CHAPTER IV

LINEAR TRANSFORMATIONS AS Γ–SEMIGROUPS

Let V be an infinite–dimensional vector space over a division ring, $L(V)$ the semigroup under composition of all linear transformations on V and 1_V the identity map on V. The image of v under $\alpha \in L(V)$ is written by $v\alpha$. For $\alpha \in L(V)$, let $\ker\alpha$ and $\operatorname{im}\alpha$ denote the kernel and the image of α, respectively. The followings are linear transformation subsemigroups of $L(V)$ and the details of the proof can be seen in [1] and [2]:

$$G(V) = \{\ \alpha \in L(V) \mid \alpha \text{ is an isomorphism }\},$$
$$AI(V) = \{\ \alpha \in L(V) \mid \dim(V/F(\alpha)) < \infty\ \}, \text{ where } F(\alpha) = \{\ v \in V \mid v\alpha = v\ \},$$
$$M(V) = \{\ \alpha \in L(V) \mid \ker\alpha = \{0\}\ \},$$
$$E(V) = \{\ \alpha \in L(V) \mid \operatorname{im}\alpha = V\ \},$$
$$AM(V) = \{\ \alpha \in L(V) \mid \dim\ker\alpha < \infty\ \},$$
$$AE(V) = \{\ \alpha \in L(V) \mid \dim(V/\operatorname{im}\alpha) < \infty\ \},$$
$$OM(V) = \{\ \alpha \in L(V) \mid \dim\ker\alpha \text{ is infinite }\},$$
$$OE(V) = \{\ \alpha \in L(V) \mid \dim(V/\operatorname{im}\alpha) \text{ is infinite }\},$$

for each $k \in \mathbb{N}$

$$K(V,k) = \{\ \alpha \in L(V) \mid \dim\ker\alpha \geq k\ \}$$
$$K'(V,k) = \{\ \alpha \in L(V) \mid \dim\ker\alpha > k\ \},$$
$$CI(V,k) = \{\ \alpha \in L(V) \mid \dim(V/\operatorname{im}\alpha) \geq k\ \},$$
$$CI'(V,k) = \{\ \alpha \in L(V) \mid \dim(V/\operatorname{im}\alpha) > k\ \},$$

$$I(V,k) = \{\ \alpha \in L(V) \mid \dim\ \operatorname{im}\alpha \le k\ \},$$
$$I'(V,k) = \{\ \alpha \in L(V) \mid \dim\ \operatorname{im}\alpha < k\ \}.$$

The following remarks are the facts which will be used later.

Remark 4.1. *For any nonempty subset Γ of $L(V)$, $L(V)$ is a $\Gamma-$semigroup.*

Remark 4.2. ([2]) *The following statements hold.*
(i) $OM(V)$ is a right ideal of $L(V)$.
(ii) $OE(V)$ is a left ideal of $L(V)$.

This chapter deals with linear transformation semigroups on V. We will find the necessary and sufficient conditions for a nonempty subset Γ of V which the linear transformation subsemigroups mentioned above are $\Gamma-$subsemigroups.

The following proposition is also necessary for our consideration in the next results.

Proposition 4.3. *Let S be a subsemigroup of $L(V)$ containing 1_V. Then S is a $\Gamma-$subsemigroup of $L(V)$ if and only if $\Gamma \subseteq S$.*

Proof. First, assume that S is a $\Gamma-$subsemigroup of $L(V)$. Let $\alpha \in \Gamma$. Then $\alpha = (1_V)\alpha(1_V) \in S\Gamma S \subseteq S$.

Conversely, assume $\Gamma \subseteq S$. Then $S\Gamma S \subseteq SS \subseteq S$. Thus S is a $\Gamma-$subsemigroup of $L(V)$. □

By Proposition 4.3, the following subsemigroups of $L(V)$ are $\Gamma-$subsemigroups of $L(V)$ if and only if they contain Γ:

$$G(V) = \{ \alpha \in L(V) \mid \alpha \text{ is an isomorphism } \},$$

$$AI(V) = \{ \alpha \in L(V) \mid \dim(V/F(\alpha)) < \infty \}, \text{ where } F(\alpha) = \{ v \in V | v\alpha = v \},$$

$$M(V) = \{ \alpha \in L(V) \mid \ker \alpha = \{0\} \},$$

$$E(V) = \{ \alpha \in L(V) \mid \operatorname{im} \alpha = V \},$$

$$AM(V) = \{ \alpha \in L(V) \mid \dim \ker \alpha < \infty \},$$

$$AE(V) = \{ \alpha \in L(V) \mid \dim(V/\operatorname{im} \alpha) < \infty \}.$$

Theorem 4.4. *For all nonempty subsets* Γ *of* $L(V)$*,* $OM(V)$ *and* $OE(V)$ *are* $\Gamma-$*subsemigroups of* $L(V)$*.*

Proof. This is obtained from Remark 4.2. □

Theorem 4.5. *For all nonempty subsets* Γ *of* $L(V)$*,* $K(V,k)$ *and* $K'(V,k)$ *are* $\Gamma-$*subsemigroups of* $L(V)$*.*

Proof. Let Γ be a nonempty subset of $L(V)$. Let $\alpha, \gamma \in K(V,k)$ and $\beta \in \Gamma$. Then $\ker \alpha \subseteq \ker \alpha\beta\gamma$. Thus $k \leq \dim \ker \alpha \leq \dim \ker \alpha\beta\gamma$ implies that $\alpha\beta\gamma \in K(V,k)$. Therefore $K(V,k)$ is a $\Gamma-$subsemigroup of $L(V)$, so is $K'(V,k)$. □

Theorem 4.6. *For all nonempty subsets* Γ *of* $L(V)$*,* $CI(V,k)$ *and* $CI'(V,k)$ *are* $\Gamma-$*subsemigroups of* $L(V)$*.*

Proof. Let Γ be a nonempty subset of $L(V)$. Let $\alpha, \gamma \in CI(V,k)$ and $\beta \in \Gamma$. Then $\operatorname{im} \alpha\beta\gamma \subseteq \operatorname{im} \gamma$ and $\dim(V/\operatorname{im} \gamma) \leq \dim(V/\operatorname{im} \alpha\beta\gamma)$. Thus $k \leq \dim(V/\operatorname{im} \gamma) \leq \dim(V/\operatorname{im} \alpha\beta\gamma)$ implies that $\alpha\beta\gamma \in CI(V,k)$. Therefore $CI(V,k)$ is a $\Gamma-$subsemigroup of $L(V)$, so is $CI'(V,k)$. □

Theorem 4.7. *For all nonempty subsets* Γ *of* $L(V)$*,* $I(V,k)$ *and* $I'(V,k)$ *are* $\Gamma-$*subsemigroups of* $L(V)$*.*

Proof. Let Γ be a nonempty subset of $L(V)$. Let $\alpha, \gamma \in I(V,k)$ and $\beta \in \Gamma$. Then $\operatorname{im} \alpha\beta\gamma \subseteq \operatorname{im} \gamma$. Thus $\dim \operatorname{im} \alpha\beta\gamma \leq \dim \operatorname{im} \gamma \leq k$ implies that $\alpha\beta\gamma \in I(V,k)$. Therefore $I(V,k)$ is a $\Gamma-$subsemigroup of $L(V)$, so is $I'(V,k)$. $\square$