$ is the displacement vector relative to the base; $\ddot{q}_f$ is the specified base acceleration and $\{m\}$ is the mass vector pertaining to the corresponding degree-of-freedom. Eq. (2.45) can be simply transformed to be the equations of motion in the frequency domain by the similar procedure of deriving Eq. (2.30). Substituting $q_f = u_f e^{i\omega t}$ and $\{q_r\} = \{u\}e^{i\omega t}$ incorporating with their derivatives into Eq. (2.45) yields

$$[S]\{u\} = \omega^2\{m\}u_f \tag{2.46}$$

where u_f is the displacement amplitude of the base motion at each forcing frequency, ω. Complex displacement amplitude, $\{u\}$, can be obtained by solving Eq. (2.46). Ratio of the absolute value of complex displacement amplitude to the displacement amplitude of the base motion is defined to be 'the transfer function'.