CHAPTER III

THE ORTHOGONAL TRANSFORMATIONS AND THE QUATERNION EQUATIONS.

3.1 The Quaternion Equation $q' = p.q.\bar{p}$, where $|p| = 1$, represents an Orthogonal Transformation in three dimensions.

The equations (8) in chapter I which come from the quaternion equation, can be written in the matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2(ab-cd) & 2(ac+bd) \\ 2(ba+cd) & d^2+b^2-a^2-c^2 & 2(bc-ad) \\ 2(ac-bd) & 2(bc+ad) & d^2+c^2-a^2-b^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

or in brief , it is represented by $X' = AX$.

We consider the case where modulus of p, $|p|$ is equal to 1, or

$$|p| = \sqrt{d^2+a^2+b^2+c^2} = 1.$$

We will show that the matrix of the transformation is the general orthogonal matrix. If it is the general orthogonal matrix, we can express it in the form

$$J\,(I - S)(I + S)^{-1} \qquad \ldots\ldots\ldots\ldots\ (1)$$

where S is a skew symmetric matrix, J is the matrix having ± 1 in each diagonal place and zero elsewhere, and I is the identity matrix. (2, p.164)

Let

$$J_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let

$$A_1 = J_1 A$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d^2+a^2-b^2-c^2 & 2(ab - cd) & 2(ac + bd) \\ 2(ba + cd) & d^2+b^2-a^2-c^2 & 2(bc - ad) \\ 2(ac - bd) & 2(bc + ad) & d^2+c^2-a^2-b^2 \end{bmatrix}$$

$$= \begin{bmatrix} -(d^2+a^2-b^2-c^2) & -2(ab - cd) & -2(ac + bd) \\ -2(ba + cd) & -(d^2+b^2-a^2-c^2) & -2(bc - ad) \\ 2(ac - bd) & 2(bc + ad) & d^2+c^2-a^2-b^2 \end{bmatrix}$$

Then, finding the inverse of the matrix $(I + A_1)$, we obtain :-

$$(I + A_1)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & \frac{d}{c} & \frac{a}{c} \\ \frac{d}{c} & 1 & \frac{b}{c} \\ -\frac{a}{c} & -\frac{b}{c} & 1 \end{bmatrix}$$

Let

$$S = (I - A_1)(I + A_1)^{-1}$$

$$= \begin{bmatrix} 1+(d^2+a^2-b^2-c^2) & 2(ab - cd) & 2(ac + bd) \\ 2(ab + cd) & 1+(d^2+b^2-a^2-c^2) & 2(bc - ad) \\ -2(ac - bd) & -2(bc + ad) & 1-(d^2+c^2-a^2-b^2) \end{bmatrix} \cdot \frac{1}{2} \cdot \begin{bmatrix} 1 & \frac{d}{c} & \frac{a}{c} \\ \frac{d}{c} & 1 & \frac{b}{c} \\ -\frac{a}{c} & -\frac{b}{c} & 1 \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} 0 & -d & a \\ d & 0 & b \\ -a & -b & 0 \end{bmatrix}$$

We can see that S is a skew symmetric matrix as required . Then, finding the inverse of the matrix $(I + S)$, we obtain

$$(1 + S)^{-1} = \begin{bmatrix} b^2 + c^2 & -(ab - cd) & -(ac + bd) \\ -(ab+cd) & a^2 + c^2 & ad - bc \\ ac + bd & ad + bc & c^2 + d^2 \end{bmatrix};$$

$$\text{also } (I - S) = \begin{bmatrix} 1 & \frac{d}{c} & -\frac{a}{c} \\ -\frac{d}{c} & 1 & -\frac{b}{c} \\ \frac{a}{c} & \frac{b}{c} & 1 \end{bmatrix}.$$

Then we have

$$(I-S)(I+S)^{-1} = \begin{bmatrix} 1 & \frac{d}{c} & -\frac{a}{c} \\ -\frac{d}{c} & 1 & -\frac{b}{c} \\ \frac{a}{c} & \frac{b}{c} & 1 \end{bmatrix} \begin{bmatrix} b^2+c^2 & -(ab-cd) & -(ac+bd) \\ -(ab+cd) & a^2+c^2 & ad-bc \\ ac+bd & ad+bc & c^2+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} -(d^2+a^2-b^2-c^2) & -2(ab-cd) & -2(ac+bd) \\ -2(ab+cd) & -(d^2+b^2-a^2-c^2) & -2(bc-ad) \\ 2(ac - bd) & 2(bc + ad) & d^2+c^2-a^2-b^2 \end{bmatrix} = A_1,$$

therefore $J_1 A = A_1 = (I - S)(I + S)^{-1}$,

and $A = J(I - S)(I + S)^{-1}$,

where $J = J_1^{-1}$.

Thus we can express the matrix of the transformation in the form (1), and so the matrix of the transformation is the general orthogonal matrix. It follows that the quaternion equation (5) in chapter (I) represents the general orthogonal transformation in three dimensions.

3.2 The Quaternion Equation $q' = p.q.\Pi$, where $|\Pi| = |p| = 1$, represents an Orthogonal Transformation in four dimensions.

The quaternion equation (1) in chapter II is written in expanded form :-

$$
\begin{aligned}
(w' + x'i + y'j + z'k) &= (d+ai+bj+ck)(w+xi+yj+zk)(\delta + \alpha i + \beta j + \gamma k) \\
&= \Big\{ (dw-ax-by-cz)+(aw+dx+bz-cy)i+(bw+dy+cx-az)j \\
&\qquad +(cw+dz+ay-bx)k \Big\} \cdot (\delta + \alpha i + \beta j + \gamma k)
\end{aligned}
$$

After performing the multiplication and comparing each component we find

$$
\begin{aligned}
w' &= (\delta d - \alpha a - \beta b - \gamma c)w + (-\delta a - \alpha d - \beta c + \gamma b)x \\
&\quad + (-\delta b + \alpha c - \beta d - \gamma a)\, y + (-\delta c - \alpha b + \beta a - \gamma d)\, z \;, \\
x' &= (\alpha d + \delta a + \gamma b - \beta c)\, w + (-\alpha a + \delta d + \gamma c + \beta b)\, x \\
&\quad + (-\alpha b - \delta c + \gamma d - \beta a)\, y + (-\alpha c + \delta b - \gamma a - \beta d)\, z, \\
y' &= (\beta d + \alpha c + \delta b - \gamma a)\, w + (-\beta a - \alpha b + \delta c - \gamma d)\, x \\
&\quad + (-\beta b + \alpha a + \delta d + \gamma c)y + (-\beta c + \alpha d - \delta a - \gamma b)\, z, \\
z' &= (\gamma d + \delta c + \beta a - \alpha b)\, w + (-\gamma a - \delta b + \beta d - \alpha c)\, x \\
&\quad + (-\gamma b + \delta a - \beta c - \alpha d)y + (-\gamma c + \delta d + \beta b + \alpha a)\, z \;.
\end{aligned}
$$

We can write it in matrix form thus

$$\begin{bmatrix} w' \\ \\ x' \\ \\ y' \\ \\ z' \end{bmatrix} = \begin{bmatrix} \delta d-\alpha a & -\delta a-\alpha d & -\delta b+\alpha c & -\delta c-\alpha b \\ -\beta b-\gamma c & -\beta c+\gamma b & -\beta d-\gamma a & +\beta a-\gamma d \\ \alpha d+\delta a & -\alpha a+\delta d & -\alpha b-\delta c & -\alpha c+\delta b \\ +\gamma b-\beta c & +\gamma c+\beta b & +\gamma d-\beta a & -\gamma a-\beta d \\ \beta d+\alpha c & -\beta a-\alpha b & -\beta b+\alpha a & -\beta c+\alpha d \\ +\delta b-\gamma a & +\delta c-\gamma d & +\delta d+\gamma c & -\delta a-\gamma b \\ \gamma d+\delta c & -\gamma a-\delta b & -\gamma b+\delta a & -\gamma c+\delta d \\ +\beta a-\alpha b & +\beta d-\alpha c & -\beta c-\alpha d & +\beta b+\alpha a \end{bmatrix} \begin{bmatrix} w \\ \\ x \\ \\ y \\ \\ z \end{bmatrix} \quad \ldots\ldots(2)$$

or in brief, it is represented by

$$Q' = B\,Q\,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

When the modulus, $|p|$ of the quaternion p, and the modulus, $|q|$ of the quaternion q are both equal to 1, or

$$|p| = \sqrt{d^2+a^2+b^2+c^2} = 1,$$

$$|\pi| = \sqrt{\delta^2+\alpha^2+\beta^2+\gamma^2} = 1,$$

we can show that the matrix B can be expressed in the form (1), by the same method as that in section 3.1. The calculation is very long. Then B is the general orthogonal transformation ; and the quaternion equation (1) in chapter II, represents the general orthogonal transformation in four dimensions when $|p|$ and $|\pi|$ are both equal to 1.