CHAPTER II

MOLECULAR ORBITAL THEORY IN QUAMTUM PHARMACOLOGICAL STUDIES

Molecular orbital theory offers a great promise in calculating electronic structures and predicting properties of drug molecules. The success of molecular orbital theory in predicting chemical and physical events has been encouraging, as witnessed by a steady increase in its use in the literature (36-38). Within the past decade, molecular orbital studies have been introduced into drug research to investigate mechanisms of action and to guide the design of more potent agents.

Quantum mechanics provides a description of matter that is superior to all previous theories. It is a consistent set of principles that has no proof but afford a satisfactory description of natural phenomena. A basic principle in quantum mechanics is that the position and momentum of particles cannot be determined simultaneously. Our knowledge of these conjugate quantities is thus statistical.

2.1 Quantum Mechanical Principles

A mathematical formalism has been structured to incorporate these principles. The probability and quantized properties of matter are treated as though the particles were waves. It is really a convenient fiction, yet an extremely good one, that closely reproduces the properties of matter. The mathematics are known as quantum mechanics or wave mechanics. The fundamental equation is the Schroedinger wave equation.

2.1.1 The Schroedinger Wave Equation

No proof of the Schroedinger wave equation is possible. It is a fundamental relationship that gives logical coherence to most experimental observations. Some analogies between quantum theory and classical mechanics can serve as a point of departure for the development of the wave equation.

From classical mechanics, the law of conservation of energy can be written as

$$T + V = E \tag{2.1}$$

where E is the total energy, and T and V are the kinetic and potential energies, respectively. Schroedinger suggested that the proper way to describe the wave character of particles was to replace the classical kinetic-and potential energy functions T, V by linear operators $\hat{T}$, $\hat{V}$ and set up a wave equation of the form

$$\{\hat{T} + \hat{V}\} \Psi = E \Psi \tag{2.2}$$

The solution to equation (2.2), the so-called wavefunction Ψ, would describe the spatial motion of all the particles of the system moving in the field of force specified by the potential-energy operator $\hat{V}$.

In simple one-electron systems, such as the hydrogen atom, the problem is essentially to describe the motion of the electron in the Coulombic force field of the nucleus. In this case the classical potential-energy function and the quantum-mechanical potential-energy operator are identical, and for an electron moving in the field of a nucleus of charge Ze,

$$\hat{V} = -Ze^2r^{-1} \tag{2.3}$$

where r is the distance of the electron from the nucleus and e is the unit of electronic charge. With the coordinate system centered on the atomic nucleus, one needs consider only the kinetic energy of the electron. Schroedinger's prescription required that the classical kinetic-energy expression for a single particle,

$$T = p^2/2m \tag{2.4}$$

where p and m are the momentum and mass of the particle, respectively, be replaced by the linear differential operator

$$\hat{T} = \frac{-h^2}{8\pi^2 m}\nabla^2 \tag{2.5}$$

where h is Planck's constant, m the electronic mass, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.6}$$

with ∇^2 is the Laplace operator in cartesian coordinates. Thus the Schroedinger equation for the hydrogen atom in quantum mechanical expression become:

$$\left\{ \frac{-h^2}{8\pi^2 m} \nabla^2 - \frac{Ze^2}{r} \right\} \Psi(1) = E\Psi(1) \tag{2.7}$$

where $\Psi(1)$ represents the amplitude of the particle wave and is called the wavefunction. It is a useful convention for later considerations to denote wavefunctions depending on the coordinates of only one electron. Such functions are called orbitals and are the quantum-mechanical counterpart of planetary orbits in classical mechanics. Similarly, one-electron energies will be denoted by ε. The linear operator in bracket in Equation 2.7 is known as the hamiltonian operator for the system under consideration, and is denoted by H . Thus the Schroedinger equation for the hydrogen atom may be written in the form

$$H(1)\Psi(1) = \varepsilon\Psi(1) \tag{2.8}$$

The Schroedinger equation for a larger system consisting of a set of interacting electrons and nuclei is formulated in a similar manner. This first requires specification of the full hamiltonian for the system. The hamiltonian is again the sum of kinetic-energy operators for the nuclei and for the electrons together with the potential energy terms representing the various coulombic interactions. These are repulsive for electron-electron and nucleus-nucleus pairs, but attractive between electrons and nuclei. If there are N nuclei and n electrons, the many-particle hamiltonian operator H^{total} is

$$H^{total}(1,2,\ldots,N;1,2,\ldots,n) = \frac{-h^2}{8\pi^2}\sum_{A}^{N}\bar{M}_A^{1}\nabla_A^2 + \sum_{A<B} e^2 Z_A Z_B r_{AB}^{-1} - \frac{h^2}{8\pi^2 m}\sum_{P}^{n} \nabla_p^2$$

$$- \sum_{A}\sum_{p} e^2 Z_A r_{Ap}^{-1} + \sum_{p<q} e^2 r_{pq}^{-1} \qquad (2.9)$$

Here M_A is the mass of nucleus A; m and e are the electronic mass and charge, respectively; $Z_A e$ is the charge on nucleus A; and r_{ij} is the distance between particles i and j. Summations involving indices A and B are over atomic nuclei and those involving p and q are over electrons.

The Schrodinger equation for the entire system is thus

$$H^{total}(1,2,\ldots,N;1,2,\ldots n)\ \Psi(1,2,\ldots,N;1,2,\ldots,n)$$

$$= E\ \Psi(1,2,\ldots,N;1,2,\ldots,n) \qquad (2.10)$$

where Ψ is now a complete wavefunction for all particles in the molecule and E is the total energy of the system.

The exact solution of Schroedinger equation which includes the hamiltonian is mathematically impossible. However,its success is truly remarkable, but because of the intractability of the many-electron equations, it is limited to very simple and molecular systems for exact or nearly exact descriptions. More approximate, hence, less rigorous, treatment of larger molecules must be introduced.

2.1.2 The Born-Oppenheimer Approximation

Since nuclei are much heavier than electrons, they move more slowly. Hence, to a good approximation, one can consider the electrons in a molecule to be moving in the field of fixed nuclei. Within this approximation, the first term of equation (2.9), the kinetic energy of the nuclei, can be neglected and the second term, the repulsion between the nuclei, can be considered to be constant. In more quantitative terms, the Born-Oppenheimer approximation amounts to separate off the nuclear kinetic energy and nuclear-nuclear replusion terms from H^{total}, and consider only the part of the hamiltonian which depends on the positions but not the momenta of the nuclei. This is the electronic hamiltonian operator,

$$H^{el} = - \sum_{i=1}^{n} \frac{1}{2} \nabla_i^2 - \sum_{i=1}^{n} \sum_{i=1}^{N} \frac{Z_A}{r_{iA}} + \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{1}{r_{ij}} \qquad (2.11)$$

The solution to a Schroedinger equation involving the electronic hamiltonian,

$$H^{el}\Psi^{el} = E^{el}\Psi^{el} \qquad (2.12)$$

are purely electronic wavefunctions Ψ^{el}, describing the motion of the electrons in the field of the fixed nuclei. The total energy E of the system of a given internuclear distance is then given as

$$E^{total} = E^{el} + \sum_{A=1}^{N} \sum_{B>A}^{N} \frac{Z_A Z_B}{r_{AB}} \qquad (2.13)$$

Molecular orbital theory is concerned with electronic wavefunctions only, and we hence drop the superscript el on the hamiltonian operator and the wavefunctions without ambiguity.

2.1.3 The Antisymmetry or Pauli Exclusion Principle

To completely describe electrons, which are essentially indistinguishable particles, it is necessary, however, to specify its spin. In the context of nonrelativistic theory, two spin functions $\alpha(\omega)$ and $\beta(\omega)$ corresponding to spin up and down, respectively are introduced. These are functions of an unspecified spin variable w; from the operational point of view we need only specify that the two spin functions are complete and that they are orthonormal,

$$\int d\tau\alpha^*(\omega)\alpha(\omega) = \int d\tau\beta^*(\omega)\beta(\omega) = 1 \tag{2.14}$$

$$< \alpha|\alpha > = < \beta|\beta > = 1 \tag{2.15}$$

and

$$\int d\tau\alpha^*(\omega)\beta(\omega) = \int d\tau\beta^*(\omega)\alpha(\omega) = 0 \tag{2.16}$$

$$\text{or} \quad < \alpha|\beta > = < \alpha|\beta > = 0 \tag{2.17}$$

where the integration has been used in a formal way. In this formalism an electron is described not only by the three spatial coordinates r but also by one spin coordinate ω. We denote these four coordinates collectively by χ,

$$\chi = \{ \mathbf{r}, \omega \} \tag{2.18}$$

The wavefunction for an n-electron system is then a function of $x_1,x_2,\ldots,x_n$. That is $\Psi(x_1,x_2,\ldots,x_n)$.

If we consider the many electron density function $\rho(1,2,\ldots,n)$,

$$\rho(1,2,\ldots,n) = \psi^2(1,2,\ldots,n) \qquad (2.19)$$

It has no effect if we interchange the coordinates of any two electron. Ψ must be changed only by a factor of +1 or -1 under such an interchange,

$$\Psi(1,2,\ldots,i,j,\ldots,n) = \pm\Psi(1,2,\ldots,j,i,\ldots n) \qquad (2.20)$$

In the former case Ψ is said to be symmetric with respect to the interchange, and in the latter case Ψ is an antisymmetric, and these are the only two possibilities compatible with the invariance of Ψ^2. In fact, the antisymmetric property is appropriate for electrons, since it leads naturally to the Pauli exclusion principle (39) in orbital theory which states that no two electrons may be assigned to identical spin orbitals.

The antisymmetry principle may be formulated as

$$P_{ij}\Psi(1,2,\ldots,n) = -\Psi(1,2,\ldots,n) \qquad (2.21)$$

where P_{ij} is a permutation operator which interchanges all the coordinates (including spin coordinates) of electrons i and j. The properly antisymmetrized form of the orbital approximation to the two-electron wavefunction may be generated from the original Hartree product by operating on the Hartree product with a linear combination of permutation operator P_k, each resulting in one of the 2! distinct permutations of i and j possible, in the example under consideration,

$$\psi(1,2) = \mathcal{N} \sum_{k}^{2}(-1)^{P}P_k\psi_1(1)\alpha(1)\psi_1(2)\beta(2) \tag{2.22}$$

where $(-1)^P$ is +1 for an even permutation and -1 for an odd permutation.

As we shall see, the requirment of antisymmetry is easily enforced by determinant. The many-electron wavefunction for a 2n electron system, with two electrons per spatial orbital as a determinant of the 2n spin orbitals involved, is

$$\Psi(1,2,\ldots,n) = \mathcal{N} \begin{vmatrix} \psi_1(1)\alpha(1) & \psi_1(1)\beta(1) & \psi_2(1)\alpha(1) & \ldots\psi_n(1)\beta(1) \\ \psi_1(2)\alpha(2) & \psi_1(2)\beta(2) & \psi_2(2)\alpha(2) & \ldots \quad . \\ \vdots & \vdots & \vdots & \ldots \quad \vdots \\ \psi_1(2n)\alpha(2n) & \psi_1(2n)\beta(2n) & . & \ldots\ \psi_n(2n)\beta(2n) \end{vmatrix} \tag{2.23}$$

with the normalization constant, $\mathcal{N}$, appropriately adjusted. Equation (2.23) is often abbreviated as the product of the diagonal elements of the matrix enclosed in bars

$$\Psi(1,2,\ldots,n) = \left|\psi_1(1)\alpha(1)\psi_1(2)\beta(2)\ \ldots\psi_n(2n)\beta(2n)\right| \tag{2.24}$$

where the appropriate normalzation implied. Such determinants of spin orbitals are known as Slater determinants (40). A single Slater determinant is the simplest orbital wavefunction which satisfies the antisymmetry principle, that often written in the literature as:

$$\Psi(1,2,...,n) = |\psi_1(1)\bar{\Psi}_1(2) \ldots \psi_n(2n-1)\bar{\Psi}_n(2n)| \qquad (2.25)$$

or ever more simply,

$$\Psi(1,2,...,n) = \left|\psi_1\bar{\Psi}_1 \ldots \Psi_n\bar{\Psi}_n\right| \qquad (2.32)$$

2.1.4 Molecular Orbital and Linear Combination of Atomic Orbital (LCAO) Approximation

The concept of molecular orbitals is a natural extension of consideration of atomic orbitals. It is assumed that the orbitals of electrons in a molecule extend over the entire molecule and are subject to the potentials of all the nuclei and of the charge distributions of all other electrons. Electrons occupy these molecular orbitals according to the same restrictions imposed on electrons in atomic orbitals. Each electron in the molecule can then be described by use of a particular wavefuction ψ . The orbitals are polycentric (over all nuclei). If solutions of ψ are normalized, ψ^2 indicates the probability of finding the electrons at some point in space. This permits us to draw contours of constant probability around the molecule for the electrons and to conceptualize the location of regions of high and low probability (or charge).

Each molecular orbital, ψ, is associated with a particular energy value, which corresponds approximately to the energy necessary to remove that electron from the molecule.

For example, the molecular orbital for H_2 contains the two electrons, one from each atom, contributing to the bond between the atoms. A resonable assumption can be made, however, that when one electron is in the vicinity of one nucleus, such as A, the forces acting on it are approximately those found in a single hydrogen atom. Thus, the wavefunction for the isolated atom, ψ_a, should be a contributor to the ψ for the H_2 molecule. Likewise, if the electron is near nucleus B, then ψ_b should approximate that contribution to the molecular orbital. If the molecular orbital is characterized by contributions from ψ_a and ψ_b, it is reasonable to assume that a linear combination of atomic orbitals (LCAO approximation) could represent the molecular orbital. The general LCAO equation is

$$\Psi = C_a \psi_a + C_b \psi_b \qquad (2.27)$$

where the coefficients, C_a and C_b, weight the contributions of the two atomic orbitals.

A general LCAO expression for any molecule becomes

$$\Psi = C_a \psi_a + C_b \psi_b + \ldots + C_n \psi_n \qquad (2.28)$$

The energies associated with the orbitals ψ_a and ψ_b must be about the same magnitude, or the LCAO equation reduces to the wavefunctions of two isolated atomic orbitals and ψ is then not representative of a bond or molecule. The electron density domains, or charge clouds, of ψ_a and ψ_b must overlap, and the overlapping regions must be of identical sign. Solutions to the wave equation for the molecule must now be sought, specifically for the coefficients to give the charge distribution, and for the energies of the molecular orbitals. The approach to solution of the coefficients is based on the variation theorem, which states that, for a particular molecular orbital, the best values of the coefficients are those that give minimal value to the energy, or

$$\frac{\partial E}{\partial C_{i\ ,\ i=1,2,..n}} = 0 \qquad (2.29)$$

The kinetic and potential terms in the Schrodinger equation can be expressed in unspecified form by a single operator, the hamiltonian, H. The Schroedinger equation therefore reduces to

$$H\,\Psi = \varepsilon\,\Psi \qquad (2.30)$$

where Ψ is a solution for the molecular orbital. Rewriting we obtain,

$$\int \Psi (\, H - E\,) \Psi \;=\; 0 \qquad (2.31)$$

Expanding the equations (2.27) and (2.31) in terms of the atomic atomic orbitals, we obtain

$$C_a(H - E)\psi_a + C_b(H - E)\psi_b \;=\; 0 \qquad (2.32)$$

Since Ψ^2 is the probability of finding the electron in a region of space and for over all space this probability is unity,

$$\int \Psi^2 d\tau = 1 \qquad (2.33)$$

the LCAO equation should be multiplied through by the two wavefunctions for the two atomic orbitals ψ_a and ψ_b and integrated over all space. The results are

$$C_a \int \psi_a (H-E) \psi_a d\tau + C_b \int \psi_a (H-E) \psi_b d\tau = 0 \qquad (2.34)$$

$$C_a \int \psi_b (H-E) \psi_a d\tau + C_b \int \psi_b (H-E) \psi_b d\tau = 0 \qquad (2.35)$$

A set of standard notations is used to represent certain integrals in equations (2.34) and (2.35), these are

$$\alpha_i = \int \psi_i H \psi_i d\tau \qquad (2.36)$$

$$\beta_{ij} = \int \psi_i H \psi_j d\tau \qquad (2.37)$$

$$\delta_{ij} = \int \psi_i \psi_j d\tau \qquad (2.38)$$

$$E = \int \psi_i E \psi_i d\tau \qquad (2.39)$$

Equations (2.34) and (2.35) are thus became

$$C_a(\alpha_a - E) + C_b(\beta_{ab} - ES_{ab}) = 0 \qquad (2.40)$$

$$C_a(\beta_{ba} - ES_{ba}) + C_b(\alpha_b - E) = 0 \qquad (2.41)$$

which are a set of simultaneous equation in two unknowns, the C's and E. The solution to the simultaneous equations gives two value of E. The lower energy molecular orbital is a bonding orbital and the higher energy molecular orbital is an antibonding orbital.

The solution of the wave function of the two-atom molecule can be extended to multiatom molecules by similar treatment of the general expression for the LCAO equation.

2.2 Self-Consistent Field Molecular Orbital Theory

As result of the Born-Oppenheimer approximation, the total energy with fixed nuclei can thus be written as

$$E = \sum_{A<B}\sum \frac{Z_A Z_B}{r_{AB}} + \langle \Psi | H_{el} | \Psi \rangle \qquad (2.42)$$

The first term represents the nuclear repulsion, the second term represents the electronic energy. It is useful at this stage, to further break up the hamiltonian operator into two terms, H_1 and H_2. The term H_1 includes all single-electron terms, nuclear attraction operators, and kinetic energy operator, and H_2 includes all two-electron terms, the electron-electron repulsion operators.

$$H^{el} = H_1 + H_2 \qquad (2.43)$$

where $$H_1 = -\sum_i \left(\sum_A \frac{Z_A}{r_{Ai}} + \frac{1}{2} \nabla_i^2 \right) \qquad (2.44)$$

$$H_2 = \sum_{i<j}\sum \frac{1}{r_{ij}} \qquad (2.45)$$

Substituting H_{el} in equation (2.42) by its value of equation (2.43), we obtain

$$E = \sum_A \sum_B Z_A Z_B / R_{AB} + < \Psi | H_1 | \Psi > + < \Psi | H_2 | \Psi > \qquad (2.46)$$

The expression for this equation can be defined as a sum of three terms: nuclear-nuclear repulsions, one electron interactions and electron-electron repulsion. The latter two terms thus represent the total electronic energy of the system.

By the LCAO MO approximation, each electron is described by a normalized molecular orbital constructed from a linear combination of atomic orbitals, we obtain

$$E_1 = 2 \sum_i [< \sum_k C_{ki} \psi_k(1) | H_1 | \sum_l C_{li} \psi_l(1) >]$$

$$= 2 \sum_k \sum_l \sum_i C_{ki} C_{li} < \psi_k(1) | H_1 | \psi_l(1) >$$

$$= \sum_k \sum_l P_{kl} < \psi_k(1) | H_1 | \psi_l(1) > \qquad (2.47)$$

where $$P_{kl} = 2 \sum_i^{occ} (C_{ki} C_{li}) \qquad (2.48)$$

The integral $<\psi_k(1) | H_1 | \psi_l(1)>$, hereafter denoted by H_{kl}, represents the kinetic and nuclear attraction energy of an electron residing in the k-l overlap region. Equation (2.47) may now be expressed as

$$E_1 = \sum_k \sum_l P_{kl} H_{kl} \qquad (2.49)$$

We obtain a similar way for the electron-electron repulsion:

$$E_2 = \frac{1}{2}\sum_k \sum_l \sum_m \sum_n P_{kl}P_{mn}[<\psi_k(1)\psi_m(2)|H_2|\psi_l(1)\psi_n(2)>$$

$$- \frac{1}{2}<\psi_k(1)\psi_m(2)|H_2|\psi_l(2)\psi_n(1)>] \qquad (2.50)$$

The first two-electron integral is the coulomb repulsion between the electrons residing in the overlap regions k-l and m-n, respectively and often written as <kl|mn>. The second two-electron integral is the exchange integral, which arises because of the antisymmetric nature of the determinantal wavefunction, and written as <kn|lm>.

$$E_2 = \frac{1}{2}\sum_k \sum_l P_{kl}\sum_m \sum_n P_{mn} <kl|mn> - \frac{1}{2}<kn|lm> \qquad (2.51)$$

The expression for the total energy can be written as

$$E = \sum_A \sum_{A<B} \frac{Z_A Z_B}{r_{AB}} + \sum_k \sum_l P_{kl}[H_{kl} + \frac{1}{2}\sum_m \sum_n P_{mn}[<kl|mn> - \frac{1}{2}<kn|lm>] \qquad (2.52)$$

The nuclear repulsion terms as well as the one electron term H_{kl} and the electron attractions can either be calculated, as in the ab initio method, or approximated, as described in the third part of this chapter on CNDO method.

2.2.1 Hartree Fock Equation

For our restricted case of a closed-shell single-determinantal wavefunction, the variation method leads to

$$\hat{F}\psi_i = \varepsilon_i \psi_i \tag{2.53}$$

These equations are well known as the Hartree-Fock equations, and F is often called the Fock operator. The expression for F is

$$\hat{F}(1) = -\frac{1}{2}\nabla_1^2 - \sum_u (Z_u/r_{u1}) + \sum_{j=1}^{n} (2\hat{J}_j - \hat{K}_j) \tag{2.54}$$

The symbols $\hat{J}_j$ and $\hat{K}_j$ stand for operators related to the $1/r_{ij}$ operators in H. $\hat{J}_j$ is called a coulomb operator because it leads to energy terms corresponding to charge cloud repulsions. It is possible to write $\hat{J}_j$, explicitly:

$$\hat{J}_j = \int \psi_j^*(2)(1/r_{12})\, \psi_j(2)d\tau(2) \tag{2.54}$$

$\hat{K}_j$ leads ultimately to the production of exchange integrals, and so it is called an exchange operator. It is written explicitly in conjunction with a function on which it is operating.

$$\hat{K}_j \psi_i(1) = \int \psi_j^*(2)(1/r_{12})\, \psi_i(2)d\tau(2)\, \psi_j(1) \tag{2.56}$$

Given any trial function ψ_i, the expectation value ε_i of the hamiltonian operator H is a number given by

$$\varepsilon_i = <\psi_i | F_i | \psi_i> = H_{ii}^{core} + (2J_{ij} - K_{ij}) \tag{2.57}$$

$$<\psi_i | \hat{J}_j | \psi_i> = <\psi_i(1)\,\psi_j(2) | 1/r_{12} | \psi_i(1)\,\psi_j(2)> = \hat{J}_{ij}$$

$$<\psi_i | \hat{K}_j | \psi_i> = <\psi_i(1)\,\psi_j(2) | 1/r_{12} | \psi_i(2)\,\psi_j(1)> = \hat{K}_{ij} \tag{2.58}$$

2.2.2 The Roothaan Equation

To complete the description of the general LCAO MO method, we try to determine the best molecular orbitals, yield the lowest possible energy.

In the resulting variational procedure, the total energy is minimized with respect to each of the coefficient, C:

$$dE/dC_{ki} = 0 \qquad \text{for each index k and i} \tag{2.59}$$

According to equation (2.48) we obtain

$$\frac{d(\sum_k \sum_l P_{kl})}{dC_{ki}} = 2\sum_l C_{li}(1 - S_{kl}\sum_m \sum_n C_{mi}C_{ni}) \tag{2.60}$$

and a set of linear homogeneous equations of the type

$$dE = 0 = 2\sum_l C_{li}(1 - S_{kl}\sum_m \sum_n C_{mi}C_{ni})$$

$$\times [H_{kl} + \sum_m \sum_n P_{mn}(<kl|mn> -\frac{1}{2}<kn|lm>)] \tag{2.61}$$

The Fock matrix element can be defined as

$$F_{kl} = H_{kl} + \sum_m \sum_n P_{mn}(<kl|mn> - \frac{1}{2}<kn|lm>) \tag{2.62}$$

and the molecular orbital energy as

$$E_i = \sum_m \sum_n (C_{mi}C_{ni}) F_{kl} \tag{2.63}$$

The set of equations are then reduced to

$$\frac{dE}{dC_{kl}} = \sum_{l} C_{li}(F_{kl} - E_i S_{kl}) = 0 \qquad \text{for each index k} \qquad (2.64)$$

These are the Roothaan equations, which can be written more compactly as the single matrix equation

$$F\,C = S\,C\,E \qquad (2.65)$$

where C is a kxk square matrix of the expansion coefficients C_{li} and E is a diagonal matrix of the orbital energies ϵ_i.

Consider new C'and F' which related to old matrice C and F by

$$C' = S^{1/2}C\ , \qquad F' = S^{-1/2}FS^{-1/2} \qquad (2.66)$$

where $S^{1/2}$ is the square root of S

Then equation (2.65) becomes

$$F'C' = C'E \qquad (2.67)$$

and in the form of a standard eigenvalue ab initio computation. The elements ε_i of E will be the roots of the determinant equation

$$| F'_{kl} - \varepsilon_i \delta_{kl} | = 0 \qquad (2.68)$$

the lowest roots corresponding to the cccupied molecular orbitals.

For each root ε_i, the coefficient C_{li} can be found from the linear equations.

$$\sum_{l} C_{il}(F'_{kl} - \varepsilon_i \delta_{kl}) = 0 \tag{2.69}$$

and the coefficient is then determined from

$$C = S^{-1/2}C' \tag{2.70}$$

2.2.3 The Charge Density and Population Analysis

If we have an electron described by the spatial wavefunction $\psi_a(r)$, then the probability of finding that electron in a volumn element dr at a point r is $|\psi_a(r)|^2 dr$. The probability distribution function (charge density) is $|\psi_a(r)|^2$. If we have a closed-shell molecule described by a single determinant wavefunction with each occupied molecular orbital ψ_a containing two electrons, then the total charge density is

$$\rho(r) = 2\sum_{a}^{occ} |\psi_a(r)|^2 \tag{2.71}$$

such that $\rho(r)dr$ is the probability of finding an electron in dr at r. The integral of this charge density is just the total number of electrons,

$$\int dr\rho(r) = 2\sum_{a}^{occ} \int dr |\psi_a(r)|^2 = n \tag{2.72}$$

Inserting the molecular orbital expansion, $\psi_i = \sum_{u=1}^{k} C_{ui}\phi_u$, into the expression (2.71), we obtain

$$\rho(r) = 2\sum_{a}^{occ} \psi_a^*(r)\psi_a(r)$$

$$= 2\sum_{a}^{occ}\sum_{v} C_{va}^* \phi_v^*(r) \sum_{u} C_{ua} \phi_u(r)$$

$$= \sum_{uv} [\ 2\sum_{a}^{occ} C_{ua}\ C_{ua}^*\ ]\ \phi_u(r)\phi_v^*(r)$$

$$= \sum_{u} P_{uv}\ \phi_u(r)\ \phi_v^*(r) \qquad (2.73)$$

where we have defined a density matrix or, as it is somtimes called, a charge-density bond-order matrix

$$P_{uv} = 2\sum_{a}^{occ} C_{ua}\ C_{va}^* \qquad (2.74)$$

There is no unique definition of the number of electrons to be associated with a given atom or nucleus in a molecule, but it is useful to perform such population analysis. Since

$$n = 2\sum_{a}^{occ} \int dr|\psi_a(r)|^2 \qquad (2.75)$$

Devides the total number of electrons, n, into two electrons per molecular orbital, by substituting the basis expansion of ψ_a into equation (2.76), we have

$$n = \sum_{u}\sum_{v} P_{uv}\ S_{vu} = \sum_{u} (PS)_{uu} = trPS \qquad (2.76)$$

and it is possible to interpret $(PS)_{uu}$ as the number of electrons to be associated with ϕ_u. This is called a Mulliken Population

Analysis (41). Assumming the basis functions are centered on atomic nuclei, the corresponding number of electrons to be associated with a given atom in a molecule are obtained by summing over all basis functions centered on that atom. The net charge associated with an atom is then given by

$$q_A = Z_A - \sum_{u \in A} (PS)_{uu} \qquad (2.77)$$

when Z_A is the charge of atomic nucleus A; the index of summation indicates that we only sum over the basis functions centered on A.

2.2.4 The Central Field Approximation and the Self-Consistent Field Procedure

With the background of the previous sections we are now in a position to describe the actual computational procedure. The difficulty of solving the many multicenter integrals of the Fock matrix element provides the main obstacle to solving the Hartree-Fock problem.

In order to solve the Roothaan equation, the Fock matrix elements, F_{kl} which depend on the coefficient C_n, is required by guessing an initial value of C_n, allowing the determination of an initial set of P_{mn} values. The resulting Fock matrix elements are then calculated (equation 2.62) and the secular determinant (equation 2.68) is solved. The procedure results to the molecular energies ε_i. The resulting coefficients C_n are then used to establish a new set of P_{mn} values and the procedure is repeated iteratively until the difference in Pmn values between successive calculations is minimal, or "self-consistency" is achieved.

The directly SCF procedure can be written as

C'----> P ----> F ----> E ----> C' ----> P' ----> F'----> E'--->,etc

In the semiempirical approach to determining the electronic energy, many of the multicenter integrals are neglected, and the remaining integrals are evaluated semiempirically as shown in the remain part of this chapter.

2.2.5 The Form of the Basis Sets

The best solution to the Schroedinger wave equation could theoretically be obtained by the use of an infinite basis set (Hartree-Fock limit). A more realistic approach to the problem is the use of smaller numbers of functions which are good descriptions of the individual atomic orbitals in the molecule.

The most often used mathematical expressions for the atomic wavefunctions are Slater-type orbitals (STO) or Gaussian-type orbitals (GTO). The Slater functions are extremely close in form to the hydrogenic orbitals and have been extensively utilized in LCAO-MO procedures. The Gaussian functions used to describe the atomic orbitals provide the theoretical chemist with the simplest form of function for evaluation that are similar in appearance to the Gaussian distribution functions. For non-empirical calculations, Gaussian-lobe orbitals (GLO) which are more convenient than Slater-type orbitals in solving the integrals are often used.

The general form of a GTO is

$$X_{n,l,m}(r,\zeta) = N. e^{-\zeta r^2}. Y_{l,m}(\Theta,\Phi) \tag{2.78}$$

and GLO's are simply given by

$$X(r) = N.e^{-\zeta r^2} \tag{2.79}$$

The basis sets for ab initio calculations should describe the actual wavefunction well enough to give chemically useful results and lead to integrals which can be evaluated reasonably accurately and cheaply on a computer.

In this work minimal basis sets of Gaussian-lobe orbitals (GLO) have been used in ab initio calculations. The advantage of this basis set is the decrease of the C.P.U. time. And the relative energy obtained are also reasonable (42,43).

2.3 Approximate Molecular Orbital Theories

From the previous section, it was stated that ab initio calculations require exact calculation of all integrals contributing to the elements of the Fock matrix. For the large molecules the number of two-electron integrals becomes enermous. This has led to the development of a more approximate approach to the LCAO-MO-SCF method. Approximate molecular orbital theories are by nature semiempirical, in that one no longer attempts to derive molecular preperties directly from the principles of quantum mechanics, but rather seeks to interpret correlations within experimental data.

2.3.1 The Nature of Semiempirical Theory

Semiempirical molecular orbital theory,occupies a unique position in the canon of theoretical chemistry. It models the ab initio single determinant, minimal Slater basis set theory, which is purely mathematical approximation to a test and accepted fundamental quantity, the groud state solution to the electronic Schroedinger equation. On the other hand, it has the trappings of pure empiricism - the fitting of function to a discrete set of data points by varying adjustable parameters. The adjustable parameters are the weakness of the theory in the sense that their presence allows the violation of fundamental principle of quantum mechanics. For example, it is possible by adjusting parameters to obtain total energies lower than the exact ground-state energy, which by the variational principle is impossible for wavefunctions satisfying the Pauli exclusion principle. Yet, paradoxically, the parameters strengthen the theory in the sense that they allow sufficient flexibility for the theory to potentially transcend in accuracy the results obtained by its ab initio counterpart.

Throughout semiempirical quantum chemistry, with rare exceptions, it is customary to use the minimum basis sets, which include only the "s" and "p" orbitals of first row elements. Substitution of these LCAO expansions into the eigenvalue problem procedures which is known as the Fock matrix, then must be recognized. They are, expressed in terms of a series of integrals; some of these are treated as adjustable parameters, others are treated as systematic parameters, and yet others are neglected according to the ZDO scheme or some variant thereof. Another different approximate method such as CNDO, INDO, MINDO, etc., differ principally in the approximations model at this point. In this study we will deal with the CNDO/2 method which are simple enough to be applied to a wide range of chemical problems with less computational work. A general feature of all of these methods is the consideration of valence-shell electrons only.

A number of variants of a systematic approach meeting the above criterion have been developed by Pople and co-workers (44), and these are now widely used. The approximations are based on the idea of neglect of differential overlap between atomic orbitals in molecules. Before discussing the details of the CNDO approximation, we should consider the important invariant properties of the SCF equation, which any approximate molecular orbital theories must maintain.

2.3.2 Invariant Levels of Approximation

The most difficult and time-consuming part of LCAO self-consistent molecular orbital calculations is the evaluation and handling of large number of electronic repulsion integrals. It is well known that many of these electron repulsion integrals have values near zero, especially those involving the overlap distribution $\psi_u(1)\ \psi_v(1)$, with $u \neq v$. Thus, in developing approximate self-consistent field molecular orbital schemes, a useful approach is the systematic neglect of electron repulsion integrals having uniformly small values.

According to equation (2.28), the molecular orbital can be written as the linear combination of the atomic orbital, also a new basis set of functions t_1 could be formulated from the original set ψ_k by the matrix equation

$$T = O\Psi \qquad (2.80)$$

Here O is any nonsingular square matrix, any square matrices M acquire the new value

$$M = OMO^{-1} \qquad (2.81)$$

Thus, the whole calculation could be formulated in term of the new basis function. To solve the Roothaan equation, the Fock matrix has been seperated into one electron and two-electron terms,

$$FC = (H + G)C = SCE \qquad (2.82)$$

from the equation (2.82), we obtain

$$(O^{-1}H'O + O^{-1}G'O)C = O^{-1}S'OCE \qquad (2.83)$$

wher H', G' and S' correspond to the new basis functions T. If we multiply equation (2.83) from the left by O. It will become

$$[H'+G'](OC) = S'(OC)E \qquad (2.84)$$

that (OC) are the orbital coefficeints for the solution of the Roothaan equation using a new basis set. The diagonal matrix E equals to E in equation (2.82) that use the basis Ψ and trace E is therefore invariant.

To find the effects of the transformation generally, we need not know the way in which the various integrals transform. Under the transformation, the overlap matrix, the core hamiltonian, and the two-electron integral become

$$S'_{uv} = O_{uk}O_{vl}S_{kl} \qquad (2.85)$$

$$H'_{uv} = O_{uk}O_{vl}H_{kl} \qquad (2.86)$$

and

$$\langle uv|st\rangle' = \sum_k \sum_l \sum_m \sum_n O_{uk}O_{vl}O_{sm}O_{tn}\langle kl|mn\rangle \qquad (2.87)$$

where u, v, s, t are used for the new basis functions. An approximate LCAO-MO-SCF calculation will only be invariant in so far as they satisfy these three transformation conditions.

1. Transformation which only mix $2p_x$, $2p_y$ and $2p_z$ on the same atom which have the same principal and azimuthal quantum numbers
2. Transformation which mix any atomic orbitals on the same atom (the atomic hybrid orbitals), for example, the four atomic orbitals 2s, $2p_x$, $2p_y$, $2p_z$ may be replaced by four sp^3
3. Transformations which mix atomic orbitals centered on different atoms (lead to nonatomic orbital)

Transformation of type 1 and 2 are required to be invariant for any of the approximate LCAO-MO-SCF calculations. The following section deals with complete neglect of differential overlap (CNDO) which be applied in a manner invariant under the two transformation.

2.3.3 Approximate with Complete Neglect of Differential Overlap (CNDO)

The least sophisticated of the neglect of differential overlap methods is the complete neglect of differential overlap or CNDO method. Two version have been proposed, a preliminary version called CNDO/1, seldom used nowadays, and a reparametrized one, CNDO/2 (34). The main approximations of CNDO method are the Neglect of Overlap (NO), the Zero Differential Overlap approximation (ZDO) and the Core- Hamiltonian approximation.

2.3.3.1 The Neglect of Overlap (NO)

In this approximation, it is assumed that the repulsion between electrons belonging to orbitals of the same atom is dependent only upon the nature of the atom and not upon the nature of the orbitals involed.

$$S_{kl} = \delta_{kl} \qquad (2.88)$$

2.3.3.2 The Zero-Differential-Overlap (ZDO) Approximation

The second main approximation of CNDO is the zero differential-overlap (ZDO) approximation. The ZDO approximation is used for all products of different atomic orbitals and applied to the two-electron integrals of the Roothaan equation:

$$\langle kl|mn\rangle = \delta_{km}\,\delta_{kl}\,\delta_{mn} \qquad (2.89)$$

where δ_{kl} and δ_{mn} are the Kronecker delta functions.

The ZDO approximation which is used for atomic orbitals must satisfy the invariance of transformation that we have discussed before. For the monatomic differential overlap ψ_k, ψ_l involving the product of two different orbitals on the same atom, the arbitary transformation of them will not be invariant. In order to restore the rotational invariance, we assume that the remaining two-electron integrals depend only on the nature of the atom and not on the actual type of orbitals. Thus

$$\gamma_{km} = \langle kk|mm\rangle = \gamma_{AB} \qquad (2.90)$$

where k is an orbital of atom A and m is an orbital of atom B.

2.3.3.3 The Core Hamiltonian Approximation

Let us now examine the consequences of the ZDO approximation on the Fock matrix elements. The first part of one-electron terms, H_{kl} represents the interaction of each valence shell electron with the effective cores of the various atoms and its kinetic energy.

$$H_{kl} = \langle \psi_k(1)| - 1/2\, \nabla^2 - \sum_A V_A(r)|\psi_1(1)\rangle \qquad (2.91)$$

We distinguish equation (2.91) to three cases, depending cn whether k is equal to l or not, and whether they are located on the same atom or not.

<u>k = l</u>: In this case no term is cancelled by the ZDO. We group together the terms representing the attraction of an electron

in the orbital k by its own core A and its kinetic energy, and the attraction of electron i by the core of other atoms B :

$$H_{kk} = \langle\psi_k(1)\,|-1/2\,\nabla^2 - V_A(r)|\psi_k(1)\rangle$$

$$\sum_{A\neq B}\sum \langle\psi_k(1)\,|\,V_B\,|\,\psi_k(1)\rangle \tag{2.92}$$

The expression of H_{kk} thus becomes :

$$H_{kk} = U^A_{kk} - \sum_{A\neq B}\sum V_{kk} \tag{2.93}$$

<u>k ≠ l and k,l ∈ A:</u> by the ZDO approximation, the matrix elements H_{kl} will be set to zero.

$$H_{kk} = 0 \quad , \psi_k \neq \psi_l \qquad \text{(both on A)} \tag{2.94}$$

<u>k ≠ l and k ∈ A, l ∈ B:</u> In this case, where ψ_k and ψ_l are on different atoms A and B, the NO approximation does not apply. On the contrary these elements correspond to the basic bonding capacity due to the overlap between the orbitals. we can have the separation term of atoms A, B and all other C:

$$H_{kl} = \langle\psi_k(i)\,|-1/2\nabla^2 - V_A - V_B|\,\psi_l(i)\rangle$$

$$- \sum_{c\neq A,B} \langle\psi_k(i)|V_c|\,\psi_l(i)\rangle \tag{2.95}$$

where the second part gives the interaction of the electron distribution with the core of third atoms C. These integrals will be neglected, since they are comparable to the three-center, two-electron integrals which have already been omitted. The first part depends only on the local environment and is a measure of the possible lowering of energy levels by an electron being in the electrostatic field of two atoms simultaneously. It is commonly referred to as a resonance integral under the symbol β_{kl}.

$$H_{kl} = \beta_{kl} \tag{2.96}$$

The β_{kl} terms are then approximated according to Mulliken and Wolfsberg and Helmholtz (47) by:

$$\beta_{kl} = S_{kl}\beta_{AB} \tag{2.97}$$

where β_{AB} consists of the atom specific parameters β_A and β_B. The total electronic density of atom A defines as

$$P_{AA} = P_{kk} \tag{2.98}$$

The summation applies to all valence orbitals k of atom A;

$$P_{BB} = P_{mm} \tag{2.99}$$

The Fock matrix elements become

$$F_{kk} = U_{kk}^{A} + (P_{AA} - 1/2 \, P_{kk}) \, \gamma_{AA} + \sum_{B\neq A} \sum (P_{BB} \, \gamma_{AB} - V_{A}^{B}) \tag{2.100}$$

$$F_{kl} = H_{kl} - \frac{1}{2} P_{kl} \, \gamma_{AB} \tag{2.101}$$

The total energy derived from CNDO can be expressed in terms of the energies associated with monatomic contributions and that associated with diatomic contributions

$$E = \sum_A E_A + \sum_A \sum_B E_{AB} \qquad (2.102)$$

where

$$E_A = \sum_k^A P_{kk} U_{kk}^A + 1/2 \sum_k^A \sum_m^A (P_{kk} P_{mm} - 1/2\, P_{km}^2)\, \gamma_{AA} \qquad (2.103)$$

$$E_{AB} = \sum_k^A \sum_l^B (2P_{kl} \beta_{AB} - 1/2\, P_{kl}^2\, \gamma_{AB})$$

$$+ (Z_A Z_B R_{AB}^{-1} - P_{AA} V_a^B - P_{BB} V_B^A + P_{AA} P_{BB} \gamma_{AB}) \qquad (2.104)$$

2.3.4 The CNDO Parametrization

The CNDO/2 parametrization had been modified by Pople and Segal (34) in order to improve its ability to reproduce observed equilibrium bond lengths and dipole moments. It contains the following elements

2.3.4.1. The Local Core Matrix Element U_{kk}

In the first of this modification, the local core matrix element U_{kk} represents the attraction of an electron in the orbital k by its own core A and its kinetic energy. According to the importance of the electronegativity in determining the relative ability of an atom to depart from neutrality by losing electrons, the U_{kk} term was defined as the average of the experimental and calculated values of the ionization potential IP and electron affinity EA :

$$U_{kk}^{A} = 1/2 \left[-IP_k - (Z_A - 1) \gamma_{AA} - EA_k - Z_A \gamma_{AA} \right]$$

$$= -1/2 \left[IP_k + EA_k \right] - \left[Z_A - 1/2 \right] \gamma_{AA} \qquad (2.105)$$

From the Goeppert-Mayer and Sklar approximation, we obtain

$$- IP_k = U_{kk}^{A} + (Z_A - 1) \gamma_{AA} \qquad (2.106)$$

$$- EA_k = U_{kk}^{A} + Z_A \gamma_{AA} \qquad (2.107)$$

2.3.4.2 The Electron - Core Potential Intergrals V_B

The second modification concerned the electron repulsion integrals ,V_{AB}. This term is now set equal to

$$V_B^A = Z_B \gamma_{AB} \qquad (2.108)$$

2.3.4.3 The Bonding Parameters B_{AB}

$$B_{AB} = 1/2 (B_A + B_B) \qquad (2.109)$$

where B_A and B_B are parameters characteristic of atoms A and B and determined empirically by comparing them with reference to ab initio calculations on some simple molecules.

The Fock matrix in the CNDO/2 framework can be written as

$$F_{kk} = -1/2 (IP_k + EA_k) + [(P_{AA} - Z_A) - 1/2 (P_{kk} - 1)]\gamma_{AA}$$

$$+ (P_{BB} - Z_B) \gamma_{AB} \qquad (2.110)$$

$$F_{kl} = B_{AB}S_{kl} - 1/2\ p_{kl}\,\gamma_{AB} \tag{2.111}$$

2.3.5 The CNDO Basis Set

The CNDO basis set consists of Slater-type orbitals (STO) for the valence shell. These are:

$$1s : (\zeta^3 / \P)^{1/2} \exp (-\zeta r)$$

$$2s : (\zeta^5 / \P)^{1/2}\ r \exp (-\zeta r/2)$$

$$2p_x : (\zeta^5 / \P)^{1/2}\ x \exp (-\zeta r/2)$$

$$2p : (\zeta^5 / \P)^{1/2}\ y \exp (-\zeta r/2)$$

$$2p_z : (\zeta^5 / \P)^{1/2}\ z \exp (-\zeta r/2)$$

$$3s : (\zeta^5 / \P)^{1/2}\ r \cos\ \exp (-\zeta r/2)$$

The overlap integrals S_{kl} are all calculated explicitly using formulae given by Mulliken, Ricke, and Orloff (49). The electron repulsion integrals γ_{AB}, γ_{AA} and the potentials V_B^A, in order to maintain invariance, will involve valences functions only.

2.3.6. The SCF Procedure for CNDO

Solving the Roothaan equation in CNDO approximation, the iterative procedure is needed by a series of steps comparable to those used in ab initio calculation. The steps of solving are:

1. An initial guess is made at the molecular orbital coefficients C by a Hückel-type calculation in which the diagonal elements F_{kk} are replaced by the appropriate ionization potentials and electron affinities and F_{kl} are replaced by $B_{AB}S_{kl}$

2. Electrons are assigned in pairs to the molecular orbitals with lowest energies.

3. The density matrix P is calculated from the coefficients of the occupied molecular orbitals and then used to form a new Fock matrix F

4. A new set of coefficients C are derived from this F

5. The process is repeated from step 2 until self-consistency is achieved.