เสถียรภาพของสมการเชิงฟังก์ชันเฟรเชแบบมีเงื่อนไข

นางสาวทิพย์ภาภรณ์ อึ้งรัศมี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2554

STABILITY OF CONDITIONAL FRÉCHET FUNCTIONAL EQUATION

Miss Tippaporn Eungrasamee

A Dissertation Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy Program in Mathematics

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2011

Thesis Title	STABILITY OF CONDITIONAL FRÉCHET FUNCTIONAL EQUATION
By	Miss Tippaporn Eungrasamee
Field of Study	Mathematics
Thesis Advisor	Associate Professor Paisan Nakmahachalasint, Ph.D.
Thesis Co-advisor	Associate Professor Patanee Udomkavanich, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

....Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

....Chairman
(Assistant Professor Nataphan Kitisin, Ph.D.)

....Thesis Advisor
(Associate Professor Paisan Nakmahachalasint, Ph.D.)

....Thesis Co-advisor
(Associate Professor Patanee Udomkavanich, Ph.D.)

....Examiner
(Keng Wiboontan, Ph.D.)

....Examiner
(Khamron Mekchay, Ph.D.)

....External Examiner
(Professor Vichian Laohakosol, Ph.D.)

ทิพย์ภาภรณ์ อึ้งรัศมี : เสถียรภาพของสมการเชิงฟังก์ชันเฟรเชแบบมีเงื่อนไข. (STABILITY OF CONDITIONAL FRÉCHET FUNCTIONAL EQUATION) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร. ไพศาล นาคมหาชลาสินธุ์, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : รศ. ดร. พัฒนี อุดมกะวานิช, 38 หน้า.

ในวิทยานิพนธ์เล่มนี้ เราจะพิสูจน์เสถียรภาพแบบไฮเออร์ส-อูแลมของสมการเชิงฟังก์ชันเฟรเชก่อน แล้วจึงพิสูจน์เสถียรภาพนั้นเมื่อกำกัดช่วงแผ่

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..
สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก................................
ปีการศึกษา 2554 ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม................................

5173815723 : MAJOR MATHEMATICS
KEYWORDS : HYERS-ULAM STABILITY / FRÉCHET FUNCTIONAL EQUATION / FINITE DIFFERENCE EQUATION / CONDITIONAL FUNCTIONAL EQUATION

TIPPAPORN EUNGRASAMEE : STABILITY OF CONDITIONAL FRÉCHET FUNCTIONAL EQUATION. ADVISOR : ASSOC. PROF. PAISAN NAKMAHACHALASINT, Ph.D., CO-ADVISOR : ASSOC. PROF. PATANEE UDOMKAVANICH, Ph.D., 38 pp.

In this dissertation, we will first prove the Hyers-Ulam stability of the Fréchet functional equation, and then we will prove such stability when the span is restricted.

Department	: Mathematics and Computer Science	Student's Signature	....
Field of Study	: Mathematics	Advisor's Signature	....
Academic Year:	2011	Co-advisor's Signature	....

ACKNOWLEDGEMENTS

This dissertation could not have been completed without Associate Professor Dr. Paisan Nakmahachalasint, who not only served as my thesis advisor but also encouraged and guided me through the dissertation process. I also would like to acknowledge and extend my heartfelt gratitude to my advisor, Associate Professor Dr. Patanee Udomkavanich, for her vital encouragement and helpful advice throughout my academic programme. Besides, I would like to express my gratitude to all dissertation committee: Assistant Professor Dr. Nataphan Kitisin, Dr. Keng Wiboontan, Dr. Khamron Mekchay, and Professor Dr. Vichian Laohakosol. Furthermore, I thank all teachers who have instructed and taught me all valuable knowledge.

A special thank to the Development and Promotion of Science and Technology Talents (DPST) project for the financial support throughout my study.

Most especially to my family and friends, words alone cannot express what I owe them for their understanding and whose patient love enabled me to complete this dissertation.

CONTENTS

CHAPTER I

INTRODUCTION

This chapter will provide some background in functional equations, stability problems of functional equations, as well as the motivation of our proposed problem.

1.1 Functional Equations

A functional equation is simply an equation whose unknowns are functions. To solve a functional equation is to seek all possible functions satisfying the equation.

Example 1.1. In order to find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4 \text{ for all } x \in \mathbb{R}, \tag{1.1}$$

we will start with replacing x by $1-x$ to get

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4 \text{ for all } x \in \mathbb{R}.$$

Substituting $f(1-x) = 2x - x^4 - x^2 f(x)$ from (1.1) into the above equation and solving for $f(x)$, we obtain $f(x) = 1 - x^2$ for all $x \in \mathbb{R}$.

On the contrary, if $f(x) = 1 - x^2$, then

$$x^2 f(x) + f(1-x) = x^2(1-x^2) + (1-(1-x)^2) = 2x - x^4,$$

which asserts that the functional equation (1.1) holds for all $x \in \mathbb{R}$. Therefore, the solution of (1.1) is the function f given by $f(x) = 1 - x^2$ for all $x \in \mathbb{R}$. □

One of the most studied functional equations is the additive functional equation:

$$f(x+y) = f(x) + f(y). \tag{1.2}$$

In the pioneering paper by A.L.Cauchy [4] in 1821, it was shown that all continuous solutions of (1.2) on $\mathbb{R}$ are linear functions given by $f(x) = cx$ for all $x \in \mathbb{R}$, where c is a real constant. This additive functional equation was later known as the *Cauchy functional equation.* An existence of a nonlinear additive function on $\mathbb{R}$ was not realized before 1905, when G. Hamel [9] constructed the general solution of (1.2) using a Hamel basis over $\mathbb{Q}$. A remarkable result regarding the nonlinear additive functions is that the graph $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ is a dense subset of $\mathbb{R}^2$ (see [10]). In other words, for all $\varepsilon > 0$ and for all $(x, y) \in \mathbb{R}^2$, there exists a point $(a, f(a)) \in G(f)$ such that $(x - a)^2 + (y - f(a))^2 < \varepsilon^2$, which indicates that the graph of a nonlinear additive functions must consist of points scattered all over the plane $\mathbb{R}^2$.

A functional equation closely related to (1.2) is the Jensen functional equation:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}. \tag{1.3}$$

It should be noted that (1.3) is invariant under an introduction of a constant; i.e. if we define a function g by $g(x) = f(x) - c$, where c is a constant, then g still satisfies (1.3). It is known (see [5]) that the general solution of (1.3) is of the form $f(x) = c + A(x)$, where A is a solution of (1.2).

We may think of $c + A(x)$ as a generalized polynomial function of degree 1. In an attempt to generalize the result to higher degrees, M. Fréchet [7] studied a functional equation written in terms of a difference operator with a span h:

$$\Delta_h^{n+1} f(x) = 0, \tag{1.4}$$

where n is a nonnegative integer. More precisely, Fréchet showed that a continuous solution $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.4) if and only if f is a polynomial function of degree at most n, i.e. $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. Hence, (1.4) will be referred to as the *Fréchet functional equation.*

1.2 Stability Problems of Functional Equations

In 1940, the problem of stability of functional equations was first introduced by S.M. Ulam [25] during his talk delivered to the Mathematics Club of the University of Wisconsin. Ulam proposed the following question: *"Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(f(x), H(x)) < \varepsilon$ for all $x \in G_1$?"*

In the following year, this stability problem was affirmatively answered by D.H. Hyers [11] that for a mapping f between Banach spaces E_1 and E_2, if f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$ and for some $\varepsilon > 0$, then there will exist a unique additive mapping $A : E_1 \to E_2$ satisfying the inequality

$$\|A(x) - f(x)\| \leq \varepsilon.$$

The mapping A may be formed from $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$. The stability in this sense was later recognized as the *Hyers-Ulam stability* and became one of fundamental concepts of the stability theory of functional equations.

Another inspiring work regarding the stability problem of functional equation is the work by Th.M. Rassias [19] in 1978 where he proved that if f satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \in [0, \infty)$ and $p \in [0, 1)$, then there is a unique additive mapping $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p .$$

This type of stability is known as the *Hyers-Ulam-Rassias stability.* The stability theorem then has been generalized to the case that the bound of the Cauchy difference is a function with some certain conditions by Gãvruta [8] in 1994.

1.3 Motivation and Proposed Problem

The stability problem of functional equations has gained wide popularity from researchers over the past few decades. Nonetheless, to the best of our knowledge, stability of polynomial-type functional equations were studied up to those of order not exceeding 4. This motivates us to develop a new technique to prove the stability of a functional equation which its solution is a generalized polynomial of an arbitrary order.

The main purpose of this dissertation is to investigate the Hyers-Ulam stability of the Fréchet functional equation (1.4) and its stability when the span h is restricted to the region $\|h\| > a$ for a given positive real number a.

CHAPTER II

PRELIMINARIES

This chapter covers some basic theorems and lemmas concerning the difference operator and multi-additive functions. Throughout this chapter, we let X and Y be linear spaces and $f : X \to Y$ be an arbitrary function.

We shall begin with some basic definitions related to our work.

Definition 2.1. If a function $A : X \to Y$ satisfies the property

$$A(x+y) = A(x) + A(y) \tag{2.1}$$

for all $x, y \in X$, then A is called an *additive function*.

Definition 2.2. Define the *difference operator* Δ_h with the span $h \in X$ by

$$\Delta_h f(x) = f(x+h) - f(x) \quad \text{for all } x \in X.$$

For each positive integer n, we define the iterates Δ_h^n by the recurrence

$$\Delta_h^{n+1} f = \Delta_h(\Delta_h^n f).$$

We may also write the iterated operators $\Delta_{h_1} \cdots \Delta_{h_n}$ shortly as $\Delta_{h_1 \ldots h_n}$.

Lemma 2.3. *[5] The difference operators commute; that is, for all $h_1, h_2 \in X$,*

$$\Delta_{h_1}\Delta_{h_2} f = \Delta_{h_2}\Delta_{h_1} f.$$

Lemma 2.4. *[5] For all $h_1, h_2 \in X$,*

$$\Delta_{h_1+h_2} f = \Delta_{h_1} f + \Delta_{h_2} f + \Delta_{h_1}\Delta_{h_2} f.$$

We can also express the term $\Delta_h^n f(x)$ in the form of the summation of f.

Lemma 2.5. *[5] Let n be a positive integer. For all $x, h \in X$,*

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh). \tag{2.2}$$

In a similar way, an n-additive function can be defined as in the following.

Definition 2.6. Let n be a positive integer. A function $A_n : X \to Y$ will be called an *n-additive function* if it is additive in each of its arguments; i.e., for each $1 \leq i \leq n$ and for all $x_1, \ldots, x_n, y_i \in X$,

$$A_n(x_1, \ldots, x_i + y_i, \ldots, x_n) = A_n(x_1, \ldots, x_i, \ldots, x_n) + A_n(x_1, \ldots, y_i, \ldots, x_n).$$

Lemma 2.7. *[5] Let $A_n : X^n \to Y$ be an n-additive function, where n is a positive integer, and let r be a rational number. For all $x_1, \ldots, x_n \in X$.*

$$A_n(x_1, \ldots, rx_i, \ldots, x_n) = rA_n(x_1, \ldots, x_i, \ldots, x_n). \tag{2.3}$$

In particular when $r = 0$, $A_n(x_1, \ldots, 0, \ldots, x_n) = 0$.

Definition 2.8. The *diagonalization* of an n-additive function $A_n : X^n \to Y$, where n is a positive integer, is the function $A^n : X \to Y$ defined by

$$A^n(x) = A_n(x, \ldots, x) \quad \text{for all } x \in X. \tag{2.4}$$

Lemma 2.9. *[5] Let $A^n : X \to Y$ be the diagonalization of an n-additive function, where n is a positive integer, and let r be a rational number. For all $x \in X$,*

$$A^n(rx) = r^n A^n(x).$$

Definition 2.10. A function $f : X^n \to Y$ will be called *symmetric* if it is invariant under a permutation of its arguments; that is,

$$f(x_1', \ldots, x_n') = f(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$ and $(x_1', \ldots, x_n')$ denotes any permutation of $(x_1, \ldots, x_n)$.

Theorem 2.11. *[5] Let $A^n : X \to Y$ be the diagonalization of a symmetric n-additive function, where n is a positive integer, and let $m \geq n$ be an integer. Then, for all $x, h_1, \ldots, h_m \in X$,*

$$\Delta_{h_1 \cdots h_m} A^n(x) = \begin{cases} n! A_n(h_1, \ldots, h_n) & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

The following remarkable theorem by M. Kuczma [16] shows that $\Delta_{h_1 \ldots h_n} f(x)$ can be rewritten in terms of $\Delta_h^n f(x)$ for various values of x and h.

Theorem 2.12. *[16] Let $f : X \to Y$ be a function and let $h_1, \ldots, h_n \in X$ be arbitrary. For $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, define*

$$\alpha_{\varepsilon_1 \ldots \varepsilon_n} = -\sum_{r=1}^{n} \frac{\varepsilon_r h_r}{r}$$

and

$$b_{\varepsilon_1 \ldots \varepsilon_n} = -\sum_{r=1}^{n} \varepsilon_r h_r.$$

Then for every $x \in X$,

$$\Delta_{h_1, \ldots, h_n} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} (-1)^{\varepsilon_1 + \ldots + \varepsilon_n} \Delta^n_{\alpha_{\varepsilon_1 \ldots \varepsilon_n}} f\left(x + b_{\varepsilon_1 \ldots \varepsilon_n}\right).$$

Definition 2.13. Let n be a nonnegative integer. A function $f : X \to Y$ which satisfies

$$\Delta_h^{n+1} f(x) = 0,$$

for all $x, h \in X$, will be called a *polynomial function of order n*.

Theorem 2.14. *[16] If $f : X \to Y$ is a polynomial function of order n, then, for all $x, h_1, \ldots, h_{n+1} \in X$,*

$$\Delta_{h_1, \ldots, h_{n+1}} f(x) = 0.$$

The following theorem give us the general solution of the Fréchet functional equation.

Theorem 2.15. *[5] Let n be a nonnegative integer. A function $f : X \to Y$ is a polynomial function of order n, i.e. $\Delta_h^{n+1} f(x) = 0$ for all $x \in X$ and for all $h \in X$,* *then there exist symmetric k-additive functions $A_k : X^k \to Y,\ k = 0, 1, \ldots, n$ whose diagonalizations $A^k : X \to Y$ satisfy*

$$f(x) = A_0 + \sum_{k=1}^{n} A^k(x).$$

CHAPTER III
STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

In this chapter, we will investigate the stability of an n-dimensional quadratic functional equation. The first section proves its general solution and the following section will demonstrate its stability which elucidates the fundamental concepts in our main work.

3.1 General Solution of Quadratic Functional Equations

One way to generalize a functional equation is to generalize the number of arguments appearing in the functional equation. Here we take the classical quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{3.1}$$

where its solution is the diagonalization of a bi-additive function, and we attempt to generalize it to the following form:

$$\sum_{1\leq i<j\leq n}\left(f(x_i+x_j)+f(x_i-x_j)\right)=2(n-1)\sum_{i=1}^{n}f(x_i) \tag{3.2}$$

where $n>1$ is a given integer.

In the following theorem, we will show that the functional equation (3.1) and (3.2) are equivalent; i.e. they possess the same set of solutions.

Theorem 3.1. *Let X and Y be vector spaces. A mapping $f:X\to Y$ satisfies the functional equation (3.2) where $n>1$, for all $x_1,\ldots,x_n\in X$, if and only if it satisfies the quadratic functional equation (3.1) for all $x,y\in X$.*

Proof. (Necessity) Putting $x_1 = x_2 = \ldots = x_n = 0$ in (3.2) yields

$$(n-1)\, nf(0) = 2\,(n-1)\, nf(0)\,.$$

Since $n > 1$, we have

$$f(0) \quad = \quad 0 \tag{3.3}$$

Then, setting $x_1 = x,\ x_2 = y$, and $x_3 = x_4 = \ldots = x_n = 0$ in (3.2), we have

$$\begin{aligned} f(x+y) + f(x-y) + 2\,(n-2)\, f(x) + 2\,(n-2)\, f(y) + (n-2)\,(n-3)\, f(0) \\ = 2\,(n-1)\,(f(x) + f(y))\,. \end{aligned}$$

Using (3.3) ensures the validity of (3.1).

(Sufficiency) Assume (3.1) holds. Then,

$$\begin{aligned} \sum_{1\le i<j\le n} (f(x_i + x_j) + f(x_i - x_j)) &= \sum_{1\le i<j\le n} (2f(x_i) + 2f(x_j)) \\ &= 2\,(n-1) \sum_{i=1}^{n} f(x_i). \end{aligned}$$

This completes the proof. $\square$

3.2 Generalized Stability

Throughout this section X and Y will be a real normed vector space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x_1, x_2, \ldots, x_n) := \sum_{1\le i<j\le n} (f(x_i + x_j) + f(x_i - x_j)) \ - \ 2\,(n-1) \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \ldots, x_n \in X$. At once, we prove the stability theorem of an n-dimensional quadratic functional equation.

Theorem 3.2. *Let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\begin{cases} \sum\limits_{i=0}^{\infty} 4^{-i} \phi(2^i x, 2^i x, 0, \ldots, 0) \quad \textit{converges for all } x \in X, \textit{ and} \\ \lim\limits_{m\to\infty} 4^{-m} \phi(2^m x_1, 2^m x_2, \ldots, 2^m x_n) \ = \ 0 \quad \textit{for all } x_1, \ldots, x_n \in X, \end{cases} \tag{3.4}$$

or

$$\begin{cases} \sum\limits_{i=0}^{\infty} 4^{i}\phi\left(2^{-i}x, 2^{-i}x, 0, \ldots, 0\right) & converges\, for\, all\, x \in X,\, and \\ \lim\limits_{m\to\infty} 4^{m}\phi\left(2^{-m}x_1, 2^{-m}x_2, \ldots, 2^{-m}x_n\right) \;=\; 0 & for\, all\, x_1, \ldots, x_n \in X. \end{cases} \tag{3.5}$$

If a function $f : X \to Y$ satisfies the inequality

$$\|Df\left(x_1, x_2, \ldots, x_n\right)\| \leq \phi\left(x_1, x_2, \ldots, x_n\right) \tag{3.6}$$

for all $x_1, \ldots, x_n \in X$, and, in addition, $f(0) = 0$ if (3.5) holds, then there is a unique function $Q : X \to Y$ such that Q satisfies (3.2) and, for all $x \in X$,

$$\left\| f\left(x\right) + \tfrac{n^2-n-3}{3} f\left(0\right) - Q\left(x\right) \right\| \leq \begin{cases} \frac{1}{4}\sum\limits_{i=0}^{\infty} 4^{-i}\phi\left(2^{i}x, 2^{i}x, 0, \ldots, 0\right) & if\, (3.4)\, holds \\ \frac{1}{4}\sum\limits_{i=1}^{\infty} 4^{i}\phi\left(2^{-i}x, 2^{-i}x, 0, \ldots, 0\right) & if\, (3.5)\, holds \end{cases} \tag{3.7}$$

The function Q is given by

$$Q\left(x\right) = \begin{cases} \lim\limits_{m\to\infty} 4^{-m} f\left(2^{m}x\right) & if\, (3.4)\, holds \\ \lim\limits_{m\to\infty} 4^{m} f\left(2^{-m}x\right) & if\, (3.5)\, holds \end{cases} \tag{3.8}$$

for all $x \in X$.

Proof. We will first prove the case when condition (3.4) holds. Let $g : X \to Y$ be the function defined by $g\left(x\right) := f\left(x\right) + \frac{n^2-n-3}{3} f\left(0\right)$ for all $x \in X$. Putting $x_1 = x_2 = x$ and $x_3 = \ldots = x_n = 0$ in (3.6) yields

$$\begin{aligned} \|g\left(2x\right) - 4g\left(x\right)\| &= \|f\left(2x\right) - 4f\left(x\right) - \left(n^2 - n - 3\right) f\left(0\right)\| \\ &= \|Df\left(x, x, 0, \ldots, 0\right)\| \\ &\leq \phi\left(x, x, 0, \ldots, 0\right) \end{aligned}$$

for all $x \in X$. Dividing the above relation by 4 yields

$$\left\| \frac{g\left(2x\right)}{4} - g\left(x\right) \right\| \leq \frac{1}{4}\phi\left(x, x, 0, \ldots, 0\right). \tag{3.9}$$

Therefore,

$$\begin{aligned} \left\| \tfrac{g\left(2^{m}x\right)}{4^{m}} - g\left(x\right) \right\| &= \left\| \sum_{i=0}^{m-1} \left(\tfrac{g\left(2^{i+1}x\right)}{4^{i+1}} - \tfrac{g\left(2^{i}x\right)}{4^{i}} \right) \right\| \\ &\leq \sum_{i=0}^{m-1} \tfrac{1}{4^{i}} \left\| \tfrac{g\left(2\cdot 2^{i}x\right)}{4} - g\left(2^{i}x\right) \right\| \\ &\leq \tfrac{1}{4} \sum_{i=0}^{m-1} 4^{-i}\phi\left(2^{i}x, 2^{i}x, 0, \ldots, 0\right) \end{aligned} \tag{3.10}$$

for any positive integer m and for all $x \in X$.

We will show that the sequence $\left\{\frac{g(2^m x)}{4^m}\right\}$ converges for all $x \in X$. For every positive integer l and m, we consider

$$\begin{aligned}\left\|\frac{g\left(2^{l+m}x\right)}{4^{l+m}} - \frac{g(2^m x)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{g\left(2^l \cdot 2^m x\right)}{4^l} - g\left(2^m x\right)\right\| \\ &\leq \frac{1}{4^m}\sum_{i=0}^{l-1} 4^{-i}\frac{1}{4}\phi\left(2^i \cdot 2^m x, 2^i \cdot 2^m x, 0, \ldots, 0\right) \\ &= \frac{1}{4}\sum_{i=0}^{l-1} 4^{-(i+m)}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right).\end{aligned}$$

By condition (3.4), the right-hand side approaches 0 when m tends to infinity. Thus, the sequence $\left\{\frac{g(2^m x)}{4^m}\right\}$ is a Cauchy sequence. Since a Banach space is complete, we can define

$$Q(x) \quad := \quad \lim_{m\to\infty}\frac{g\left(2^m x\right)}{4^m}$$

for all $x \in X$. Consequently, by passing to the limit in (3.10) when m goes to infinity, it follows that

$$\|Q(x) - g(x)\| \leq \frac{1}{4}\sum_{i=o}^{\infty} 4^{-i}\phi\left(2^i x, 2^i x, 0, \ldots, 0\right)$$

for all $x \in X$. This inequality implies the validity of (3.7). Moreover, let $x_1, \ldots, x_n$ be any points in X. We have

$$\begin{aligned}\|DQ(x_1, \ldots, x_n)\| &= \lim_{m\to\infty} 4^{-m}\left\|Dg\left(2^m x_1, \ldots, 2^m x_n\right)\right\| \\ &\leq \lim_{m\to\infty} 4^{-m}\left(\left\|Df\left(2^m x_1, \ldots, 2^m x_n\right)\right\| + \left|\frac{n(n-1)\left(n^2-n-3\right)}{3}\right|\|f(0)\|\right) \\ &\leq \lim_{m\to\infty} 4^{-m}\phi\left(2^m x_1, \ldots, 2^m x_n\right).\end{aligned}$$

Using condition (3.4) the right-hand side tends to 0. Hence, Q satisfies (3.2) for all $x_1, \ldots, x_n \in X$ which implies that Q is a quadratic function. It should be noted that $Q(ax) = a^2 Q(x)$ for every positive integer a and for all $x \in X$. [5]

Now, we prove the uniqueness of Q. Let $Q' : X \to Y$ be another function satisfying (3.2) and (3.4). Therefore,

$$\begin{aligned}\|Q'(x) - Q(x)\| &= \frac{1}{4^m}\left\|Q'\left(2^m x\right) - Q\left(2^m x\right)\right\| \\ &\leq \frac{1}{4^m}\left\|Q'\left(2^m x\right) - g\left(2^m x\right)\right\| + \frac{1}{4^m}\left\|g\left(2^m x\right) - Q\left(2^m x\right)\right\| \\ &\leq 2 \cdot \frac{1}{4^m}\frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right) \\ &\leq \frac{1}{2}\sum_{i=0}^{\infty} 4^{-(i+m)}\phi\left(2^{i+m}x, 2^{i+m}x, 0, \ldots, 0\right)\end{aligned}$$

for all $x \in X$. By condition (3.4), the right-hand side goes to 0 as m tends to infinity, and it follows that $Q'(x) = Q(x)$ for all $x \in X$. Hence, Q is unique.

For the case that condition (3.5) holds, we can state the proof in a similar manner as in the case which the condition (3.4) holds with the additional condition, $f(0) = 0$. Starting by replacing x with $\frac{x}{2}$ in (3.9) and multiplying by 4, we get

$$\left\| g(x) - 4g\left(\frac{x}{2}\right) \right\| \leq \phi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

for all $x \in X$. This inequality can be extended by mathematical induction to

$$\begin{aligned} \left\| g(x) - 4^m g\left(\tfrac{x}{2^m}\right) \right\| &\leq \sum_{i=0}^{m-1} 4^i \phi\left(\tfrac{x}{2^{i+1}}, \tfrac{x}{2^{i+1}}, 0, \ldots, 0\right) \\ &\leq \tfrac{1}{4} \sum_{i=1}^{m} 4^i \phi\left(2^{-i}x, 2^{-i}x, 0, \ldots, 0\right) \end{aligned}$$

for any positive integer m and for all $x \in X$.

We can show that a sequence $\left\{4^m g\left(\frac{x}{2^m}\right)\right\}$ converges for all $x \in X$ and let

$$Q(x) := \lim_{m \to \infty} 4^m f\left(2^{-m}x\right)$$

for all $x \in X$. The rest of the proof is similar to the corresponding part of the proof of the previous case. Thus, it will be omitted. □

Therefore, we now obtain the stability of an n-dimensional quadratic functional equation in the Hyers-Ulam and the Hyers-Ulam-Rassias senses, respectively.

Corollary 3.3. *If a function $f : X \to Y$ satisfies the inequality*

$$\| Df(x_1, \ldots, x_n) \| \leq \varepsilon$$

for all $x_1, \ldots, x_n \in X$ and for some real number $\varepsilon > 0$, then there exists a unique function $Q : X \to Y$ such that Q satisfies (3.2) and

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{\varepsilon}{3}$$

for all $x \in X$.

Proof. We choose $\phi(x_1, \ldots, x_n) = \varepsilon$ for all $x_1, \ldots, x_n \in X$. Being in condition (3.4) in Theorem 3.2, it follows that

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{\varepsilon}{4^i} = \frac{\varepsilon}{3}$$

for all $x \in X$ as desired. □

Corollary 3.4. *Given positive real number ε and p with $p \neq 2$. If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \ldots, x_n)\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$, then there exists a unique function $Q : X \to Y$ such that Q satisfies (3.2) and

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{\varepsilon}{|2 - 2^{p-1}|} \|x\|^p$$

for all $x \in X$.

Proof. We choose $\phi(x_1, \ldots, x_n) = \varepsilon \sum\limits_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. If $0 < p < 2$, then condition (3.4) in Theorem 3.2 is fulfilled, and consequently

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \varepsilon \cdot 2 \left\| 2^i x \right\|^p = \frac{\varepsilon}{2 - 2^{p-1}} \|x\|^p$$

for all $x \in X$. If $p > 2$, the condition (3.5) in Theorem 3.2 is fulfilled, and consequently

$$\left\| f(x) + \tfrac{n^2-n-3}{3} f(0) - Q(x) \right\| \leq \frac{1}{4} \sum_{i=1}^{\infty} 4^{i} \varepsilon \cdot 2 \left\| 2^{-i} x \right\|^p = \frac{\varepsilon}{2^{p-1} - 2} \|x\|^p$$

for all $x \in X$. $\square$

CHAPTER IV

STABILITY OF FRÉCHET FUNCTIONAL EQUATIONS

This chapter begins with the Hyers-Ulam stability of the Fréchet functional equation and will move on to the stability problem with a restricted span.

4.1 Hyers-Ulam Stability of Fréchet Functional Equations

In this section, we will explore the stability of the Fréchet functional equation

$$\Delta_y^{n+1} f(x) = 0 \tag{4.1}$$

where n is a nonnegative integer.

To improve the readability, the first subsection will prove some lemmas that will be helpful for the stability theorem and the following subsection will complete the proof of the stability.

4.1.1 Auxiliary Lemmas

Throughout the section, we shall let X be a linear space and let Y be a Banach space.

Lemma 4.1. *Let $\varepsilon > 0$. Let $f : X \to Y$ be a function such that $\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon$ for all $x, h \in X$. For arbitrary $h_1, \ldots, h_{n+1} \in X$ and for every $x \in X$,*

$$\left\|\Delta_{h_1,\ldots,h_{n+1}} f(x)\right\| \leq 2^{n+1}\varepsilon. \tag{4.2}$$

Proof. Utilize Theorem 2.12 and apply the bound $\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon$, we immediately get the desired result. $\square$

Lemma 4.2. *Let n be a positive integer and let $\varepsilon > 0$. Let $f : X \to Y$ be a function such that $\left\|\Delta_h^{n+1} f(x)\right\| \le \varepsilon$ for all $x, h \in X$. Define a function $g : X^n \to Y$ by*

$$g(x_1, \ldots, x_n) = \frac{1}{n!}\Delta_{x_1,\ldots,x_n} f(0) \quad \textit{for all } x_1, \ldots, x_n \in X.$$

Then g is symmetric and, for every $x_1, \ldots, x_n, y \in X$ and for each $1 \le i \le n$,

$$\|g(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\| \\ \le \frac{2^{n+1}}{n!}\varepsilon.$$

Moreover, for every $x_1, \ldots, x_n \in X$ and for each positive integer m,

$$\left\|\frac{g(2^m x_1, \ldots, 2^m x_n)}{2^{mn}} - g(x_1, \ldots, x_n)\right\| \le \frac{2^{n+1}}{n!}\varepsilon.$$

Proof. Since the difference operators commute, it immediately follows that g is symmetric. By the definition of g along with Lemma 2.4, for every $x_1, \ldots, x_n, y \in X$,

$$\begin{aligned} &\|g(x_1 + y, x_2, \ldots, x_n) - g(x_1, x_2, \ldots, x_n) - g(y, x_2, \ldots, x_n)\| \\ &= \frac{1}{n!}\left\|\Delta_{x_2 \ldots x_n}(\Delta_{x_1+y} f(0) - \Delta_{x_1} f(0) - \Delta_y f(0))\right\| \\ &= \frac{1}{n!}\left\|\Delta_{x_2 \ldots x_n}\Delta_{x_1}\Delta_y f(0)\right\|. \end{aligned}$$

From Lemma 4.1, $\|\Delta_{x_2 \ldots x_n}\Delta_{x_1}\Delta_y f(0)\| \le 2^{n+1}\varepsilon$. Hence,

$$\|g(x_1 + y, x_2, \ldots, x_n) - g(x_1, x_2, \ldots, x_n) - g(y, x_2, \ldots, x_n)\| \le \frac{2^{n+1}}{n!}\varepsilon. \tag{4.3}$$

Recalling the symmetry of g, we can conclude that, for every $x_1, \ldots, x_n, y \in X$ and for each $1 \le i \le n$,

$$\|g(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\| \\ \le \frac{2^{n+1}}{n!}\varepsilon.$$

If we let $y = x_i$, the above inequality turns to

$$\|g(x_1, \ldots, x_{i-1}, 2x_i +, x_{i+1}, \ldots, x_n) - 2g(x_1, \ldots, x_n)\| \le \frac{2^{n+1}}{n!}\varepsilon. \tag{4.4}$$

Now consider

$$\|g(2x_1, \ldots, 2x_n) - 2^n g(x_1, \ldots, x_n)\|$$
$$= \left\| \sum_{k=1}^{n} \left(2^{n-k} g(2x_1, \ldots, 2x_k, x_{k+1}, \ldots, x_n) - 2^{n-(k-1)} g(2x_1, \ldots, 2x_{k-1}, x_k, \ldots, x_n) \right) \right\|$$
$$\leq \sum_{k=1}^{n} 2^{n-k} \|g(2x_1, \ldots, 2x_k, x_{k+1}, \ldots, x_n) - 2g(2x_1, \ldots, 2x_{k-1}, x_k, \ldots, x_n)\|$$

Applying (4.4), we will have, for each k,

$$\|g(2x_1, \ldots, 2x_k, x_{k+1}, \ldots, x_n) - 2g(2x_1, \ldots, 2x_{k-1}, x_k, \ldots, x_n)\| \leq \frac{2^{n+1}}{n!} \varepsilon.$$

Therefore,

$$\|g(2x_1, \ldots, 2x_n) - 2^n g(x_1, \ldots, x_n)\| \leq \sum_{k=1}^{n} 2^{n-k} \left(\frac{2^{n+1}}{n!} \varepsilon \right) = \frac{2^{n+1}(2^n - 1)}{n!} \varepsilon.$$

That is, for every $x_1, \ldots, x_n \in X$,

$$\left\| \frac{g(2x_1, \ldots, 2x_n)}{2^n} - g(x_1, \ldots, x_n) \right\| \leq \frac{2(2^n - 1)}{n!} \varepsilon. \tag{4.5}$$

Let m be a positive integer. Consider

$$\left\| \frac{g(2^m x_1, \ldots, 2^m x_n)}{2^{mn}} - g(x_1, \ldots, x_n) \right\|$$
$$= \left\| \sum_{i=0}^{m-1} \left(\frac{g(2^{i+1} x_1, \ldots, 2^{i+1} x_n)}{2^{(i+1)n}} - \frac{g(2^i x_1, \ldots, 2^i x_n)}{2^{in}} \right) \right\|$$
$$\leq \sum_{i=0}^{m-1} \frac{1}{2^{in}} \left\| \frac{g(2^{i+1} x_1, \ldots, 2^{i+1} x_n)}{2^k} - g(2^i x_1, \ldots, 2^i x_n) \right\|.$$

Apply the bound (4.5), we finally have

$$\left\| \frac{g(2^m x_1, \ldots, 2^m x_n)}{2^{mn}} - g(x_1, \ldots, x_n) \right\| \leq \frac{2(2^n - 1)\varepsilon}{n!} \sum_{i=0}^{m-1} \frac{1}{2^{in}} \leq \frac{2^{n+1}\varepsilon}{n!}.$$

□

Lemma 4.3. *Let n be a nonnegative integer. If $F, \tilde{F} : X \to Y$ are polynomial functions of order n with $F(0) = \tilde{F}(0)$, and there exists a constant $\varepsilon > 0$ such that*

$$\left\| F(x) - \tilde{F}(x) \right\| \leq \varepsilon \quad \textit{for all } x \in X, \tag{4.6}$$

then $F(x) = \tilde{F}(x)$ for all $x \in X$.

Proof. We will prove by mathematical induction on the order n. For the basis step $n = 0$, F and $\tilde{F}$ simply are constant functions and thus $F(x) = \tilde{F}(x)$ for all $x \in X$.

For the inductive step, we assume that the Lemma holds for $n = k - 1$. Here $k \geq 1$. We will prove that the lemma also holds for $n = k$. Assume the assumptions in the lemma when $n = k$. In addition, F and $\tilde{F}$ are polynomial functions of order k; that is, by Theorem 2.15,

$$F(x) = F(0)+A^1(x)+\cdots+A^k(x) \quad \text{and} \quad \tilde{F}(x) = F(0)+\tilde{A}^1(x)+\cdots+\tilde{A}^k(x), \tag{4.7}$$

where $A^i(x)$ and $\tilde{A}^i(x)$, for each $i = 1, \ldots, k$, are the diagonalization of some symmetric i-additive functions from X^n to Y. Note that we have also taken into account $F(0) = \tilde{F}(0)$. By Lemma 2.9, for every positive integer m,

$$F(2^m x) = F(0) + A^1(2^m x) + \cdots + A^k(2^m x) = F(0) + 2^m A^1(x) + \cdots + 2^{mk} A^k(x).$$

Therefore, for every $x \in X$,

$$\lim_{m\to\infty} \frac{F(2^m x)}{2^{mk}} = A^k(x).$$

Similarly, for every $x \in X$,

$$\lim_{m\to\infty} \frac{\tilde{F}(2^m x)}{2^{mk}} = \tilde{A}^k(x).$$

Therefore,

$$\left\| A^k(x) - \tilde{A}^k(x) \right\| = \left\| \lim_{m\to\infty} \frac{F(2^m x) - \tilde{F}(2^m x)}{2^{mk}} \right\| = \lim_{m\to\infty} \frac{1}{2^{mk}} \left\| F(2^m x) - \tilde{F}(2^m x) \right\|$$

Since $\left\| F(x) - \tilde{F}(x) \right\| \leq \varepsilon$ for all $x \in X$, we have

$$\lim_{m\to\infty} \frac{1}{2^{mk}} \left\| F(2^m x) - \tilde{F}(2^m x) \right\| = 0.$$

That is $A^k(x) = \tilde{A}^k(x)$ for all $x \in X$. If we let $G(x) = F(0)+A^1(x)+\cdots+A^{k-1}(x)$ and $\tilde{G}(x) = F(0)+\tilde{A}^1(x)+\cdots+\tilde{A}^{k-1}(x)$, then G and $\tilde{G}$ are polynomial functions of order $k-1$ with $G(0) = \tilde{G}(0)$. Moreover, for every $x \in X$,

$$\left\| G(x) - \tilde{G}(x) \right\| = \left\| F(x) - \tilde{F}(x) \right\| \leq \varepsilon,$$

which fulfils all assumptions of the lemma when $n = k - 1$. Thus, the induction hypothesis gives $G(x) = \tilde{G}(x)$ for all $x \in X$, which in turn concludes that $F(x) = \tilde{F}(x)$ for all $x \in X$. $\square$

4.1.2 Stability of Fréchet Functional Equation

We now state the stability theorem of the Fréchet functional equation as follows.

Theorem 4.4. *Let X be a linear space and let Y be a Banach space. Let $\varepsilon > 0$ and n be a positive integer. If a function $f : X \to Y$ satisfies the inequality*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \textit{for all } x, h \in X, \tag{4.8}$$

then there exists a symmetric n-additive function where its diagonalization $A^n : X \to Y$ satisfies

$$\|\Delta_h^n (f(x) - A^n(x))\| \leq 2^{n+2}\varepsilon \quad \textit{for all } x, h \in X. \tag{4.9}$$

Proof. Suppose a function $f : X \to Y$ satisfies the condition in the theorem. Define a function $g : X^n \to Y$ by

$$g(x_1, \ldots, x_n) := \frac{1}{n!}\Delta_{x_1 \ldots x_n} f(0) \quad \text{for all } x_1, \ldots, x_n \in X.$$

From Lemma 4.2, we have, for every $x_1, \ldots, x_n$ and for each positive integer m,

$$\left\|\frac{g(2^m x_1, \ldots, 2^m x_n)}{2^{mn}} - g(x_1, \ldots, x_n)\right\| \leq \frac{2^{n+1}\varepsilon}{n!}. \tag{4.10}$$

We will show that $\left\{\frac{g(2^m x_1, \ldots, 2^m x_n)}{2^{mn}}\right\}_{m=1}^{\infty}$ is a Cauchy sequence. Let $p, q > 0$ be integers.

$$\begin{aligned}
&\left\|\frac{g(2^{p+q} x_1, \ldots, 2^{p+q} x_n)}{2^{(p+q)n}} - \frac{g(2^p x_1, \ldots, 2^p x_n)}{2^{pn}}\right\| \\
&\qquad = \frac{1}{2^{pn}}\left\|\frac{g(2^q \cdot 2^p x_1, \ldots, 2^q \cdot 2^p x_n)}{2^{qn}} - g(2^p x_1, \ldots, 2^p x_n)\right\| \leq \frac{2^{n+1}\varepsilon}{2^{pn} n!}.
\end{aligned}$$

Taking the limit $p \to \infty$, the term on the right-hand side tends to zero. Hence, the sequence is a Cauchy sequence as desired. This allows us to define a function

$A_n : X^n \to Y$ by

$$A_n(x_1, \dots, x_k) := \lim_{m\to\infty} \frac{g(2^m x_1, \dots, 2^m x_k)}{2^{mk}}.$$

Next we will show that A_n is n-additive. From Lemma 4.2, g is symmetric and so is A_n. Thus, it suffices to prove the additivity only for the first component of A_n.

$$\begin{aligned}
&\|A_n(x_1+y, x_2, \dots, x_n) - A_k(x_1, x_2, \dots, x_n) - A_k(y, x_2, \dots, x_n)\| \\
&= \left\| \lim_{m\to\infty} \frac{1}{2^{mn}} \left(g(2^m(x_1+y), 2^m x_2, \dots, 2^m x_n) - g(2^m x_1, \dots, 2^m x_n) - g(2^m y, 2^m x_2, \dots, 2^m x_n)\right) \right\| \\
&= \lim_{m\to\infty} \frac{1}{2^{mn}} \|(g(2^m(x_1+y), 2^m x_2, \dots, 2^m x_n) - g(2^m x_1, \dots, 2^m x_n) - g(2^m y, 2^m x_2, \dots, 2^m x_n))\|
\end{aligned}$$

Using Lemma 4.2 again, we can see that the limit goes to zero, which concludes the additivity of A_n. Now if we take the limit $m \to \infty$ in (4.10), then, for every $x_1, \dots, x_n$,

$$\|A_n(x_1, \dots, x_n) - g(x_1, \dots, x_n)\| \le \frac{2^{n+1}\varepsilon}{n!}. \tag{4.11}$$

Setting $x_1 = \cdots = x_n = h$ in (4.11) and recalling that $g(h) = \frac{1}{n!}\Delta_h^n f(0)$, we have

$$\left\| A^n(h) - \frac{1}{n!}\Delta_h^n f(0) \right\| \le \frac{2^{n+1}\varepsilon}{n!},$$

where $A^n : X \to Y$ is the diagonalization of A_n. From Theorem 2.11, $\Delta_h^n A^n(x) = n! A^n(h)$. Therefore,

$$\|\Delta_h^n A^n(x) - \Delta_h^n f(0)\| \le 2^{n+1}\varepsilon.$$

Consider

$$\begin{aligned}
\|\Delta_h^n (f(x) - A^n(x))\| &= \|\Delta_h^n (f(x) - f(0)) + \Delta_h^n f(0) - A^n(x))\| \\
&\le \|\Delta_h^n \Delta_x f(0)\| + \|\Delta_h^n f(0) - A^n(x))\|.
\end{aligned}$$

From Lemma 4.1, $\|\Delta_h^n \Delta_x f(0)\| \le 2^{n+1}\varepsilon$. Hence,

$$\|\Delta_h^n (f(x) - A^n(x))\| \le 2^{n+2}\varepsilon$$

as desired. □

Theorem 4.5. *Let X be a linear space and let Y be a Banach space. Let $\varepsilon > 0$ and n be a positive integer. If a function $f : X \to Y$ satisfies the inequality (4.8),* *then there exists a unique function $F : X \to Y$ such that F satisfies the Fréchet functional equation, $\Delta_h^{n+1} F(x) = 0$, with $F(0) = f(0)$, and*

$$\|f(x) - F(x)\| \leq 2^{(n^2+5n-6)/2}\varepsilon \quad \text{for all } x \in X. \tag{4.12}$$

Moreover, $F(x) = f(0) + A^1(x) + \cdots + A^n(x)$, where, for all $x \in X$,

$$A^n(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$$

and, for each $k = 1, \ldots, n-1$,

$$A^k(x) = \lim_{m\to\infty} \frac{1}{2^{mk}} \left(f(2^m x) - A^{k+1}(2^m x) - \cdots - A^n(2^m x)\right).$$

Proof. We will prove by mathematical induction on n. For $n = 1$, we have the stability problem for Jensen functional equation which has already been proved ([13], for example). For the inductive step, assume that the theorem holds for $n = k-1$, we shall prove that the theorem also holds for $n = k$. Assume that, for every $x, h \in X$,

$$\left\|\Delta_h^{k+1} f(x)\right\| \leq \varepsilon.$$

By Theorem 4.4, there exists the diagonalization $A^k : X \to Y$ of a symmetric k-additive function such that

$$\left\|\Delta_h^k (f(x) - A^k(x))\right\| \leq 2^{k+2}\varepsilon.$$

If we let $h(x) = f(x) - A^k(x)$ for all $x \in X$, then $\left\|\Delta_h^k h(x)\right\| \leq 2^{k+2}\varepsilon$. Applying the induction hypothesis, there exists a function $H : X \to Y$ such that

$$\Delta_h^k H(x) = 0 \quad \text{and} \quad H(0) = h(0) \tag{4.13}$$

$$\text{and} \quad \|h(x) - H(x)\| \leq 2^{(k^2+3k-10)/2} \cdot 2^{k+2}\varepsilon = 2^{(k^2+5k-6)/2}\varepsilon. \tag{4.14}$$

Define $F(x) = H(x) + A^k(x)$. Using (4.13) and Theorem 2.15, we see that

$$\Delta_h^{k+1} F(x) = \Delta_h^{k+1}(H(x) + A^k(x)) = 0;$$

and $F(0) = H(0) + A^k(0) = f(0)$. In addition,

$$\|f(x) - F(x)\| = \|h(x) - H(x)\| \leq 2^{(k^2+5k-6)/2}\varepsilon,$$

which completes the induction.

The uniqueness of F is guaranteed by Lemma 4.3. □

It is worth noting that the generalized stability problem with the bound $\|\Delta_h^{n+1} f(x)\| \leq \phi(x,h)$ for some general distance function ϕ (with certain conditions) is possible by essentially the same method as what we have shown, but the result might appears awkward. Therefore, it will be of advantage to treat only the stability in the sense of Hyers and Ulam here.

4.2 Stability Problems of Conditional Functional Equations

Generally, functional equations will be stated without additional conditions on the arguments, except for those violating the validity of the function values. It is usually more challenging to solve functional equations with some additional restrictions. The restricted domain of the arguments is sometimes called *domains of validity* ([18] and [22]). Recall the classical example of Cauchy functional equation,

$$f(x+y) = f(x) + f(y), \tag{4.15}$$

when x, y are restricted to a region Ω. Its solutions may depend on the domain Ω as well as regularity assumptions of f such as the continuity and the boundedness.

Example 4.6. As already noted, when $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function and the domain of validity, Ω, is the entire domain $\mathbb{R}^2$, all continuous solutions will be linear functions. On the other hand, consider all functions f satisfying

$$f(x+y) = f(x) + f(y) \text{ for all } (x,y) \in \mathbb{R} \times \{0\}. \tag{4.16}$$

That is, the variable y is restricted to be only zero. It is obvious that all solutions of Eq.(4.15) when $\Omega = \mathbb{R}^2$ are also solution of Eq.(4.16). However, there exists a function, for instance, $f(x) = x^2$, which satisfies Eq.(4.16), but does not satisfy Eq.(4.15) when $\Omega = \mathbb{R}^2$. As a consequence, it can be concluded that the set of solutions of Eq.(4.15) when $\Omega = \mathbb{R}^2$ is contained in the set of solutions of Eq.(4.16). $\square$

Due to the fact shown in the previous example, we found that new solutions may generally occur if the domain of variables is restricted. It is worth noting that some authors ([3], [18], for example) used the term *conditional functional equations* to describe the functional equation with restricted domains. Furthermore, there is a number of researches ([3], [6], [15], [18], and [20], for example) that address this type of problems. The Fréchet functional equation with restricted domain is one of interesting framework. One example of such works belongs to W. Towanlong and P. Nakmahachalasint [24].

Accordingly, it is also merit to call attention to such functional equations in the sense of stability problems. The bound of stability of a functional equation with restricted domain may be affected due to emerging new solutions. Some appealing works dealing with such stability problems belong to F. Skof [23] and Z. Kominek [14].

4.3 Stability of Conditional Fréchet Functional Equation

In this section, we will determine the Hyers-Ulam stability of the Fréchet functional equation, $\Delta_h^{n+1} f(x) = 0$, when the spans, h, is restricted with the condition $\|h\| > a$, for a positive real number a.

Lemma 4.7. *Let n be a nonnegative integer. Let $f : X \to Y$ be a function. Then*

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \Delta_{(k+1)z}^{n+1} f(x_0 + kh_0) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \Delta_{h_0+jz}^{n+1} f(x_0 + jz)$$

for all $x_0, h_0, z \in X$.

Proof. Let $x_0, h_0, z \in X$. By Lemma 2.5, we have

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \Delta_{(k+1)z}^{n+1} f(x_0 + kh_0) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x_0 + kh_0 + j(k+1)z). \quad (4.17)$$

Swapping the order of the summations in (4.17) and applying Lemma 2.5, we get

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\Delta_{(k+1)z}^{n+1}f(x_0+kh_0)\\
&\quad=\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}f(x_0+jz+k(h_0+jz))\\
&\quad=\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Delta_{h_0+jz}^{n+1}f(x_0+jz).
\end{aligned}$$

□

Lemma 4.8. *Let a be a positive real number and let n be a nonnegative integer. If a function $f : X \to Y$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1}f(x)\right\| \le \varepsilon \quad \textit{for all } x \in X \textit{ and } \|h\| > a, \tag{4.18}$$

then

$$\left\|\Delta_h^{n+1}f(x)\right\| \le (2^{n+2}-1)\varepsilon \quad \textit{for all } x, h \in X.$$

Proof. Let a function $f : X \to Y$ satisfy (4.18). Let $x_0, h_0 \in X$. Choose $z \in X$ such that $\|z\| > a + \|h_0\|$. We then obtain that for $k = 0, \ldots, n+1$, $(k+1)\|z\| \ge \|z\| > a + \|h_0\| \ge a$. Thus, by (4.18), we have

$$\begin{aligned}
\left\|\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\Delta_{(k+1)z}^{n+1}f(x_0+kh_0)\right\| &\le \sum_{k=0}^{n+1}\binom{n+1}{k}\left\|\Delta_{(k+1)z}^{n+1}f(x_0+kh_0)\right\|\\
&\le 2^{n+1}\varepsilon. \qquad (4.19)
\end{aligned}$$

Similarly, for all $j = 1, \ldots, n+1$, we also have that $\|h_0 + jz\| \ge \|z\| - \|h_0\| > a$ and then

$$\left\|\sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Delta_{h_0+jz}^{n+1}f(x_0+jz)\right\| \le (2^{n+1}-1)\varepsilon. \tag{4.20}$$

From Lemma 4.7,

$$\begin{aligned}
(-1)^{n+1}\Delta_{h_0}^{n+1}f(x_0) &= \sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\Delta_{(k+1)z}^{n+1}f(x_0+kh_0)\\
&\quad-\sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Delta_{h_0+jz}^{n+1}f(x_0+jz).
\end{aligned}$$

That is

$$\begin{aligned}\left\|\Delta_{h_0}^{n+1} f(x_0)\right\| \leq & \left\|\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\Delta_{(k+1)z}^{n+1} f(x_0+kh_0)\right\| \\ & + \left\|\sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Delta_{h_0+jz}^{n+1} f(x_0+jz)\right\| \\ \leq & 2^{n+1}\varepsilon + (2^{n+1}-1)\varepsilon = (2^{n+2}-1)\varepsilon.\end{aligned}$$

□

We now reach the settlement of the stability of the Fréchet functional equation with restricted spans.

Theorem 4.9. *Let a be a positive real number and let n be a nonnegative integer. If a function $f : X \to Y$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \textit{for all } x \in X \textit{ and } \|h\| > a,$$

then there exists a unique function $F : X \to Y$ such that F satisfies the Fréchet functional equation with $F(0) = f(0)$ and

$$\|f(x) - F(x)\| \leq 2^{\frac{n^2+5n-6}{2}}(2^{n+2}-1)\varepsilon \quad \textit{for all } x \in X.$$

Proof. The desired result follows directly from Theorem 4.5 and Lemma 4.8. □

For the case of functions defined on the set of real numbers, instead of restricting the span h to $\|h\| > a$, we may separately consider the restrictions $h > a$ and $h < a$ as in the following two lemmas.

Lemma 4.10. *Let $a \in \mathbb{R}$ and let n be a nonnegative integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \textit{for all } x \in \mathbb{R} \textit{ and } h \in (a, \infty), \tag{4.21}$$

then

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq (2^{n+2}-1)\varepsilon \quad \textit{for all } x, h \in \mathbb{R}.$$

Proof. Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy (4.21). Let $x_0, h_0 \in \mathbb{R}$. To prove this lemma, we will choose $z > \max\{0, a, a - h_0\}$. The rest of the proof is similar to Lemma 4.8. $\square$

Lemma 4.11. *Let $a \in \mathbb{R}$ and let n be a nonnegative integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \text{for all } x \in \mathbb{R} \text{ and } h \in (-\infty, a), \tag{4.22}$$

then

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq (2^{n+2} - 1)\varepsilon \quad \text{for all } x, h \in \mathbb{R}.$$

Proof. Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy (4.22). It is the fact that, for all $x_0, h_0 \in \mathbb{R}$, $\Delta_{h_0}^{n+1} f(x_0) = (-1)^{n+1} \Delta_{-h_0}^{n+1} f(x_0 + (n+1)h_0)$. Due to this fact and (4.22), we obtain

$$\left\|\Delta_{-h}^{n+1} f(x + (n+1)h)\right\| = \left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon.$$

Since $-h \in (a, \infty)$, we now can employ Lemma 4.10 to obtain the result as desired. $\square$

Theorem 4.12. *Let $a \in \mathbb{R}$ and let n be a positive integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \text{for all } x \in \mathbb{R} \text{ and } h \in (a, \infty),$$

then there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ such that F satisfies the Fréchet functional equation with $F(0) = f(0)$ and

$$\|f(x) - F(x)\| \leq 2^{\frac{n^2+5n-6}{2}} (2^{n+2} - 1)\varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Proof. The desired result follows directly from Theorem 4.5 and Lemma 4.10. $\square$

Theorem 4.13. *Let $a \in \mathbb{R}$ and let n be a positive integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies, for $\varepsilon > 0$,*

$$\left\|\Delta_h^{n+1} f(x)\right\| \leq \varepsilon \quad \text{for all } x \in \mathbb{R} \text{ and } h \in (-\infty, a),$$

then there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ such that F satisfies the Fréchet functional equation with $F(0) = f(0)$ and

$$\|f(x) - F(x)\| \leq 2^{\frac{n^2+5n-6}{2}} (2^{n+2} - 1)\varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Proof. The desired result follows directly from Theorem 4.5 and Lemma 4.11. $\square$

REFERENCES

[1] Adam M.: On the Stability of Some Quadratic Functional Equation, *J. Nonlinear Sci. Appl.* **4** (2011), 50-59.

[2] Arunkumar M., Ravi K., Rassias M.J.: Stability of a Quartic and Orthogonally Quartic Functional Equation, *Bull. Math. Anal. Appl.* **3** (2011), 13-24.

[3] Brzdek J., Sikorska J.: A Conditional Exponential Functional Equation and Its Stability, *Nonlinear Analysis.* **72** (2010), 2923-2934.

[4] Cauchy A.L.: *Course d' analyse de l' Ecocle Polytechnique*, l. Analyse Algébrique, V. Paris, 1821.

[5] Czerwik S.: *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.

[6] Fochi M.: An Alternative Function Equation on Restricted Domain, *Aequations Math.* **70** (2005), 201-212.

[7] Fréchet M.: Une definition fonctionelle des polynomes, *Nouv Ann* **9** (1909), 145-162.

[8] Gãvruta P.: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[9] Hamel G.: Einer basis aller Zahlen und die unstertigen L-sungen der Funktionalgleichung $f(x+y) = f(x) + f(y)$, *Math. Ann.* **60** (1905), 459-472.

[10] Hewitt E., Zukerman H.S.: Remarks on the Functional Equation $f(x+y) = f(x) + f(y)$, *Math. Mag.* **42** (1970), 121-123.

[11] Hyers D.H.: On the Stability of the Linear Functional Equations, *Proc. Natl. Acad. Sci.* **27** (1941), 222-224.

[12] Jun K.W., Kim H.M.: Ulam stability problem for a mixed type of cubic and additive functional equation,*Bull. Belg. Math. Soc. simon Stevin* **13** (2006), 271-285.

[13] Jung S.-M.: Hyers-Ulam-Rassias stability of Jensen's equation and its application, *Proc. Amer. Math. Soc.* **126** (1998), 3137-3143.

[14] Kominek Z.: On a local stability of the Jensen functional equation, *Demonstratio Math.* **22** (1989), 499-507.

[15] Kuczma M.: Function Equations on Restricted Domains, *Aequations Math.* **18** (1978), 1-34.

[16] Kuczma M.: *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Publishers and Silesian University Press, Warszawa-Kraków-Katowice, 1985.

[17] Lee S.-B., Park W.-G., Bae J.-H.: On the stability of a Mixed Type Functional Equation, *J. chungcheong Math. Soc.* **19** (2006), 69-77.

[18] Paganoni L., Rätz J.: Conditional Function Equations and Orthogonal Additivity, *Aequations Math.* **50** (1995), 135-142.

[19] Rassias Th.M.: On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society*, **72**(2) (1978), 297-300.

[20] Rätz J.: On Orthogonally Additivite Mappings, *Aequations Math.* **28** (1985), 35-49.

[21] Ravi K., Rassias J.M., Murali R.: Orthogonal Stability of a Mixed Type Additive and Quadratic Functional Equation, *Mathematica Aeterna* **1** (2011), 185-199.

[22] Shalit O.M.: The Overdeterminedness of a Class of Functional Equations, *Aequations Math.* **74** (2007), 242-248.

[23] Skof F.: Sull'approssimazione delle applicazioni localmente δ-additive, *Atti Accad. Sci. Torino* **117** (1983), 377-389.

[24] Towanlong W., Nakmahachalasint P.: Fréchet Functional Equations with Restricted Spans: Solution on Real Numbers, *ScienceAsia* **37**(2) (2011), 125-129.

[25] Ulam S.M.: *Problems in Modern Mathematics*, Chapter 6, John Wiley & Sons, New York, NY, USA, 1964.

[26] Wiwatwanich A., Nakmahachalasint P.: On the Stability of a Cubic Functional Equation, *Thai Journal of Mathematics* **3** (2008), 69-76.

VITA

Name	:	Miss Tippaporn Eungrasamee
Date of Birth	:	3 June, 1980.
Place of Birth	:	Songkhla, Thailand
Education	:	B.Sc.(Mathematics), Kasetsart University, 2000, M.Sc.(Applied Mathematics), Mahidol University, 2005.
Scholarship	:	Development and Promotion of Science and Technology Talents Project (DPST)