CHAPTER IV

THE QUATERNIONS REPRESENT A ROTATION IN FOUR DIMENSIONS

Consider the geometrical rotations in four dimensions as in chapter II.

<u>Rotation 1</u> rotates the WOX plane about the YOZ plane through an angle θ_{12} . The matrix R_1 of chapter II is the matrix of the transformation. It has been stated in chapter II that the quaternion equation (1) in chapter II can represent the rotation in four dimensions. Hence, the matrix R_1 of chapter II must be equal to the matrix B of chapter III. By equating each element we obtain sixteen equations :-

$$\delta d + \alpha a - \beta b - \gamma c = \cos\theta_{12} \quad \ldots\ldots\ldots\ldots(1)$$

$$-\delta a - \alpha d - \beta c + \gamma b = \sin\theta_{12} \quad \ldots\ldots\ldots\ldots(2)$$

$$-\delta b + \alpha c - \beta d - \gamma a = 0 \quad \ldots\ldots\ldots\ldots(3)$$

$$-\delta c - \alpha b + \beta a - \gamma d = 0 \quad \ldots\ldots\ldots\ldots(4)$$

$$\delta a + \alpha d - \beta c + \gamma b = -\sin\theta_{12} \quad \ldots\ldots\ldots\ldots(5)$$

$$\delta d - \alpha a + \beta b + \gamma c = \cos\theta_{12} \quad \ldots\ldots\ldots\ldots(6)$$

$$-\delta c - \alpha b - \beta a + \gamma d = 0 \quad \ldots\ldots\ldots\ldots(7)$$

$$\delta b - \alpha c - \beta d - \gamma a = 0 \quad \ldots\ldots\ldots\ldots(8)$$

$$\delta b + \alpha c + \beta d - \gamma a = 0 \quad \ldots\ldots\ldots\ldots(9)$$

$$\delta c - \alpha b - \beta a - \gamma d = 0 \quad \ldots\ldots\ldots\ldots(10)$$

$$\delta d + \alpha a - \beta b + \gamma c = 1 \quad \ldots\ldots\ldots\ldots(11)$$

$$-\delta a + \alpha d - \beta c - \gamma b = 0 \quad \ldots\ldots\ldots\ldots(12)$$

$$\delta c - \alpha b + \beta a + \gamma d = 0 \quad \ldots\ldots\ldots\ldots(13)$$

$$\delta b - \alpha c + \beta d - \gamma a = 0 \qquad \ldots\ldots\ldots\ldots (14)$$

$$\delta a - \alpha d - \beta c - \gamma b = 0 \qquad \ldots\ldots\ldots\ldots (15)$$

$$\delta d + \alpha a + \beta b - \gamma c = 1 \qquad \ldots\ldots\ldots\ldots (16)$$

From (1) and (6) $$\delta d - \alpha a = \cos\theta_{12} \qquad \ldots\ldots\ldots\ldots (17)$$

and $$-\gamma c - \beta b = 0 \qquad \ldots\ldots\ldots\ldots (18)$$

From (2) and (5) $$\gamma b - \beta c = 0 \qquad \ldots\ldots\ldots\ldots (19)$$

$$-\delta a - \alpha d = \sin\theta_{12} \qquad \ldots\ldots\ldots\ldots (20)$$

From (3) and (9) $$\alpha c - \gamma a = 0 \qquad \ldots\ldots\ldots\ldots (21)$$

$$-\beta d - \delta b = 0 \qquad \ldots\ldots\ldots\ldots (22)$$

From (4) and (13) $$\beta a - \alpha b = 0 \qquad \ldots\ldots\ldots\ldots (23)$$

$$-\gamma d - \delta c = 0 \qquad \ldots\ldots\ldots\ldots (24)$$

From (7) and (10) $$-\alpha b - \beta a = 0 \qquad \ldots\ldots\ldots\ldots (25)$$

$$-\delta c + \gamma d = 0 \qquad \ldots\ldots\ldots\ldots (26)$$

From (8) and (14) $$-\alpha c - \gamma a = 0 \qquad \ldots\ldots\ldots\ldots (27)$$

$$\delta b - \beta d = 0 \qquad \ldots\ldots\ldots\ldots (28)$$

From (11) and (16) $$\delta d + \alpha a = 1 \qquad \ldots\ldots\ldots\ldots (29)$$

$$\gamma c - \beta b = 0 \qquad \ldots\ldots\ldots\ldots (30)$$

From (12) and (15) $$-\beta c - \gamma b = 0 \qquad \ldots\ldots\ldots\ldots (31)$$

$$\alpha d - \delta a = 0 \qquad \ldots\ldots\ldots\ldots (32)$$

From (17) and (29) $$\delta d = \frac{1}{2}(1+\cos\theta_{12}) \qquad \ldots\ldots (33)$$

$$\alpha a = -\frac{1}{2}(\cos\theta_{12}-1) \qquad \ldots\ldots (34)$$

From (18) and (30) $$\beta b = 0 \qquad \ldots\ldots\ldots\ldots (35)$$

$$\gamma c = 0 \qquad \ldots\ldots\ldots\ldots (36)$$

From (19) and (31) $$\beta c = 0 \qquad \ldots\ldots\ldots\ldots (37)$$

$$\gamma b = 0 \qquad \ldots\ldots\ldots\ldots (38)$$

From (20) and (32) $\delta a = -\frac{1}{2}\sin\theta_{12}$ .... (39)

$\alpha d = -\frac{1}{2}\sin\theta_{12}$ .... (40)

From (21) and (27) $\gamma a = 0$ .... (41)

$\alpha c = 0$ .... (42)

From (22) and (28) $\beta d = 0$ .... (43)

$\delta b = 0$ .... (44)

From (23) and (25) $\alpha b = 0$ .... (45)

$\beta a = 0$ .... (46)

From (24) and (26) $\delta c = 0$ .... (47)

$\gamma d = 0$ .... (48)

From (33) and (40) $\delta\alpha d^2 = -\frac{1}{4}\sin\theta_{12}(1+\cos\theta_{12})$

From (34) and (39) $\delta\alpha a^2 = \frac{1}{4}\sin\theta_{12}(\cos\theta_{12}-1)$

From (44) and (45) $\delta\alpha b^2 = 0$

From (42) and (46) $\delta\alpha c^2 = 0$

Then $\delta\alpha(d^2+a^2+b^2+c^2) = -\frac{1}{2}\sin\theta_{12}$ .... (49)

Since the modulus of p is equal to 1,

$$d^2+a^2+b^2+c^2 = 1,$$

and equation (49) becomes

$$\delta\alpha = -\frac{1}{2}\sin\theta_{12}$$

or

$$\alpha = -\frac{\sin\theta_{12}}{2\delta} \qquad \text{.......... (50)}$$

From (33) and (43) $\delta\beta d^2 = 0$

From (39) and (46) $\delta\beta a^2 = 0$

From (35) and (44) $\delta\beta b^2 = 0$

From (37) and (47) $\delta\beta c^2 = 0$

Then $\delta\beta(d^2 + a^2 + b^2 + c^2) = 0$,

or $\delta\beta = 0$

But by (50), $\delta \neq 0$ therefore $\beta = 0i$

From (33) and (48) $\delta\gamma d^2 = 0$

From (39) and (41) $\delta\gamma a^2 = 0$

From (38) and (44) $\delta\gamma b^2 = 0$

From (36) and (47) $\delta\gamma c^2 = 0$

or $\delta\gamma = 0$

Hence $\gamma = 0$

Since, the modulus of p, $|p|$, is equal to 1,

we have $\delta^2 + \alpha^2 + \beta^2 + \gamma^2 = 1$

Substituting the values of α, β, γ form the results above, we get the equation for δ. Solving for the value of δ, we obtain

$$\delta = \pm\sqrt{\frac{1 \pm \cos\theta_{12}}{2}}$$

Substituing this value of δ in (50), we obtain

$$\alpha = \mp\frac{\sin\theta_{12}}{\sqrt{2(1 \pm \cos\theta_{12})}}$$

Then we get the quaternion

$$\Pi = \pm\left(\sqrt{\frac{1 \pm \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 \pm \cos\theta_{12})}}\; i\right)$$

Similarly we find that p is given by

$$p = \pm\left(\sqrt{\frac{1 \pm \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 \pm \cos\theta_{12})}}\; i\right)$$

By substituting in the original equation (1) of chapter II, we find that these two quaternions can be only

$$p = \pm\left[\sqrt{\frac{1 + \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 + \cos\theta_{12})}}\, i\right],$$

$$\pi = \pm\left\{\sqrt{\frac{1 + \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 + \cos\theta_{12})}}\, i\right\},$$

where the signs outside the brackets must be the same. For convenience we choose the plus signs only, because the plus and minus signs give the same transformation. So, the quaternions which represent the rotation 1 are

$$p = \sqrt{\frac{1 + \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 + \cos\theta_{12})}}\, i\ ,$$

and $$\pi = \sqrt{\frac{1 + \cos\theta_{12}}{2}} - \frac{\sin\theta_{12}}{\sqrt{2(1 + \cos\theta_{12})}}\, i\ .$$

<u>Rotation 2</u> rotates the XOY plane about the WOZ plane through an angle θ_{23} . The matrix R_2 of chapter II is the matrix of the transformation. By the same method as above , we obtain the quaternions which represent the rotation 2. They are

$$p = \sqrt{\frac{1 + \cos\theta_{23}}{2}} - \frac{\sin\theta_{23}}{\sqrt{2(1 + \cos\theta_{23})}}\, k\ ,$$

and $$\pi = \sqrt{\frac{1 + \cos\theta_{23}}{2}} + \frac{\sin\theta_{23}}{\sqrt{2(1 + \cos\theta_{23})}}\, k.$$

<u>Rotation 3</u> rotates the YOZ plane about the WOX plane through an angle θ_{34} . The matrix R_3 of chapter II is the matrix of the transformation. By the same method, we obtain the quaternions

which represent the rotation 3. They are

$$p = \sqrt{\frac{1 + \cos\theta_{34}}{2}} - \frac{\sin\theta_{34}}{\sqrt{2(1 + \cos\theta_{34})}}\, i ,$$

and $$\pi = \sqrt{\frac{1 + \cos\theta_{34}}{2}} + \frac{\sin\theta_{34}}{\sqrt{2(1 + \cos\theta_{34})}}\, i .$$

<u>Rotation 4</u> rotates the WOY plane about the XOZ plane through an angle θ_{13}. The matrix R_4 of chapter II is the matrix of the transformation . The quaternions which represent the rotation 4 are

$$p = \sqrt{\frac{1 + \cos\theta_{13}}{2}} - \frac{\sin\theta_{13}}{\sqrt{2(1 + \cos\theta_{13})}}\, j$$

$$\pi = \sqrt{\frac{1 + \cos\theta_{13}}{2}} - \frac{\sin\theta_{13}}{\sqrt{2(1 + \cos\theta_{13})}}\, j .$$

<u>Rotation 5</u> rotates the XOZ plane about the WOY plane through an angle θ_{24}. The matrix R_5 of chapter II is the matrix of the transformation. We get the quaternions which represent the rotation 5 by the same method. They are

$$p = \sqrt{\frac{1 + \cos\theta_{24}}{2}} + \frac{\sin\theta_{24}}{\sqrt{2(1 + \cos\theta_{24})}}\, j ,$$

$$\pi = \sqrt{\frac{1 + \cos\theta_{24}}{2}} - \frac{\sin\theta_{24}}{\sqrt{2(1 + \cos\theta_{24})}}\, j .$$

<u>Rotation 6</u> rotates the WOZ plane about XOY plane through an angle θ_{14} . The matrix R_6 of chapter II is the matrix of the transformation. We get the following quaternions for the rotation

$$p = \sqrt{\frac{1 + \cos \theta_{14}}{2}} + \frac{\sin \theta_{14}}{\sqrt{2(1 + \cos \theta_{14}}} k,$$

$$\pi = \sqrt{\frac{1 + \cos \theta_{14}}{2}} + \frac{\sin \theta_{14}}{\sqrt{2(1 + \cos \theta_{14})}} k .$$