CHAPTER II

THE BINOMIAL COEFFICIENT FUNCTION ON THE SINGULAR LINES

2.1 The singularity in $f(r,n)$ at the lattice point $(3, -4)$

We shall take the limit of $f(r,n)$ along the line with slope m that passes through the point $(3, -4)$.

Let ε be such that $0 < |\varepsilon| < 1$.

From
$$f(r,n) = \frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)}$$

we have
$$f(3+\varepsilon, -4+m\varepsilon) = \frac{\Gamma(-4+m\varepsilon+1)}{\Gamma(3+\varepsilon+1)\,\Gamma(-4+m\varepsilon-3-\varepsilon+1)}$$

$$= \frac{\Gamma(-3+m\varepsilon)}{\Gamma(4+\varepsilon)\,\Gamma(-6+(m-1)\varepsilon)}.$$

But
$$\Gamma(-3+m\varepsilon) = \frac{\Gamma(1+m\varepsilon)}{(-3+m\varepsilon)(-2+m\varepsilon)(-1+m\varepsilon)(m\varepsilon)}$$

$$\Gamma(4+\varepsilon) = (3+\varepsilon)(2+\varepsilon)(1+\varepsilon)\Gamma(1+\varepsilon),$$

and
$$\Gamma(-6+(m-1)\varepsilon) = \frac{\Gamma(1+(m-1)\varepsilon)}{(-6+(m-1)\varepsilon)(-5+(m-1)\varepsilon)\ldots\ldots.(m-1)\varepsilon}$$

Therefore
$$\lim_{\varepsilon\to 0} f(3+\varepsilon, -4+m\varepsilon) = \frac{(-1)^6\,6!}{(-1)^3\,3!\,3!}\left(\frac{m-1}{m}\right) = -20\left(1-\frac{1}{m}\right).$$

If we use $m = \infty$, we obtain $\lim\limits_{\varepsilon\to 0} f(3+\varepsilon, -4+m\varepsilon) = -20$.

The value of this limit is the same as that obtained by Mrs. Wanida (Thesis, p14). We see that the value of the limit depends on the value of m.

When $m \to 0^+$ (or when m is small and positive), we have

$$\lim_{\varepsilon \to 0} f(3+\varepsilon, -4+m\varepsilon) = -20(1 - \frac{1}{+0})$$

$$= \infty$$

And when $m \to 0^-$ (or when m is small and negative), the value of the limit of the function approaches $-\infty$ thus.

$$\lim_{\varepsilon \to 0} f(3+\varepsilon, -4+m\varepsilon) = -20(1 - \frac{1}{-0})$$

$$= -\infty$$

Therefore when the values of m vary from 0^+ to 0^-, the values of the limit decrease from $+\infty$ to $-\infty$ as shown in figure 1.

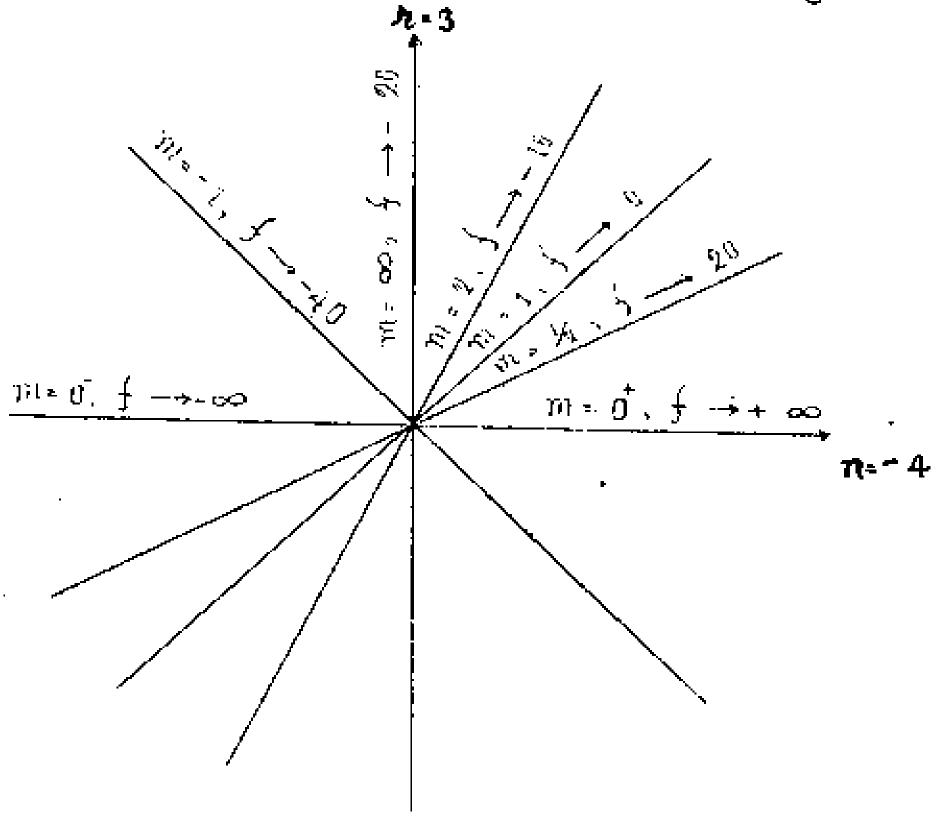

Figure 1 : Limits of $f(r,n)$ as $(r,n) \to (3, -4)$ along straight lines with various slopes.

2.2 The nature of f(r,n) at any singular lattice point.

We shall prove that at any lattice point (r,n) the limit of f(r,n) exists, where the limit is taken along a line which has the slope m to the point (r,n) from either direction.

Consider the point (r,-k) in the fourth quadrant where k is a positive integer, and r is a non-negative integer.

$$f(r+\varepsilon,\ -k+m\varepsilon) = \frac{\Gamma(-(k-1)+m\varepsilon)}{\Gamma(\ (r+1)+\varepsilon)\ \Gamma(-(k+r-1)+(m-1)\varepsilon)}\ .$$

Therefore

$$\lim_{\varepsilon \to 0} f(r+\varepsilon,\ -k+m\varepsilon) = \frac{(-1)^{k+r-1-k+1}\ (k+r-1)!}{r!\,(k-1)!}\left(\frac{m-1}{m}\right)$$

$$= (-1)^{r}\ {}^{k+r-1}C_{r}\left(1-\frac{1}{m}\right).$$

This result is illustrated in figure 2.

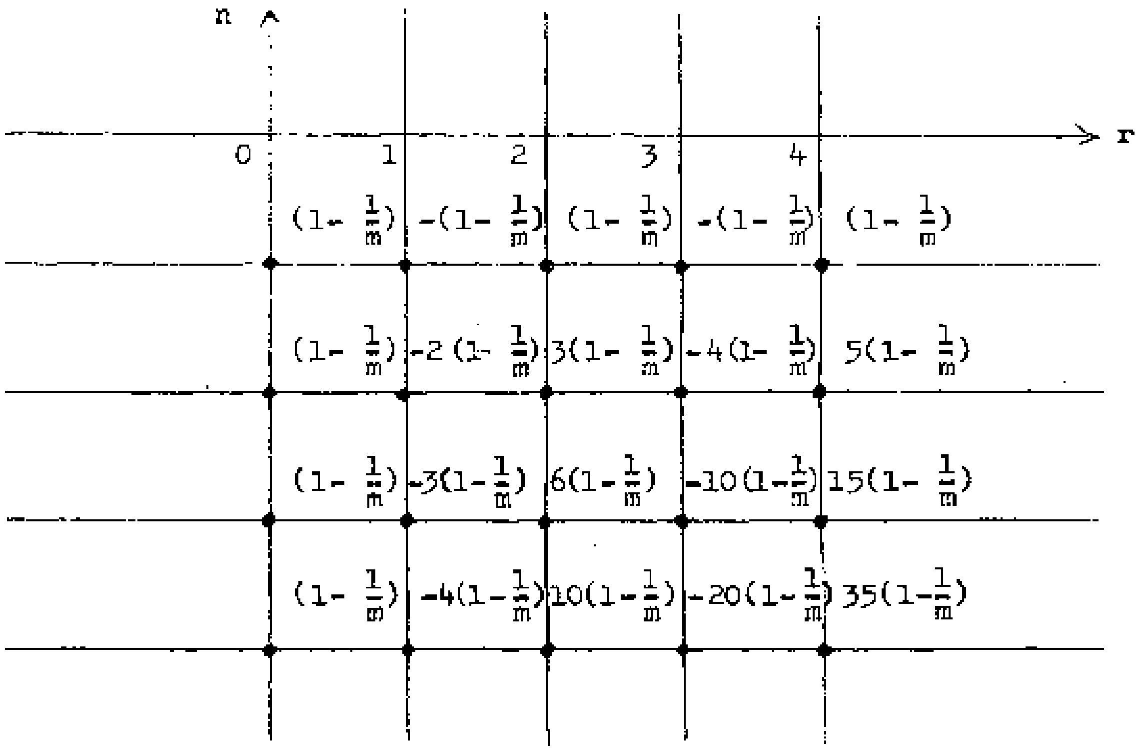

Figure 2 . The values of the limit of f(r,n) at lattice points inthe fourth quadrant.

In the same way, at the point $(-j, -k)$ in the third quadrant where k and j are positive integers,

we have $$f(-j+\varepsilon, -k+m\varepsilon) = \frac{\Gamma(-(k-1)+m\varepsilon)}{\Gamma(-(j-1)+\varepsilon)\Gamma((j-k+1)+(m-1)\varepsilon)}$$

Case 1. If $j-k \geqslant 0$, then $j \geqslant k$.

$$\Gamma(-(k-1)+m\varepsilon) = \Gamma(1+m\varepsilon)/(-(k-1)+m\varepsilon)\ldots\ldots(m\varepsilon)$$

$$\Gamma(-(j-1)+\varepsilon) = \Gamma(1+\varepsilon)/(-(j-1)+\varepsilon)\ldots\ldots(\varepsilon)$$

and

$$\Gamma((j-k+1)+(m-1)\varepsilon) = (j-k+(m-1)\varepsilon)\ldots\ldots(1+(m-1)\varepsilon)\Gamma(1+(m-1)\varepsilon)$$

Therefore $f(-j+\varepsilon, -k+m\varepsilon)$

$$= \frac{\Gamma(1+m\varepsilon)(-(j-1)+\varepsilon)\ldots\ldots(\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1+(m-1)\varepsilon)(-(k-1)+m\varepsilon)\ldots\ldots(m\varepsilon)(j-k+(m-1)\varepsilon)\ldots\ldots(1+(m-1)\varepsilon)}$$

Therefore $$\lim_{\varepsilon\to 0} f(-j+\varepsilon, -k+m\varepsilon) = \frac{(-1)^{j-1}(j-1)!}{(-1)^{k-1}(k-1)!(j-k)!}\left(\frac{1}{m}\right)$$

$$= (-1)^{j-k}\,{}^{j-1}C_{k-1}\left(\frac{1}{m}\right).$$

Case 2. If $j-k < 0$, then $j < k$. In this case we have

$$\Gamma((j-k+1)+(m-1)\varepsilon) = \Gamma(-(k-j-1)+(m-1)\varepsilon)$$

$$= \frac{\Gamma(1+(m-1)\varepsilon)}{(-(k-j-1)+(m-1)\varepsilon)\ldots\ldots(m-1)\varepsilon}$$

Then $f(-j+\varepsilon, -k+m\varepsilon)$

$$= \frac{\Gamma(1+m\varepsilon)(-(j-1)+\varepsilon)\ldots\ldots(\varepsilon)(-(k-j-1)+(m-1)\varepsilon)\ldots\ldots(m-1)\varepsilon}{\Gamma(1+\varepsilon)\Gamma(1+(m-1)\varepsilon)(-(k-1)+m\varepsilon)\ldots\ldots(m\varepsilon)}$$

This gives $\lim_{\varepsilon\to 0} f(-j+\varepsilon, -k+m\varepsilon) = 0.$

Therefore at any lattice point in the third quadrant the values of the limit of the function are

$$\lim_{\epsilon \to 0} f(-j+\epsilon,\ -k+m\epsilon) = \begin{cases} (-1)^{j-k}\ {}^{j-1}C_{k-1}\left(\frac{1}{m}\right), & \text{when } j \geqslant k \\ 0, & \text{when } j < k \end{cases}$$

and both j and k are positive integers.

These results are illustrated in figures 3 and 4.

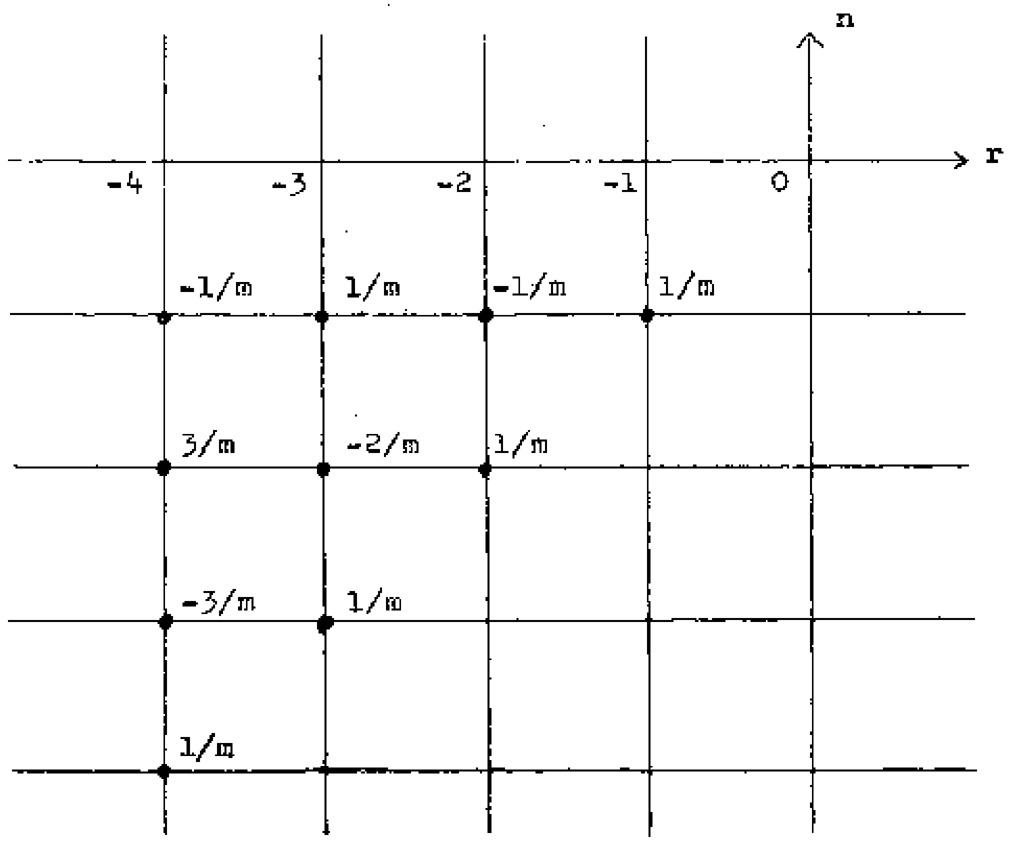

Figure 3 : Values of the limit of f(r,n) at the lattice points in the third quadrant in case 1.

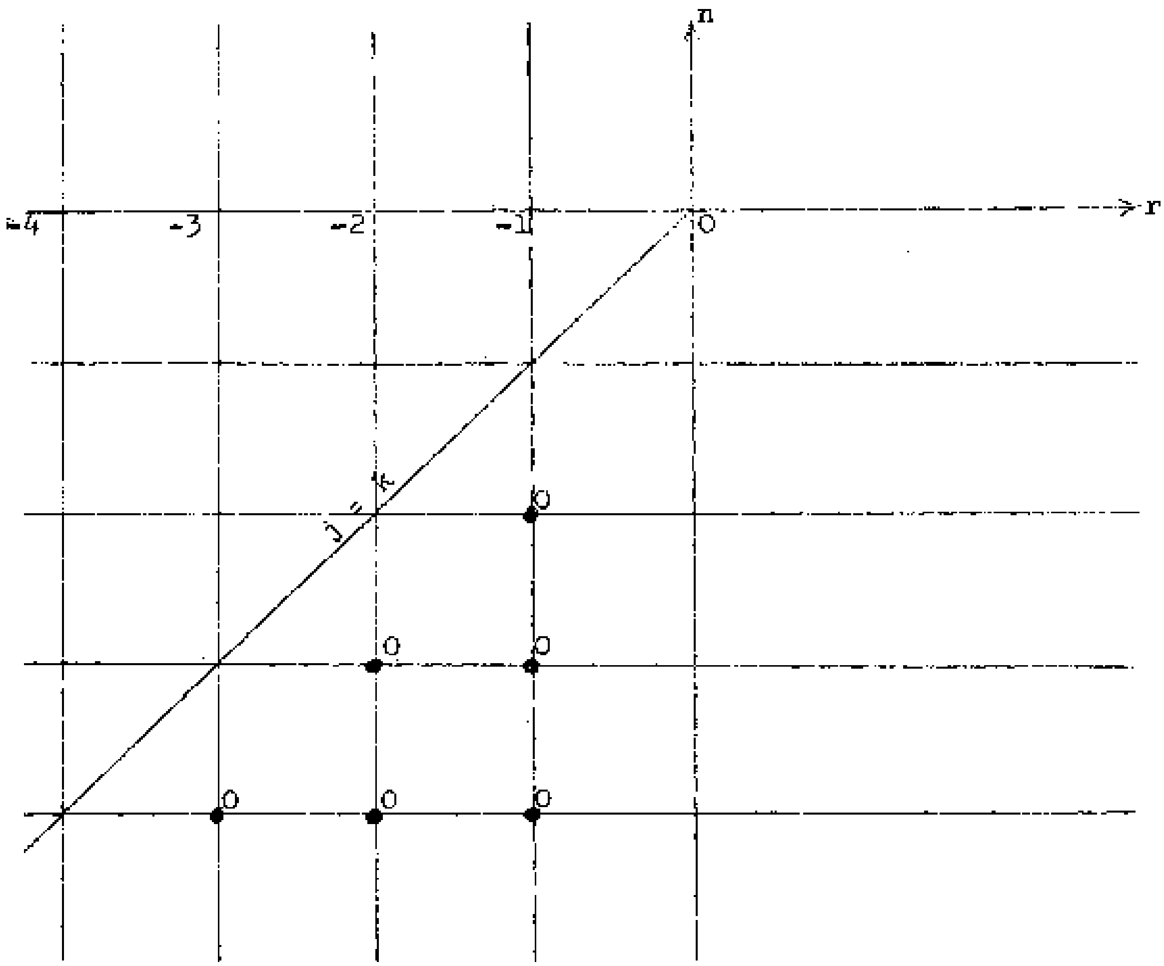

Figure 4 : Values of the limit of f(r,n) at the lattice points in the third quadrant in case 2.

Hence, at all the lattice points on singular lines we can find the limit of the function along any line with slope m passing through that point. For example, by letting $m = \frac{1}{2}$, we obtain the limits shown in figure 5.

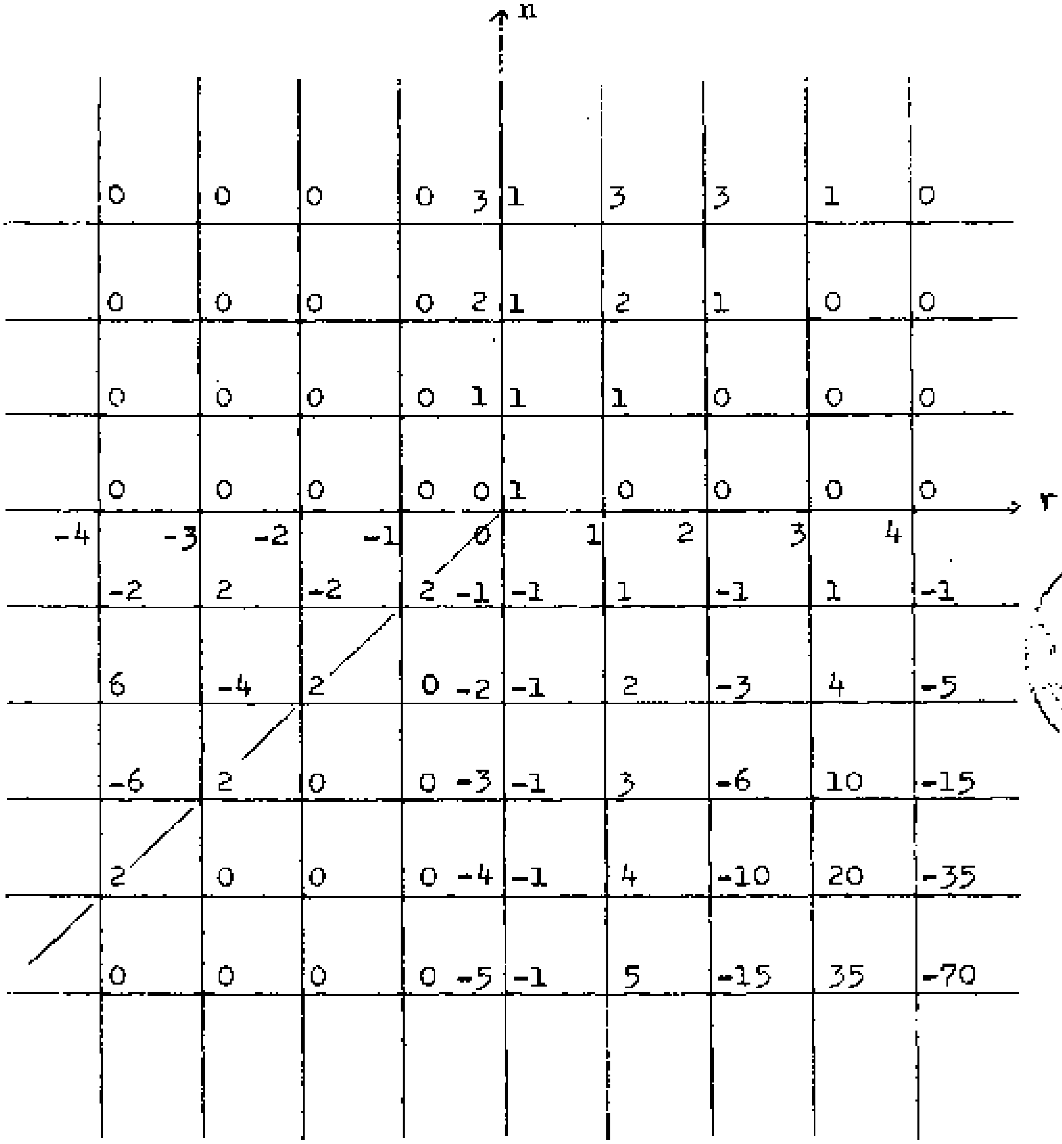

Figure 5 : The values of the limit of f(r,n) at the lattice points taked along the lines with slope $\frac{1}{2}$.

From figure 5, we easily see that these limits of the function at the lattice points over the plane satisfy the rule for constructing Pascal's triangle as explained by Wanida (Thesis). It is not therefore necessary to calculate the values of the limit at every lattice point, for once we have calculated the values on the n-axis; the others may be obtained by Pascal's rule.

The values of the limit of f(r,n) on the n-axis are,

$$\lim_{\varepsilon \to 0} f(\varepsilon, -k+m\varepsilon) = 1 - \frac{1}{m}$$

Pascal's rule at points other than lattice points will be discussed in detail in Chapter III.

2.3 The nature of f(r,n) between the lattice points on the singular lines.

The point (r+x, -k) , where $0 < x < 1$, is in between the two lattice points (r,-k) and (r+1, -k) on the singular line $n = -k$.

Here we have

$$f(r+x+\varepsilon, -k+m\varepsilon) = \frac{\Gamma(-(k-1)+m\varepsilon)}{\Gamma(r+x+1+\varepsilon)\Gamma(-(k+r+x+1)+(m-1)\varepsilon)}$$

If $r \geqslant 0$, then $r+x+1 > 0$ and $k+r+x+1 > 0$.

Therefore $$\Gamma(-(k-1)+m\varepsilon) = \frac{\Gamma(1+m\varepsilon)}{(-k+1+m\varepsilon)(-k+m\varepsilon)\ldots\ldots\ldots\ldots(m\varepsilon)}$$

$$\Gamma(r+x+1+\varepsilon) = (r+x+\varepsilon)(r+x-1+\varepsilon)\ldots(1+x+\varepsilon)\Gamma(1+x+\varepsilon)$$

and

$$\Gamma(-(k+r+x+1)+(m-1)\varepsilon) = \frac{\Gamma(2-x+(m-1)\varepsilon)}{(-k-r-x-1+(m-1)\varepsilon)\ldots\ldots\ldots\ldots(1-x+(m-1)\varepsilon)}$$

Hence $f(r+x+\varepsilon,\ -k+m\varepsilon)$

$$= \frac{\Gamma(1+m\varepsilon)(-k-r-x-1+(m-1)\varepsilon)\ldots\ldots(1-x+(m-1)\varepsilon)}{\Gamma(1+x+\varepsilon)\Gamma(2-x+(m-1)\varepsilon)(-k+1+m\varepsilon)(-k+m\varepsilon)\ldots(m\varepsilon)(r+x+\varepsilon)(r+x-1+\varepsilon)\ldots(1+x+\varepsilon)}$$

When $m\varepsilon > 0$, $\lim\limits_{\varepsilon \to 0} f(r+x+\varepsilon,\ -k+m\varepsilon) = +\infty$, and

when $m\varepsilon < 0$, $\lim\limits_{\varepsilon \to 0} f(r+x+\varepsilon,\ -k+m\varepsilon) = -\infty$.

Therefore the singularities can not be removed between the lattice points on the singular lines.