CHAPTER VI

DERIVATION OF THE TRANSFORMATION FOR VELOCITY AND ACCELERATION USING PROPER VELOCITIES AND PROPER ACCELERATIONS.

We shall suppose a body has the proper velocity u' relative to S' and the proper velocity u relative to S, and that S' has the proper velocity v relative to S, then find

$$u' = \phi(u,v).$$

Introduce a third frame of reference S" fixed in the moving body. That is the body has zero velocity in S". We have $u' = dx'/dt''$ and $u = dx/dt''$ and $v = dx/dt'$.

Differentiating equation (1), Chapter V,

$$x' = (1 + v^2/k^2)^{\frac{1}{2}}x - vt \qquad (1)$$

with respect to t" we obtain

$$u' = (1 + v^2/k^2)^{\frac{1}{2}}u - v\,dt/dt'' \qquad (2)$$

and differentiating equation (1), Chapter V,

$$t' = (-vx/k^2) + (1 + v^2/k^2)^{\frac{1}{2}}t$$

with respect to t' we obtain

$$dt/dt' = (1 + v^2/k^2)^{\frac{1}{2}}. \qquad (3)$$

Since dt/dt'' must be the same form as dt/dt', it follows that

$$dt/dt'' = (1 + u^2/k^2)^{\frac{1}{2}},$$ where u is the proper

velocity of S" relative to S.

Substituting $dt/dt'' = (1 + u^2/k^2)^{\frac{1}{2}}$ into (2) we find

$$u' = (1 + v^2/k^2)^{\frac{1}{2}}u - v(1 + u^2/k^2)^{\frac{1}{2}}. \qquad (4)$$

This equation is the transformation for velocity using the proper velocities.

We shall now find the transformation for acceleration using the proper accelerations by differentiating (4) with respect to t".

We obtain

$$a' = \left[(1 + v^2/k^2)^{\frac{1}{2}} - (uv/k^2)(1 + u^2/k^2)^{-\frac{1}{2}} \right] a, \qquad (5)$$

where $a' = du'/dt''$, is the proper acceleration of S" relative to S'

and $a = du/dt''$, is the proper acceleration of S" relative to S.

We shall now verify that when we substitute $\dfrac{\bar{v}}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}$ for v, $\dfrac{\bar{u}}{(1 - \bar{u}^2/k^2)^{\frac{1}{2}}}$ for u and $\dfrac{\bar{u}'}{(1 - \bar{u}'^2/k^2)^{\frac{1}{2}}}$ for u' where

$\bar{v}$ is the coordinate velocity of S' relative to S,

$\bar{u}$ is the coordinate velocity of S" relative to S , and

$\bar{u}'$ is the coordinate velocity of S" relative to S',

into equation (4) we obtain equation (2), Chapter I.

We get

$$\frac{\bar{u}'}{(1 - \bar{u}'^2/k^2)^{\frac{1}{2}}} = \frac{\bar{u} - \bar{v}}{(1 - \bar{u}^2/k^2)^{\frac{1}{2}}(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}$$

Squaring and writing $\bar{u}'$ in terms of $\bar{u}$ and $\bar{v}$ we have

$$\bar{u}' = \frac{\bar{u} - \bar{v}}{1 - \bar{u}\bar{v}/k^2} \quad . \tag{6}$$

Therefore equation (6) is the same as equation (2) , Chapter I when $k = c$.

Similarly, we shall verify that when we substitute $\dfrac{\bar{v}}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}$ for v and $\dfrac{\bar{u}}{(1 - \bar{u}^2/k^2)^{\frac{1}{2}}}$ for u into equation (5) we obtain (3), Chapter I, which becomes

$$a'_x = (1 - \bar{u}\bar{v}/c^2)^{-3}(1 - v^2/c^2)^{3/2} a_x, \tag{7}$$

when we replace u by $\bar{u}$ and v by $\bar{v}$.

Substituting $u = \dfrac{\bar{u}}{(1 - \bar{u}^2/k^2)^{\frac{1}{2}}}$ and $v = \dfrac{\bar{v}}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}$ into equation (5) we get

$$a' = \left(\frac{1 - \bar{u}\bar{v}/k^2}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}\right) a. \tag{8}$$

Now

$$a' = d^2x'/dt''^2 = d^2x'/dt'^2.(dt'/dt'')^2 = a_x'(dt'/dt'')^2$$
$$a = d^2x/dt''^2 = d^2x/dt^2.(dt/dt'')^2 = a_x(dt/dt'')^2 \qquad (9)$$

and differentiating the equation

$$t' = -vx/k^2 + (1 + v^2/k^2)^{\frac{1}{2}}t$$

with respect to t", we have

$$dt'/dt'' = \left[-v/k^2 dx/dt + (1 + v^2/k^2)^{\frac{1}{2}}\right] dt/dt''$$

or

$$dt'/dt'' = \left[-v\bar{u}/k^2 + (1 + v^2/k^2)^{\frac{1}{2}}\right] dt/dt''. \qquad (10)$$

Substituting $\dfrac{\bar{v}}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}$ for v into (10) we get

$$dt'/dt'' = \left(\frac{1 - \bar{u}\bar{v}/k^2}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}\right) dt/dt'' .$$

Therefore

$$(dt'/dt'')^2 = \left(\frac{1 - \bar{u}\bar{v}/k^2}{(1 - \bar{v}^2/k^2)^{\frac{1}{2}}}\right)^2 . (dt/dt'')^2. \qquad (11)$$

Substituting equations (9) and (11) into (8) we obtain

$$a_x' = (1 - \bar{u}\bar{v}/k^2)^{-3}(1 - \bar{v}^2/k^2)^{3/2} a_x . \qquad (12)$$

Hence equation (12) is the same as equation (7) when $k = c$.