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ESSENTIAL CLOSURES

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เราเสนอแนวคิดของส่วนปิดคลุมหลักทั่วไปในทางสัจพจน์และพัฒนาทฤษฎีของส่วนปิดคลุมหลัก ในกระบวนการ เราเสนอแนวคิดของความปิดหลักและความกระชับหลักในแนวทางเดียวกับแนวคิดของความปิดเชิงทอพอโลยีและความกระชับเชิงทอพอโลยีตามลำดับ นอกจากนี้ เรายังเสนอแนวคิดของความเป็นส่วนหลัก ซึ่งเป็นแกนของทฤษฎีของส่วนปิดคลุมหลัก แนวคิดนี้คือสิ่งที่แยกแยะแนวคิดของส่วนปิดคลุมหลักออกจากแนวคิดของส่วนปิดคลุมเชิงทอพอโลยี ในส่วนของการนำไปใช้ เราสร้างตัวอย่างของส่วนปิดคลุมหลักผ่านแนวคิดของซับเมเชอร์ซึ่งเป็นการกำกัดของเมเชอร์ภายนอก และใช้ส่วนปิดคลุมหลักเหล่านี้เป็นเครื่องมือศึกษาเซตค้ำจุนของเมเชอร์ต่อเนื่องแบบสัมบูรณ์และเซตค้ำจุนของโคปูลาหลายตัวแปร และเสนอแนวคิดของเซตค้ำจุนหลักของฟังก์ชัน

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We propose the notion of general essential closures in an axiomatic way and develop the theory of essential closures. In the process, we introduce the notions of essential closedness and essential compactness analogous to the notions of topological closedness and topological compactness, respectively. We also introduce the notion of essentiality which is the core of the theory of essential closures. It is what distinguishes the notion of essential closures from the notion of topological closures. For applications, we construct concrete examples of essential closures via the notion of submeasures, which are restrictions of outer measures, and use them as tools to study supports of absolutely continuous measures and supports of multivariate copulas, and to introduce the notion of essential supports of functions.

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NOTATIONS

$\mathbb{C}$	the set of complex numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{N}$	the set of natural numbers
$\mathcal{P}(X)$	the collection of subsets of the set X
$\operatorname{Int} A$	the interior of the set A
$\mathfrak{N}(x)$	the collection of open neighborhoods of the point x
λ_n	the n-dimensional Lebesgue measure
λ_n^*	the n-dimensional Lebesgue outer measure
π_W	the orthogonal projection onto the subspace W
π_j	the orthogonal projection onto the j-th axis
$\mathcal{H}^{\mathrm{s}}$	the s-dimensional Hausdorff measure
$\dim_{\mathrm{H}}$	the Hausdorff dimension
$\operatorname{gr} f$	the graph of the function f
χ_A	the indicator function of the set A
$\operatorname{diam}(A)$	the diameter of the set A
$B(x, \epsilon)$	the open ball of radius ϵ centered at the point x

CHAPTER I

INTRODUCTION

In this research, we introduce the notion of general essential closures, which is formally postulated in Chapter 2. Our notion of essential closures is modeled after the notion of topological closures and the notion of essential closure introduced by Gesztesy et al. in [13]. To see the motivation behind the notion of general essential closures, we explore basic properties and interpretations of topological closures and the essential closure.

The topological closure of a set can be viewed as the collection of points which are near (or close to) the set in a topological sense, i.e., each neighborhood of those points intersects with the set. A topological closure $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the following properties for all sets $A, B \subseteq X$:

(i) $\mathrm{cl}(\varnothing) = \varnothing$,

(ii) $A \subseteq \mathrm{cl}(A)$,

(iii) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$,

(iv) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

Let us note that this set of properties completely characterizes the topological closure, i.e., if there is a unary operation on $\mathcal{P}(X)$ satisfying the above properties, then there is a topology on X with respect to which the unary operation is the topological closure. Topological closures are extensive[1] since every point in a set is close to itself, hence included in the topological closure of that set. In this case, we can say that every point in a set is essential to the set. But this is not the case for the essential closure.

[1] A set-valued function f is said to be *extensive* if $A \subseteq f(A)$ for each A in the domain.

In their work, Gesztesy et al. introduce the notion of essential closure as a tool to study absolutely continuous spectra of some linear operators. The essential closure of a Lebesgue measurable set $A \subseteq \mathbb{R}$ is defined by

$$\overline{A}^e = \{x \in \mathbb{R} \colon \text{for all } \epsilon > 0, \lambda_1((x-\epsilon, x+\epsilon) \cap A) > 0\}.$$

Likewise, the essential closure of a set can be viewed as the collection of points which are near (or close to) a positive Lebesgue measure portion of the set in a topological sense, i.e., the intersection of each neighborhood of those points with the set is of positive Lebesgue measure. The essential closure satisfies the following properties for all Lebesgue measurable sets $A, B \subseteq \mathbb{R}$:

(i) $\overline{A}^e$ is closed,

(ii) $\overline{A}^e \subseteq \overline{A}$,

(iii) $\overline{A \cup B}^e = \overline{A}^e \cup \overline{B}^e$,

(iv) $\overline{\overline{A}^e}^e = \overline{A}^e$.

Notice that the essential closure, unlike topological closures, is not extensive. For example, $\overline{\mathbb{Q}}^e = \varnothing$. In this case, we can say that only positive Lebesgue measure subsets are essential to the set. Compared with the case of topological closures, this is exactly the motivation which gives rise to the notion of essentiality in our research. Also notice that the second property of the essential closure implies that the essential closure of the empty set is the empty set itself. Moreover, notice that the first and the second properties of the essential closure are conditional to the underlying topology which, in this case, is the standard topology on the real line. These two properties are, in fact, due to the aforementioned interpretation of the essential closure.

Our notion of essential closures is a generalization of both the notion of topological closures and the notion of essential closure introduced in [13]. Roughly speaking, a general essential closure can be viewed a unary operation mimicking topological closures but not necessarily extensive. Moreover, like in the case of the essential closure on the real line, we want to have the notion of topological nearness, hence an underlying topological structure is required. To ensure some degree of compatibility between the underlying topological structure and a general essential closure, some additional conditions are included, i.e., we require that an essential closure of a set is closed and is included in the topological closure of that set. These are exactly the first and the second properties of the essential closure on the real line.

However, an approach to postulating the notion of essential closures which concentrates only on the core properties should not assume any a priori topological structure. Such approach starts from postulating essential closure operators in the same spirit as that of topological closure operators.

Recall the definition of a topological closure operator on a space X, which is a unary operation $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following properties for all sets $A, B \subseteq X$:

(i) $\mathrm{cl}(\varnothing) = \varnothing$,

(ii) $A \subseteq \mathrm{cl}(A)$,

(iii) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$,

(iv) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

Recall that this is the set of properties which characterizes the topological closure. We call a unary operation (whose domain and codomain are reduced to some suitable collection of sets) satisfying all but the second property above an essential closure operator and is formally postulated in Chapter 2. Similar to the fact that a topological closure operator naturally induces a (unique) topological closure, an essential closure operator also induces, in a natural way, an essential closure. One difference between the notion of topological closure operators and the notion of essential closure operators is that an essential closure operator induces an essential closure, but not unique in general due to the fact that there can be several compatible topological structures. The other difference is that an essential closure can be defined on a collection of subsets of the space satisfying some conditions similar to how a measure is defined on a σ-algebra of measurable sets.

During the process of developing the theory of essential closures, we introduce a few notions related to the notion of essential closures: essential sets and nonessential sets, essential closedness, and essential compactness. We obtain that, for a certain kind of essential closure, the collection of nonessential sets acts as a generator of its corresponding essential closure. Moreover, given any collection of sets, if some conditions are met, there is an essential closure whose collection of nonessential sets is generated by the given collection of sets. In addition, if the collection of sets is closed under taking subsets and countable unions, then it is exactly the collection of nonessential sets of that essential closure. In other words, there is an essential closure which detects exactly the given collection of sets. Such collections include the collection of measure zero sets, the collection of shy sets introduced in [15], the collection of sets of first category, etc. For the notion of essential closedness, we obtain that an essentially closed set is closed and locally essential. Moreover, the notion of essential closedness and the notion of topological compactness characterizes the notion of essential compactness in a Hausdorff space equipped with a certain kind of essential closure.

It turns out that well-behaved essential closures are strongly related to the notion of submeasures introduced in Chapter 3, which are restrictions of outer measures onto some σ-algebras. Besides developing the theory of essential closures, we focus mainly on essential closures defined via submeasures, called submeasure closures, and their applications. A submeasure closure of a set is defined to be the collection of points which are near (or close to) a positive submeasure portion of the set in a topological sense, i.e., the intersection of each neighborhood of those points with the set is of positive submeasure. However, a submeasure closure is not always an essential closure. Two sufficient conditions, which guarantee that a submeasure closure is an essential closure, are given in Chapter 3. One is a condition on the topological structure of the space while the other is a condition on the submeasure. Moreover, if one of the aforementioned sufficient conditions is satisfied, the submeasure closure becomes the essential closure which detects exactly the collection of corresponding submeasure zero sets. For applications of submeasure closures, we study supports of submeasures, especially supports of absolutely continuous measures. We also introduce the notion of essential supports of functions, which is more suitable than the notion of topological supports of functions in some cases, especially functions that are only defined almost everywhere, e.g., Radon-Nikodym derivatives.

In the last chapter, we study supports of multivariate copulas. We observe that the topological closure is too coarse to use. Essential closures are almost always finer than the topological closure. There are two kinds of essential closures we pick to use in the study of supports of multivariate copulas. One is a family of submeasure closures defined via Hausdorff measures. This kind of essential closure is fit to study the local Hausdorff dimension of supports of multivariate copulas. The other is a family of submeasure closures defined via outer measures constructed according to the nature of multivariate copulas. This kind of essential closure is fit to study supports of multivariate copulas whose underlying random variables are related in a specific way, i.e., one set of random variables is completely dependent on the rest. Such multivariate dependence structure of random variables is introduce in Chapter 4. We derive explicit formulas of supports of such multivariate copulas and interpret the result to obtain a geometric necessary condition for a set to be a support of a multivariate copula.

CHAPTER II
ESSENTIAL CLOSURES

In the sequel, we often use notations $A \mapsto \widetilde{A}$ and $\mathcal{E}$ to denote both essential closures and essential closure operators. Moreover, we often use notations $A \mapsto \overline{A}$ and cl to denote both topological closures and topological closure operators.

2.1 A set of postulates for essential closures

An essential closure can be roughly viewed as a unary operation mimicking the topological closure but not necessarily extensive. Moreover, two conditions are included to establish some degree of compatibility between the essential closure and the underlying topological structure.

Postulate 2.1. Let (X, τ, Ω) be a topological space equipped with an algebra[1]. We say that a unary operation

$$A \mapsto \tilde{A} \colon \Omega \to \Omega$$

is an <u>essential closure</u> if, for each $A, B \in \Omega$, the following hold:

(i) $\widetilde{A}$ is a closed set,

(ii) $\widetilde{A} \subseteq \overline{A}$,

(iii) $\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$,

(iv) $\widetilde{\widetilde{A}} = \widetilde{A}$.

Remark 2.2. *A restriction of an essential closure to any subalgebra invariant under the essential closure is still an essential closure.*

[1] An *algebra* over a set X is a nonempty collection of subsets of X which is closed under complementation and finite unions.

Proposition 2.3. *Let $\mathcal{E}$ be an essential closure on (X, τ, Ω). Let $S \in \Omega$ be a subset of X such that $\mathcal{E}(S) \subseteq S$, i.e., S is $\mathcal{E}$-invariant. Define*

(i) *τ_S to be the subspace topology on S,*

(ii) *Ω_S to be the algebra (or σ-algebra if Ω is a σ-algebra) generated by the collection $\{A \cap S \colon A \in \Omega\}$, and*

(iii) *$\mathcal{E}_S = \mathcal{E}|_{\Omega_S}$.*

Then $\mathcal{E}_S$ is an essential closure on (S, τ_S, Ω_S).

Proof. Observe that $\Omega_S \subseteq \Omega$ and $\mathcal{E}_S$ is a unary operation on Ω_S since, for each $A \in \Omega_S$, $A \in \Omega$ and $A \subseteq S$, so $\mathcal{E}(A) \in \Omega$ and $\mathcal{E}(A) \subseteq \mathcal{E}(S) \subseteq S$, which implies that

$$\mathcal{E}_S(A) = \mathcal{E}(A) = \mathcal{E}(A) \cap S \in \Omega_S.$$

It is left to verify the four properties of essential closures. Let $A, B \in \Omega_S$. Then the following hold.

(i) $\mathcal{E}_S(A) = \mathcal{E}(A)$ is closed with respect to τ and is contained in S, hence closed with respect to τ_S.

(ii) $\mathcal{E}_S(A) = \mathcal{E}(A) = \mathcal{E}(A) \cap S \subseteq \mathrm{cl}(A) \cap S = \mathrm{cl}_S(A)$ where cl denotes the topological closure with respect to τ and cl_S denotes the topological closure with respect to τ_S.

(iii) $\mathcal{E}_S(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B) = \mathcal{E}_S(A) \cup \mathcal{E}_S(B)$.

(iv) $\mathcal{E}_S(\mathcal{E}_S(A)) = \mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A) = \mathcal{E}_S(A)$.

Hence $\mathcal{E}_S$ is an essential closure on (S, τ_S, Ω_S). □

The essential closure $\mathcal{E}_S$ can be regarded as a natural restriction of $\mathcal{E}$ onto the $\mathcal{E}$-invariant subspace S. Note that the topological closedness of S is sufficient to guarantee that S is $\mathcal{E}$-invariant because if S is closed, then $\mathcal{E}(S) \subseteq \mathrm{cl}(S) = S$ by the second property of essential closures.

Remark 2.4. *With respect to set inclusion, an essential closure is increasing. This can be shown by the fact that essential closures are distributive over finite unions. Let* $A \mapsto \widetilde{A}$ *be an essential closure on* Ω*. Assume that* $A, B \in \Omega$ *and* $A \subseteq B$*, then write* $B = A \cup B$*. Thus*

$$\widetilde{A} \subseteq \widetilde{A} \cup \widetilde{B} = \widetilde{A \cup B} = \widetilde{B}.$$

Definition 2.5. An essential closure $A \mapsto \widetilde{A}$ on Ω is said to be strong if for each $A \in \Omega$, $\widetilde{A - \widetilde{A}} = \varnothing$.

The set $A - \widetilde{A}$ can be viewed as the nonessential part of A. Thus $\widetilde{A - \widetilde{A}} = \varnothing$ can be interpreted as the nonessential part of A being small with respect to the essential closure.

Remark 2.6. *Let* $A \mapsto \widetilde{A}$ *be an essential closure on* Ω*. Then for each* $A \in \Omega$*, if* $\widetilde{A - \widetilde{A}} = \varnothing$*, we have*

$$\widetilde{A} = \widetilde{A - \widetilde{A}} \cup \widetilde{A \cap \widetilde{A}} = \widetilde{A \cap \widetilde{A}} \subseteq \widetilde{\widetilde{A}} \subseteq \overline{\widetilde{A}} = \widetilde{A}.$$

Hence, for each $A \in \Omega$*,* $\widetilde{A - \widetilde{A}} = \varnothing$ *implies* $\widetilde{\widetilde{A}} = \widetilde{A}$*. However, the converse is not true in general.*

Example 2.7. Let $X = \{0, 1\}$, $\tau = \{\varnothing, \{0\}, X\}$, and $\Omega = \mathcal{P}(X)$. Define a function $\mathcal{E} \colon \Omega \to \Omega$ by $\mathcal{E}(A) = \varnothing$ if A is empty and $\mathcal{E}(A) = \{1\}$ otherwise. Let $A, B \subseteq X$. The following are readily verified.

(i) $\mathcal{E}(A)$ is closed.

(ii) $\mathcal{E}(A) \subseteq \overline{A}$ since $\overline{\{0\}} = X$.

(iii) $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$ since both sides are $\{1\}$ unless $A = B = \varnothing$.

(iv) It is straightforward to show that $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$.

Thus $\mathcal{E}$ is an essential closure. Moreover, observe that

$$\mathcal{E}(X - \mathcal{E}(X)) = \mathcal{E}(X - \{1\}) = \{1\} \neq \varnothing.$$

Hence $\mathcal{E}$ is not strong.

Proposition 2.8. *Let $A \mapsto \widetilde{A}$ be an essential closure on $\Omega \subseteq \mathcal{P}(X)$ and suppose that $\widetilde{X} = X$. Then the following hold:*

(i) $(\widetilde{A})^c \subseteq \widetilde{A^c}$ *for each $A \in \Omega$,*

(ii) $\widetilde{G} = \overline{G}$ *for each open set $G \in \Omega$,*

(iii) $\overline{\operatorname{Int} A} \subseteq \widetilde{A}$ *for each $A \in \Omega$ such that* $\operatorname{Int} A \in \Omega$.

Proof. Recall the properties of essential closures in Postulate 2.1.

(i) Observe that $X = \widetilde{X} = \widetilde{A \cup A^c} = \widetilde{A} \cup \widetilde{A^c}$. Hence $(\widetilde{A})^c \subseteq \widetilde{A^c}$.

(ii) Observe that

$$X = \widetilde{X} = \widetilde{G \cup G^c} = \widetilde{G} \cup \widetilde{G^c} \subseteq \widetilde{G} \cup \overline{G^c} = \widetilde{G} \cup G^c.$$

Hence $G \subseteq \widetilde{G}$. Since $G \subseteq \widetilde{G} \subseteq \overline{G}$ and $\widetilde{G}$ is closed, we have $\widetilde{G} = \overline{G}$.

(iii) Since $\operatorname{Int} A \in \Omega$ is open, $\overline{\operatorname{Int} A} = \widetilde{\operatorname{Int} A} \subseteq \widetilde{A}$. □

Remark 2.9. *A topological closure restricted to an algebra containing the open sets is an essential closure on that algebra. Moreover, it is the unique extensive essential closure on the algebra.*

2.2 Nonessential sets

In this section, we introduce one of the most important notions related to essential closures: the notion of essentiality. The notion of essential closures is a tool created to detect certain types of sets. To be precise, it is a tool used to detect whether a set is essential or nonessential.

Definition 2.10. Let $\mathcal{E}$ be an essential closure on Ω. Then a set $A \in \Omega$ is said to be <u>nonessential</u> if $\mathcal{E}(A) = \varnothing$ otherwise A is said to be <u>essential</u>. The collection of nonessential sets is denoted by $\mathcal{N}_\Omega(\mathcal{E})$.

Theorem 2.11. *Let $\mathcal{E}$ be a strong essential closure on Ω. Then, for any $A \in \Omega$, $\mathcal{E}(A)$ is the intersection of closed sets $F \in \Omega$ such that $A - F$ is nonessential.*

Proof. Since $\mathcal{E}$ is strong, $\mathcal{E}(A - \mathcal{E}(A)) = \varnothing$ and hence $\mathcal{E}(A)$ is in the intersection. Moreover, for any closed set $F \in \Omega$ with $\mathcal{E}(A - F) = \varnothing$, observe that

$$\mathcal{E}(A) = \mathcal{E}(A \cap F) \cup \mathcal{E}(A - F) = \mathcal{E}(A \cap F) \subseteq \mathcal{E}(F) \subseteq \overline{F} = F.$$

This completes the proof. □

According to Theorem 2.11, one can see that the collection of nonessential sets acts as a generator of its corresponding strong essential closure. To study strong essential closures, it suffices to study their nonessential sets. The following result follows directly from Theorem 2.11.

Corollary 2.12. *Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are strong essential closures on Ω such that $\mathcal{N}_\Omega(\mathcal{E}_1) = \mathcal{N}_\Omega(\mathcal{E}_2)$. Then the two essential closures coincide.*

Example 2.13. Recall the essential closure defined in Example 2.7. One can check that the essential closure is not generated by means of Theorem 2.11. Thus the assumption that the essential closure is strong is necessary in Theorem 2.11.

Definition 2.14. An essential closure on Ω is said to be σ-nonessential if Ω is a σ-algebra[2] and every countable union of nonessential sets is nonessential.

Lemma 2.15. *Let $\mathcal{E}$ be an essential closure on an algebra Ω and $x \in \mathcal{E}(A)$. If $G \in \Omega$ is an open neighborhood of x, then $G \cap A$ is essential.*

Proof. Suppose there exists $G \in \mathfrak{N}(x) \cap \Omega$ with $\mathcal{E}(G \cap A) = \varnothing$. Observe that

$$\mathcal{E}(A) = \mathcal{E}(A \cap G^c) \subseteq \mathcal{E}(A) \cap \mathcal{E}(G^c) \subseteq \mathcal{E}(A) \cap \overline{G^c} = \mathcal{E}(A) - G,$$

contradicting the fact that $\mathcal{E}(A) - G \subseteq \mathcal{E}(A) - \{x\}$ is a proper subset of $\mathcal{E}(A)$. □

The following result requires a technical assumption that for each $G \in \mathfrak{N}(x)$, there is $O \in \mathfrak{N}(x)$ with $\overline{O} \subseteq G$. A topological space with such property is said to be regular (i.e., T_3). More information on regular spaces can be found in Munkres's book [18].

[2] A *σ-algebra* over a set X is a nonempty collection of subsets of X which is closed under complementation and countable unions.

Theorem 2.16. *Let $\mathcal{E}$ be a σ-nonessential essential closure defined on a regular measurable space*[3] *where the σ-algebra contains the Borel sets*[4]*. Then*

$$\mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}.$$

Proof. If $x \in \mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right)$ and $G \in \mathfrak{N}(x)$, then there exists $O \in \mathfrak{N}(x)$ such that $\overline{O} \subseteq G$. By Lemma 2.15,

$$\mathcal{E}\left(\bigcup_{i=1}^{\infty}(O \cap A_i)\right) = \mathcal{E}\left(O \cap \bigcup_{i=1}^{\infty} A_i\right) \neq \varnothing.$$

Since the essential closure is σ-nonessential, there exists A_j with $\mathcal{E}(O \cap A_j) \neq \varnothing$. Hence

$$\varnothing \neq \mathcal{E}(O \cap A_j) \subseteq \mathcal{E}(O) \cap \mathcal{E}(A_j) \subseteq \overline{O} \cap \mathcal{E}(A_j) \subseteq G \cap \bigcup_{i=1}^{\infty} \mathcal{E}(A_i).$$

This implies $x \in \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}$. The other inclusion follows easily from the facts that $\mathcal{E}$ is increasing with respect to set inclusion and that the images of $\mathcal{E}$ are closed. $\square$

An essential closure on a σ-algebra is not necessarily σ-nonessential. This fact is shown in the following example.

Example 2.17. Let $X = \mathbb{N}$, $\tau = \{\varnothing, \{1\}^c, X\}$ and $\Omega = \mathcal{P}(X)$. Define a unary operation $\mathcal{E}\colon \Omega \to \Omega$ by

$$\mathcal{E}(A) = \begin{cases} \varnothing & \text{if } 1 \notin A \text{ and } A \text{ is finite} \\ \{1\} & \text{otherwise.} \end{cases}$$

Observe that $\mathcal{E}$ satisfies the following for all $A, B \subseteq X$.

(i) $\mathcal{E}(A)$ is closed.

(ii) $\mathcal{E}(A) \subseteq \overline{A}$ since $\overline{A} = X$ if $A \neq \varnothing, \{1\}$.

[3] A *regular measurable space* is a regular space equipped with a σ-algebra.

[4] A *Borel set* is an element of the *Borel σ-algebra*, the smallest σ-algebra containing the open sets.

(iii) Observe that

$$\begin{aligned}\mathcal{E}(A\cup B)=\varnothing &\Leftrightarrow 1\notin A\cup B \text{ and } A\cup B \text{ is finite}\\ &\Leftrightarrow 1\notin A, 1\notin B, A \text{ is finite, and } B \text{ is finite}\\ &\Leftrightarrow \mathcal{E}(A)=\varnothing \text{ and } \mathcal{E}(B)=\varnothing\\ &\Leftrightarrow \mathcal{E}(A)\cup\mathcal{E}(B)=\varnothing.\end{aligned}$$

Hence $\mathcal{E}(A\cup B)=\mathcal{E}(A)\cup\mathcal{E}(B)$.

(iv) If $\mathcal{E}(A)=\varnothing$, then $\mathcal{E}(\mathcal{E}(A))=\varnothing$. If $\mathcal{E}(A)=\{1\}$, then $\mathcal{E}(\mathcal{E}(A))=\{1\}$. Therefore, $\mathcal{E}(\mathcal{E}(A))=\mathcal{E}(A)$.

Thus $\mathcal{E}$ is an essential closure on Ω. However,

$$\mathcal{E}(X-\{1\})=\{1\}\neq\varnothing=\bigcup_{x\neq 1}\mathcal{E}(\{x\}).$$

Hence there exists an essential closure on a σ-algebra that is not σ-nonessential. Moreover, this essential closure is not strong either since

$$\mathcal{E}(X-\mathcal{E}(X))=\mathcal{E}(\{1\}^c)=\{1\}\neq\varnothing.$$

The following two examples suggest that the notions of strong essential closures and σ-nonessential essential closures are not related in an obvious way since one does not imply the other.

Example 2.18. Let $X=\mathbb{N}$, $\tau=\{\varnothing,X\}$ and $\Omega=\mathcal{P}(X)$. Define a unary operation $\mathcal{E}\colon\Omega\to\Omega$ by

$$\mathcal{E}(A)=\begin{cases}\varnothing & \text{if } A \text{ is finite}\\ X & \text{otherwise.}\end{cases}$$

Observe that $\mathcal{E}$ satisfies the following for all $A,B\subseteq X$.

(i) $\mathcal{E}(A)$ is closed.

(ii) $\mathcal{E}(A)\subseteq\overline{A}$ since $\overline{A}=X$ if $A\neq\varnothing$.

(iii) Observe that

$$\begin{aligned}\mathcal{E}(A \cup B) = \varnothing &\Leftrightarrow A \cup B \text{ is finite} \\ &\Leftrightarrow A \text{ is finite, and } B \text{ is finite} \\ &\Leftrightarrow \mathcal{E}(A) = \varnothing \text{ and } \mathcal{E}(B) = \varnothing \\ &\Leftrightarrow \mathcal{E}(A) \cup \mathcal{E}(B) = \varnothing.\end{aligned}$$

Hence $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup E(B)$.

(iv) If $\mathcal{E}(A) = \varnothing$, then $\mathcal{E}(A - \mathcal{E}(A)) = \varnothing$. If $\mathcal{E}(A) = X$, then $\mathcal{E}(A - \mathcal{E}(A)) = \varnothing$. Therefore, $\mathcal{E}(A - \mathcal{E}(A)) = \varnothing$.

Thus $\mathcal{E}$ is a strong essential closure on Ω. However,

$$\mathcal{E}(X) = X \neq \varnothing = \bigcup_{x \in X} \mathcal{E}(\{x\}).$$

Hence, there is a strong essential closure on a σ-algebra that is not σ-nonessential.

Example 2.19. Let $X = \mathbb{R}$, $\tau = \{\varnothing, (-\infty, 0), X\}$ and $\Omega = \mathcal{P}(X)$. Define a unary operation $\mathcal{E} \colon \Omega \to \Omega$ by

$$\mathcal{E}(A) = \begin{cases} \varnothing & \text{if } A \text{ is countable} \\ [0, \infty) & \text{otherwise.} \end{cases}$$

Observe that $\mathcal{E}$ satisfies the following for all $A, B \subseteq X$.

(i) $\mathcal{E}(A)$ is closed.

(ii) $\mathcal{E}(A) \subseteq \overline{A}$ since $\mathcal{E}(A) \subseteq [0, \infty)$.

(iii) Observe that

$$\begin{aligned}\mathcal{E}(A \cup B) = \varnothing &\Leftrightarrow A \cup B \text{ is countable} \\ &\Leftrightarrow A \text{ is countable, and } B \text{ is countable} \\ &\Leftrightarrow \mathcal{E}(A) = \varnothing \text{ and } \mathcal{E}(B) = \varnothing \\ &\Leftrightarrow \mathcal{E}(A) \cup \mathcal{E}(B) = \varnothing.\end{aligned}$$

Hence $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$.

(iv) If $\mathcal{E}(A) = \varnothing$, then $\mathcal{E}(\mathcal{E}(A)) = \varnothing$. If $\mathcal{E}(A) = [0, \infty)$, then $\mathcal{E}(\mathcal{E}(A)) = [0, \infty)$. Therefore, $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$.

Thus $\mathcal{E}$ is an essential closure on Ω. Moreover, $\mathcal{E}$ is clearly σ-nonessential by definition. However, observe that

$$\mathcal{E}(X - \mathcal{E}(X)) = \mathcal{E}((-\infty, 0)) = [0, \infty) \neq \varnothing.$$

Hence, there is a σ-nonessential essential closure that is not strong.

More concrete examples of essential closures are constructed and discussed thoroughly in Chapters 3 and 4. Our next aim is to find a way to generate an essential closure from a given collection of sets.

Definition 2.20. Let $\varnothing \neq S \subseteq \Omega \subseteq \mathcal{P}(X)$ where Ω is a σ-algebra over X. Define $\mathcal{N}_\Omega(S)$ to be the smallest collection which satisfies the following conditions for all $B \in \Omega$ and $A, A_1, A_2, \dots \in \mathcal{N}_\Omega(S)$:

(i) $S \subseteq \mathcal{N}_\Omega(S) \subseteq \Omega$,

(ii) $B \subseteq A$ implies $B \in \mathcal{N}_\Omega(S)$,

(iii) $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}_\Omega(S)$.

Remark 2.21. *Note that Ω is one such collection satisfying the above conditions. Moreover, it can be readily verified that nonempty intersections of collections which satisfy the above conditions still satisfy the conditions. Hence $\mathcal{N}_\Omega(S)$ exists and is the intersection of all collections satisfying the conditions in Definition 2.20.*

Collections of sets satisfying the conditions in Definition 2.20 can be viewed as collections of small sets. Such collections appear in various fields of mathematics, e.g., the collection of measure zero sets, the collection of shy sets introduced in [15], the collection of sets of first category[5], etc.

[5] A subset of a topological space X is said to be of *first category* if it can be written as a countable union of nowhere dense subsets of X.

In the sequel, we often demand that every subset of the space be Lindelöf. Such a topological space is called hereditarily Lindelöf. Some examples of these spaces are second countable spaces, Suslin spaces, etc. More information on Suslin spaces can be found in Kechris's book [16]. More information on Lindelöf spaces and second countable spaces can be found in Munkres's book [18].

Theorem 2.22. *Let (X, τ) be a hereditarily Lindelöf space. Given any σ-algebra Ω over X containing the Borel sets and any nonempty collection $S \subseteq \Omega$, there exists a unique σ-nonessential strong essential closure whose collection of nonessential sets is exactly $\mathcal{N}_\Omega(S)$.*

Proof. The idea is from the result of Theorem 2.11. Define, for each $A \in \Omega$,

$$\widetilde{A} = \bigcap\{F \in \Omega \colon F \text{ is closed and } A - F \in \mathcal{N}_\Omega(S)\}.$$

Observe that, for each $A \in \Omega$,

$$\begin{aligned} A - \widetilde{A} &= A - \bigcap\{F \in \Omega \colon F \text{ is closed and } A - F \in \mathcal{N}_\Omega(S)\} \\ &= A - \bigcap_{n=1}^{\infty}\{F_n \in \Omega \colon F_n \text{ is closed and } A - F_n \in \mathcal{N}_\Omega(S)\} \\ &= \bigcup_{n=1}^{\infty}\{A - F_n \colon F_n \in \Omega \text{ is closed and } A - F_n \in \mathcal{N}_\Omega(S)\} \end{aligned}$$

for some countable subcollection $\{F_n\}_{n\in\mathbb{N}}$ by the Lindelöf property. Since $\mathcal{N}_\Omega(S)$ is closed under countable unions, we have $A - \widetilde{A} \in \mathcal{N}_\Omega(S)$. Now, let $A, B \in \Omega$.

(i) $\widetilde{A}$ is closed since it is an intersection of closed sets. Consequently, $A \mapsto \widetilde{A}$ is a self-mapping since the σ-algebra contains the Borel sets.

(ii) $\widetilde{A} \subseteq \overline{A}$ since, for any closed set F such that $A \subseteq F$, we have $F \in \Omega$ and $A - F = \varnothing \in \mathcal{N}_\Omega(S)$.

(iii) $\widetilde{A \cup B} \subseteq \widetilde{A} \cup \widetilde{B}$ since if $F_1, F_2 \in \Omega$ are closed sets such that $A - F_1, B - F_2 \in \mathcal{N}_\Omega(S)$, then $F = F_1 \cup F_2 \in \Omega$ is a closed set and

$$(A \cup B) - F \subseteq (A - F_1) \cup (B - F_2) \in \mathcal{N}_\Omega(S).$$

Moreover, it is straightforward to show that $A \mapsto \widetilde{A}$ is increasing with respect to set inclusion. Consequently, $\widetilde{A} \cup \widetilde{B} \subseteq \widetilde{A \cup B}$.

(iv) Since $A - \widetilde{A} \in \mathcal{N}_\Omega(S)$, we have $\widetilde{A - \widetilde{A}} = \varnothing$ by construction.

Thus $A \mapsto \widetilde{A}$ is a strong essential closure on Ω. Moreover, if $A \in \mathcal{N}_\Omega(S)$, then $\widetilde{A} = \varnothing$ by construction. On the other hand, if $\widetilde{A} = \varnothing$, then $A = A - \widetilde{A} \in \mathcal{N}_\Omega(S)$. Therefore,

$$\widetilde{A} = \varnothing \text{ if and only if } A \in \mathcal{N}_\Omega(S).$$

Thus the collections $\mathcal{N}_\Omega(A \mapsto \widetilde{A})$ and $\mathcal{N}_\Omega(S)$ coincide. In addition, since $\mathcal{N}_\Omega(S)$ is closed under countable union, the induced essential closure is σ-nonessential. The uniqueness part is obvious since the collection of nonessential sets completely determines the corresponding strong essential closure. □

In Theorem 2.22, the assumption that the σ-algebra contains the Borel sets is necessary. This is demonstrated in the following example.

Example 2.23. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$, $S = \{\varnothing\}$ and $\Omega = \{\varnothing, \{a\}, \{b, c\}, X\}$. Observe that (X, τ) is hereditarily Lindelöf, Ω does not contain τ and $\mathcal{N}_\Omega(S) = \{\varnothing\}$. Using the same construction as in Theorem 2.22, we have

$$\widetilde{\{a\}} = X \text{ while } \overline{\{a\}} = \{a, c\}.$$

Hence the induced mapping is not an essential closure since it violates the second property of essential closures.

2.3 Essential closedness

In this section, we introduce another important notion related to essential closures: the notion of essential closedness.

Definition 2.24. Let $\mathcal{E}$ be an essential closure on Ω. A set $F \in \Omega$ is said to be essentially closed if and only if $\mathcal{E}(F) = F$. We denote the collection of essentially closed sets by $\mathcal{C}_\Omega(\mathcal{E})$.

Similar to the result obtained in Theorem 2.11, we derive that an essential closure of a set A can also be written as the intersection of essentially closed sets F such that $A - F$ is nonessential.

Proposition 2.25. *Let $\mathcal{E}$ be a strong essential closure on Ω. Then, for any $A \in \Omega$,*

$$\mathcal{E}(A) = \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon A - F \in \mathcal{N}_\Omega(\mathcal{E})\}.$$

Proof. Since $\mathcal{E}$ is strong, $\mathcal{E}(A)$ is in the intersection. Moreover, for any $F \in \mathcal{C}_\Omega(\mathcal{E})$ with $A - F \in \mathcal{N}_\Omega(\mathcal{E})$, we have

$$\mathcal{E}(A) = \mathcal{E}(A \cap F) \cup \mathcal{E}(A - F) = \mathcal{E}(A \cap F) \subseteq \mathcal{E}(F) = F.$$

This completes the proof. $\square$

The following proposition gives an interpretation of the notion of essential closedness: an essentially closed set is closed and locally essential.

Proposition 2.26. *Let $\mathcal{E}$ be an essential closure on an algebra Ω and $F \in \Omega$. If F is essentially closed, then F is closed and, for any open set $G \in \Omega$ intersecting F, $G \cap F$ is essential.*

Proof. If F is essentially closed, then $F = \mathcal{E}(F)$ is closed. Observe that

$$\mathcal{E}(F - G) \subseteq \overline{F - G} = F - G \subsetneq F.$$

Moreover, $F = \mathcal{E}(F) = \mathcal{E}(F - G) \cup \mathcal{E}(F \cap G)$. Thus $\mathcal{E}(F \cap G) \neq \varnothing$. $\square$

Proposition 2.27. *Let $\mathcal{E}$ be a strong essential closure on an algebra Ω and $F \in \Omega$. If F is closed and, for any open set $G \in \Omega$ intersecting F, $G \cap F$ is essential, then F is essentially closed.*

Proof. Since F is closed, $\mathcal{E}(F) \subseteq \overline{F} = F$. Suppose $F \cap (\mathcal{E}(F))^c = F - \mathcal{E}(F) \neq \varnothing$. Then $\mathcal{E}(F - \mathcal{E}(F)) \neq \varnothing$ by assumption, contradicting the fact that $\mathcal{E}$ is strong. $\square$

Combining Propositions 2.26 and 2.27 gives a characterization of the notion of essential closedness for strong essential closures.

Corollary 2.28. *Let $\mathcal{E}$ be a strong essential closure on an algebra Ω and $F \in \Omega$. Then F is essentially closed if and only if F is closed and, for any open set $G \in \Omega$ intersecting F, $G \cap F$ is essential.*

Example 2.29. Recall the essential closure defined in Example 2.7. Choose $F = X$. One can see that, for any nonempty open set $G \in \Omega$, $\mathcal{E}(G \cap F) \neq \varnothing$. However, $\mathcal{E}(F) = \{1\} \neq F$, i.e., F is not essentially closed. Thus the assumption that the essential closure is strong is necessary in Proposition 2.27.

2.4 Essential compactness

We define the notion of essential compactness via the notion of essential covering, which is analogous to how the notion of topological compactness is defined via the notion of open covering.

Definition 2.30. Let $A \mapsto \widetilde{A}$ be an essential closure on Ω. An essential cover $\{E_\alpha\}_{\alpha\in\Lambda}$ of a set $A \in \Omega$ is an open cover of A such that, for each $\alpha \in \Lambda$, $E_\alpha \cap A$ is either empty or essential.

Definition 2.31. Let $A \mapsto \widetilde{A}$ be an essential closure on Ω. A set $K \in \Omega$ is said to be essentially compact if and only if, for each open cover of K, there exists a finite essential subcover.

Theorem 2.32. *Let $A \mapsto \widetilde{A}$ be an essential closure on Ω and $K \in \Omega$. If K is compact and essentially closed, then K is essentially compact.*

Proof. Assume that K is compact and essentially closed. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be an open cover of K. Then there exists a finite subcover: $\{E_1, \ldots, E_n\}$. By Proposition 2.26, for each i such that $E_i \cap K \neq \varnothing$, we have $\widetilde{E_i \cap K} \neq \varnothing$. Hence, K is essentially compact. $\square$

The converse of Theorem 2.32 requires an additional assumption, for example, that the space is Hausdorff (i.e., T_2). In particular, every compact set is closed. More information on Hausdorff spaces can be found in Munkres's book [18].

Theorem 2.33. *Let (X, τ) be a Hausdorff space, $A \mapsto \widetilde{A}$ be a strong essential closure on Ω, and $K \in \Omega$. If K is essentially compact, then K is compact and essentially closed.*

Proof. The case where $K = \varnothing$ is trivial since $\varnothing$ is compact, essential closed and essentially compact. Assume that $K \neq \varnothing$ and each open cover $\{E_\alpha\}_{\alpha \in \Lambda}$ of K has a finite essential subcover. Then K is compact since an essential cover is an open cover. It is left to show that K is essentially closed.

Suppose $K \neq \widetilde{K}$. Since K is compact, hence closed, $\widetilde{K} \subseteq \overline{K} = K$. Thus there exists $x \in K$ such that $x \notin \widetilde{K}$. Observe that $\{(\widetilde{K})^c, \{x\}^c\}$ is an open cover of K and any subcover of it is itself unless $\widetilde{K} = \varnothing$. (Note that $\widetilde{K}$ cannot be empty otherwise $\{X\}$ is an open cover of K and $X \cap K \neq \varnothing$ but $\widetilde{X \cap K} = \varnothing$, contradicting the fact that K is essentially compact.) Therefore, $\{(\widetilde{K})^c, \{x\}^c\}$ is an essential cover of K. Since $K - \widetilde{K} \neq \varnothing$, we have $\widetilde{K - \widetilde{K}} \neq \varnothing$, contradicting the fact that the essential closure is strong. Thus K is essentially closed. $\square$

Combining Theorems 2.32 and 2.33, we obtain a characterization of the notion of essential compactness.

Corollary 2.34. *Let X be a Hausdorff space, $A \mapsto \widetilde{A}$ be a strong essential closure on Ω, and $K \in \Omega$. Then K is essentially compact if and only if K is compact and essentially closed.*

To conclude, in a Hausdorff space equipped with a strong essential closure, the notion of essential compactness is completely characterized by the notions of essential closedness and topological compactness.

2.5 An ordering on the class of essential closures

A natural way to compare two functions on a common space is to compare them in a pointwise fashion. In this section, we introduce a natural partial ordering on the class of set-valued functions on a common space.

Definition 2.35. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be set-valued functions on a common space. Then we say that $\mathcal{E}_1$ is finer than or equal to $\mathcal{E}_2$ or $\mathcal{E}_1$ is coarser than or equal to $\mathcal{E}_2$, denoted by $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$, if $\mathcal{E}_1(A) \subseteq \mathcal{E}_2(A)$ for each element A in the domain.

Proposition 2.36. *Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are essential closures on a common space such that $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$. Then*

(i) $\mathcal{E}_2 \circ \mathcal{E}_1 = \mathcal{E}_1$ *and*

(ii) $\mathcal{E}_1 \preccurlyeq \mathcal{E}_1 \circ \mathcal{E}_2 \preccurlyeq \mathcal{E}_2$.

Proof. Observe that $\mathcal{E}_1(A) = \mathcal{E}_1(\mathcal{E}_1(A)) \subseteq \mathcal{E}_2(\mathcal{E}_1(A)) \subseteq \overline{\mathcal{E}_1(A)} = \mathcal{E}_1(A)$. Moreover, observe that $\mathcal{E}_1(A) = \mathcal{E}_1(\mathcal{E}_1(A)) \subseteq \mathcal{E}_1(\mathcal{E}_2(A)) \subseteq \mathcal{E}_2(\mathcal{E}_2(A)) = \mathcal{E}_2(A)$. $\square$

Proposition 2.37. *Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are essential closures on a common space such that $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$. If F is $\mathcal{E}_1$-essentially closed, then F is $\mathcal{E}_2$-essentially closed.*

Proof. If $\mathcal{E}_1(F) = F$, then F is closed and $F = \mathcal{E}_1(F) \subseteq \mathcal{E}_2(F) \subseteq \overline{F} = F$. $\square$

Proposition 2.38. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be strong essential closures on a common space. Then $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$ if and only if the collection of $\mathcal{E}_1$-nonessential sets contains the collection of $\mathcal{E}_2$-nonessential sets.*

Proof. The implication is obvious and the converse follows directly from Theorem 2.11. $\square$

2.6 Completions of essential closure spaces

In this section, we introduce the notion of complete essential closure spaces. The idea is similar to the notion of complete measure spaces. Our objective is to expand the domain and then extend the essential closure so that every subset of a nonessential set is also nonessential.

Definition 2.39. A quadruple $(X, \tau, \Omega, \mathcal{E})$ consisting of a nonempty set X, a topology τ, an algebra Ω, and an essential closure $\mathcal{E}$ on (X, τ, Ω) is called an essential closure space.

Theorem 2.40. *Given a σ-nonessential essential closure space $(X, \tau, \Omega, \mathcal{E})$, define*

$$\overline{\Omega} = \{E \cup F \colon E \in \Omega \text{ and } F \text{ is a subset of some } \mathcal{E}\text{-nonessential set}\}.$$

Then $\overline{\Omega}$ is a σ-algebra generated by Ω and all subsets of $\mathcal{E}$-nonessential sets.

Proof. Clearly, $\varnothing \in \overline{\Omega}$. Now, let $A, A_1, A_2, \cdots \in \overline{\Omega}$. Then there are $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set N such that $A = E \cup F$. Thus

$$A^c = E^c \cap F^c = E^c \cap (N^c \cup (F^c - N^c)) = (E \cup N)^c \cup (E^c \cap (N - F)).$$

Observe that $(E \cup N)^c \in \Omega$ and $E^c \cap (N - F) \subseteq N$. Hence $A^c \in \overline{\Omega}$. Moreover, there are $E_i \in \Omega$ and F_i a subset of an $\mathcal{E}$-nonessential set N_i such that $A_i = E_i \cup F_i$ for each i. Therefore,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (E_i \cup F_i) = \left(\bigcup_{i=1}^{\infty} E_i\right) \cup \left(\bigcup_{i=1}^{\infty} F_i\right) \in \overline{\Omega}$$

since Ω is a σ-algebra and $\mathcal{E}$ is σ-nonessential. Thus $\overline{\Omega}$ is a σ-algebra.

Observe that $\overline{\Omega}$ contains Ω and all subsets of $\mathcal{E}$-nonessential sets. Moreover, by construction, $\overline{\Omega}$ is contained in the σ-algebra generated by Ω and all subsets of $\mathcal{E}$-nonessential sets. □

Theorem 2.41. *Given a σ-nonessential essential closure space $(X, \tau, \Omega, \mathcal{E})$, define a set function $\overline{\mathcal{E}}$ on $\overline{\Omega}$ by*

$$\overline{\mathcal{E}}(E \cup F) = \mathcal{E}(E)$$

for each $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set. Then $\overline{\mathcal{E}}$ is a σ-nonessential essential closure on $\overline{\Omega}$ whose nonessential sets are exactly the subsets of $\mathcal{E}$-nonessential sets. Moreover, $\overline{\mathcal{E}}$ is the unique essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$.

Proof. First of all, $\overline{\mathcal{E}}$ is well-defined. To see this, let $A \cup B = A' \cup B'$ where $A, A' \in \Omega$ and B and B' are subsets of $\mathcal{E}$-nonessential sets C and C', respectively. Then $A' \subseteq A' \cup B' \subseteq A \cup C$, which implies $\mathcal{E}(A') \subseteq \mathcal{E}(A)$. Similarly, we have $\mathcal{E}(A) \subseteq \mathcal{E}(A')$. Moreover, $\overline{\mathcal{E}}$ is a self-mapping on $\overline{\Omega}$ since $\Omega \subseteq \overline{\Omega}$.

Now, for each $A, B \in \overline{\Omega}$, there are $E, G \in \Omega$ and subsets F and H of $\mathcal{E}$-nonessential sets N and M, respectively, such that $A = E \cup F$ and $B = G \cup H$. The following are readily verified.

(i) $\overline{\mathcal{E}}(A)$ is closed.

(ii) $\overline{\mathcal{E}}(A) = \mathcal{E}(E) \subseteq \overline{E} \subseteq \overline{A}$.

(iii) Observe that

$$\overline{\mathcal{E}}(A \cup B) = \overline{\mathcal{E}}((E \cup G) \cup (F \cup H)) = \mathcal{E}(E \cup G) = \mathcal{E}(E) \cup \mathcal{E}(G) = \overline{\mathcal{E}}(A) \cup \overline{\mathcal{E}}(B).$$

(iv) Observe that

$$\overline{\mathcal{E}}(\overline{\mathcal{E}}(A)) = \overline{\mathcal{E}}(\mathcal{E}(E)) = \mathcal{E}(\mathcal{E}(E)) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

Therefore, $\overline{\mathcal{E}}$ is an essential closure on $\overline{\Omega}$. Moreover, it is easy to see that $\overline{\mathcal{E}}(A) = \varnothing$ if and only if A is a subset of an $\mathcal{E}$-nonessential set. As a consequence, since $\mathcal{E}$ is σ-nonessential, $\overline{\mathcal{E}}$ is also σ-nonessential.

In addition, let $\mathcal{E}'$ be another essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$ and let $A \in \overline{\Omega}$. Then there are $E \in \Omega$ and F a subset of $\mathcal{E}$-nonessential set N such that $A = E \cup F$. Thus

$$\mathcal{E}'(A) = \mathcal{E}'(E \cup F) \subseteq \mathcal{E}'(E) \cup \mathcal{E}'(N) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

On the other hand,

$$\mathcal{E}'(A) = \mathcal{E}'(E \cup F) \supseteq \mathcal{E}'(E) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

Hence $\overline{\mathcal{E}}$ is the unique essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$. □

Definition 2.42. Given a σ-nonessential essential closure space $(X, \tau, \Omega, \mathcal{E})$, we define the completion of $(X, \tau, \Omega, \mathcal{E})$ to be the essential closure space $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ whose σ-algebra $\overline{\Omega}$ and essential closure $\overline{\mathcal{E}}$ are defined in Theorems 2.40 and 2.41, respectively.

Proposition 2.43. *The completion of a σ-nonessential strong essential closure space is a σ-nonessential strong essential closure space.*

Proof. Let $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ be the completion of an essential closure space $(X, \tau, \Omega, \mathcal{E})$ where $\mathcal{E}$ is strong. Then, for each $A \in \overline{\Omega}$, there exist $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set such that $A = E \cup F$. Thus

$$\begin{aligned}
\overline{\mathcal{E}}(A - \overline{\mathcal{E}}(A)) &= \overline{\mathcal{E}}((E \cup F) - \overline{\mathcal{E}}(E \cup F)) \\
&= \overline{\mathcal{E}}((E \cup F) - \mathcal{E}(E)) \\
&= \overline{\mathcal{E}}((E - \mathcal{E}(E)) \cup (F - \mathcal{E}(E)) \\
&= \mathcal{E}(E - \mathcal{E}(E)) \\
&= \varnothing.
\end{aligned}$$

Hence $\overline{\mathcal{E}}$ is strong. □

Example 2.44. Assume that (X, τ) is Lindelöf. Let Ω be a σ-algebra over X containing the Borel sets. If an essential closure space $(X, \tau, \Omega, \mathcal{E})$ is associated with a measure, then the completion $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ is the essential closure space associated with the completion of that measure. This is due to Theorem 3.10 and Corollary 3.11 in Chapter 3 that, in a Lindelöf space, if an essential closure is associated with a measure, then it is σ-nonessential and the nonessential sets are exactly the measure zero sets.

Similar to the case of σ-nonessential essential closure spaces, if we instead start with an algebra Ω and an essential closure $\mathcal{E}$, then we may take an algebraic completion of Ω to be the algebra generated by Ω and all subsets of $\mathcal{E}$-nonessential sets and take an algebraic completion of $\mathcal{E}$ to be the unique extension of $\mathcal{E}$ on the algebraic completion of Ω. Notice that we do not require any additional assumption on the algebra or on the essential closure in this case. However, since we rarely encounter algebras in practice, algebraic completions of essential closure spaces may have less practical uses compared to completions of σ-nonessential essential closure spaces. Nevertheless, it is theoretically interesting.

Theorem 2.45. *Given an essential closure space $(X, \tau, \Omega, \mathcal{E})$, define*

$$\overline{\Omega} = \{E \cup F \colon E \in \Omega \text{ and } F \text{ is a subset of some } \mathcal{E}\text{-nonessential set}\}.$$

Then $\overline{\Omega}$ is an algebra generated by Ω and all subsets of $\mathcal{E}$-nonessential sets.

Proof. Clearly, $\varnothing \in \overline{\Omega}$. Now, let $A, A_1, \ldots, A_n \in \overline{\Omega}$. Then there are $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set N such that $A = E \cup F$. Thus

$$A^c = E^c \cap F^c = E^c \cap (N^c \cup (F^c - N^c)) = (E \cup N)^c \cup (E^c \cap (N - F)).$$

Observe that $(E \cup N)^c \in \Omega$ and $E^c \cap (N - F) \subseteq N$. Hence $A^c \in \overline{\Omega}$. Moreover, there are $E_i \in \Omega$ and F_i a subset of an $\mathcal{E}$-nonessential set N_i such that $A_i = E_i \cup F_i$ for each i. Therefore,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (E_i \cup F_i) = \left(\bigcup_{i=1}^{n} E_i\right) \cup \left(\bigcup_{i=1}^{n} F_i\right) \in \overline{\Omega}$$

since Ω is an algebra and $\mathcal{E}$ is distributive over finite unions. Thus $\overline{\Omega}$ is an algebra.

Observe that $\overline{\Omega}$ contains Ω and all subsets of $\mathcal{E}$-nonessential sets. Moreover, by construction, $\overline{\Omega}$ is contained in the algebra generated by Ω and all subsets of $\mathcal{E}$-nonessential sets. □

Theorem 2.46. *Given an essential closure space $(X, \tau, \Omega, \mathcal{E})$, define a set function $\overline{\mathcal{E}}$ on $\overline{\Omega}$ by*

$$\overline{\mathcal{E}}(E \cup F) = \mathcal{E}(E)$$

for each $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set. Then $\overline{\mathcal{E}}$ is an essential closure on $\overline{\Omega}$ whose nonessential sets are exactly the subsets of $\mathcal{E}$-nonessential sets. Moreover, $\overline{\mathcal{E}}$ is the unique essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$.

Proof. First of all, $\overline{\mathcal{E}}$ is well-defined. To see this, let $A \cup B = A' \cup B'$ where $A, A' \in \Omega$ and B and B' are subsets of $\mathcal{E}$-nonessential sets C and C', respectively. Then $A' \subseteq A' \cup B' \subseteq A \cup C$, which implies $\mathcal{E}(A') \subseteq \mathcal{E}(A)$. Similarly, we have $\mathcal{E}(A) \subseteq \mathcal{E}(A')$. Moreover, $\overline{\mathcal{E}}$ is a self-mapping on $\overline{\Omega}$ since $\Omega \subseteq \overline{\Omega}$.

Now, for each $A, B \in \overline{\Omega}$, there are $E, G \in \Omega$ and subsets F and H of $\mathcal{E}$-nonessential sets N and M, respectively, such that $A = E \cup F$ and $B = G \cup H$. The following are readily verified.

(i) $\overline{\mathcal{E}}(A)$ is closed.

(ii) $\overline{\mathcal{E}}(A) = \mathcal{E}(E) \subseteq \overline{E} \subseteq \overline{A}$.

(iii) Observe that

$$\overline{\mathcal{E}}(A \cup B) = \overline{\mathcal{E}}((E \cup G) \cup (F \cup H)) = \mathcal{E}(E \cup G) = \mathcal{E}(E) \cup \mathcal{E}(G) = \overline{\mathcal{E}}(A) \cup \overline{\mathcal{E}}(B).$$

(iv) Observe that

$$\overline{\mathcal{E}}(\overline{\mathcal{E}}(A)) = \overline{\mathcal{E}}(\mathcal{E}(E)) = \mathcal{E}(\mathcal{E}(E)) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

Therefore, $\overline{\mathcal{E}}$ is an essential closure on $\overline{\Omega}$. Moreover, it is easy to see that $\overline{\mathcal{E}}(A) = \varnothing$ if and only if A is a subset of an $\mathcal{E}$-nonessential set.

In addition, let $\mathcal{E}'$ be another essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$ and let $A \in \overline{\Omega}$. Then there are $E \in \Omega$ and F a subset of $\mathcal{E}$-nonessential set N such that $A = E \cup F$. Thus

$$\mathcal{E}'(A) = \mathcal{E}'(E \cup F) \subseteq \mathcal{E}'(E) \cup \mathcal{E}'(N) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

On the other hand,

$$\mathcal{E}'(A) = \mathcal{E}'(E \cup F) \supseteq \mathcal{E}'(E) = \mathcal{E}(E) = \overline{\mathcal{E}}(A).$$

Hence $\overline{\mathcal{E}}$ is the unique essential closure on $\overline{\Omega}$ which extends $\mathcal{E}$. □

Definition 2.47. Given an essential closure space $(X, \tau, \Omega, \mathcal{E})$, we define the algebraic completion of $(X, \tau, \Omega, \mathcal{E})$ to be the essential closure space $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ whose algebra $\overline{\Omega}$ and essential closure $\overline{\mathcal{E}}$ are defined in Theorems 2.45 and 2.46, respectively.

Proposition 2.48. *The algebraic completion of a strong essential closure space is a strong essential closure space.*

Proof. Let $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ be the algebraic completion of an essential closure space $(X, \tau, \Omega, \mathcal{E})$ where $\mathcal{E}$ is strong. Then, for each $A \in \overline{\Omega}$, there exist $E \in \Omega$ and F a subset of an $\mathcal{E}$-nonessential set such that $A = E \cup F$. Thus

$$\begin{aligned}\overline{\mathcal{E}}(A - \overline{\mathcal{E}}(A)) &= \overline{\mathcal{E}}((E \cup F) - \overline{\mathcal{E}}(E \cup F)) \\ &= \overline{\mathcal{E}}((E \cup F) - \mathcal{E}(E)) \\ &= \overline{\mathcal{E}}((E - \mathcal{E}(E)) \cup (F - \mathcal{E}(E)) \\ &= \mathcal{E}(E - \mathcal{E}(E)) \\ &= \varnothing.\end{aligned}$$

Hence $\overline{\mathcal{E}}$ is strong. □

Proposition 2.49. *The algebraic completion of a σ-nonessential essential closure space is a σ-nonessential essential closure space.*

Proof. Let $(X, \tau, \overline{\Omega}, \overline{\mathcal{E}})$ be the algebraic completion of an essential closure space $(X, \tau, \Omega, \mathcal{E})$ where $\mathcal{E}$ is σ-nonessential. Let $A_1, A_2, \dots \in \overline{\Omega}$ such that A_i is a subset of an $\mathcal{E}$-nonessential set N_i for each i. Let

$$A = \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} N_i = N.$$

Thus A is a subset of an $\mathcal{E}$-nonessential set N. By construction, $\overline{\mathcal{E}}(A) = \varnothing$, which implies that $\overline{\mathcal{E}}$ is σ-nonessential. □

2.7 Essential closure operators

In this section, we provide an alternative approach to postulating the notion of essential closures. It is an approach which concentrates on the core properties of essential closures. An advantage of this approach is that we need not assume any a priori topological structure.

Similar to the notion of essential closures, an essential closure operator can be roughly viewed as a unary operation mimicking the topological closure operator but not necessarily extensive. Since there is no underlying topological structure, no additional conditions are included.

Postulate 2.50. Let X be a nonempty set and Ω be an algebra over X. An essential closure operator on (X, Ω) is a unary operation

$$A \mapsto \widetilde{A} \colon \Omega \to \Omega$$

which satisfies the following conditions for all sets $A, B \in \Omega$:

(i) $\widetilde{\varnothing} = \varnothing$,

(ii) $\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$,

(iii) $\widetilde{\widetilde{A}} = \widetilde{A}$.

Remark 2.51. *We choose the domain of an essential closure or an essential closure operator to be an algebra for our convenience in stating the conditions in our results. In fact, we may define an essential closure or an essential closure operator on a more general domain, namely, a collection of sets which is closed under finite unions. Note that every result in this section still holds.*

Remark 2.52. *A topological closure operator restricted to any algebra invariant under the topological closure operator is indeed an essential closure operator. Moreover, it is the unique essential closure operator which is extensive.*

Given an essential closure operator, our objective is to induce a topology with respect to which the essential closure operator is an essential closure. To guarantee the existence of the induced topology, we need the following lemmas.

Lemma 2.53. *Let X be a nonempty set and Ω be an algebra over X. Suppose that $A \mapsto \overline{A} \colon \Omega \to \Omega$ satisfies the following conditions for all sets $A, B \in \Omega$:*

(i) $\overline{\varnothing} = \varnothing$,

(ii) $A \subseteq \overline{A}$,

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

(iv) $\overline{\overline{A}} = \overline{A}$.

Then $A \mapsto \overline{A}$ can be extended to a topological closure operator on X.

Proof. With a slight abuse of notation, define $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C}.$$

First of all, we check that cl is indeed an extension. Suppose that $A \in \Omega$. Then we have

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C} \subseteq \overline{A} \subseteq \bigcap_{\overline{C} \supseteq \overline{A}} \overline{C} \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A)$$

where the first inclusion follows from the fact that $\overline{A}$ is in the intersection, the second inclusion is obvious, and the last inclusion follows from the fact that $A \subseteq \overline{C}$ implies $\overline{A} \subseteq \overline{C}$. Hence, cl is an extension of $A \mapsto \overline{A}$. Moreover, observe the following properties of the unary operation $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$.

(i) $\mathrm{cl}(\varnothing) = \overline{\varnothing} = \varnothing$ since $\varnothing \in \Omega$.

(ii) $A \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A)$.

(iii) Observe that

$$\bigcap_{\overline{C} \supseteq A \cup B} \overline{C} \subseteq \bigcap_{\overline{D} \supseteq A, \overline{E} \supseteq B} \overline{D} \cup \overline{E} = \Big(\bigcap_{\overline{D} \supseteq A} \overline{D}\Big) \cup \Big(\bigcap_{\overline{E} \supseteq B} \overline{E}\Big).$$

Hence $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Moreover, it is straightforward to verify that cl is increasing with respect to set inclusion. It readily follows that $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$.

(iv) If $A \subseteq \overline{C}$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(\overline{C}) = \overline{\overline{C}} = \overline{C}$ since $\overline{C} \in \Omega$. Hence

$$\mathrm{cl}(\mathrm{cl}(A)) = \bigcap_{\overline{C} \supseteq \mathrm{cl}(A)} \overline{C} \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A).$$

On the other hand, $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$ since $A \subseteq \mathrm{cl}(A)$.

Therefore, cl is a topological closure operator on X. $\square$

Remark 2.54. *An extension in Lemma 2.53 is not unique.*

Lemma 2.55. *Let $A \mapsto \widetilde{A}$ be an essential closure operator on an algebra Ω over a nonempty set X. Then there exists a topology on X with respect to which $\widetilde{A}$ is closed and contained in the topological closure of A for each $A \in \Omega$.*

Proof. Define $\overline{A} = A \cup \widetilde{A}$ for each $A \in \Omega$. Observe that, for each $A, B \in \Omega$,

(i) $\overline{\varnothing} = \varnothing \cup \widetilde{\varnothing} = \varnothing$,

(ii) $A \subseteq A \cup \widetilde{A} = \overline{A}$,

(iii) $\overline{A \cup B} = A \cup B \cup \widetilde{A \cup B} = (A \cup \widetilde{A}) \cup (B \cup \widetilde{B}) = \overline{A} \cup \overline{B}$, and

(iv) $\overline{\overline{A}} = A \cup \widetilde{A} \cup \widetilde{A \cup \widetilde{A}} = A \cup \widetilde{A} \cup \widetilde{A} \cup \widetilde{\widetilde{A}} = A \cup \widetilde{A} = \overline{A}$.

According to Lemma 2.53, the unary operation $A \mapsto \overline{A}\colon \Omega \to \Omega$ can be extended to a topological closure operator on X. Let cl be a topological closure operator extended from $A \mapsto \overline{A}\colon \Omega \to \Omega$ and let τ be the topology induced by cl. Since $\widetilde{A} \in \Omega$, we have

$$\mathrm{cl}(\widetilde{A}) = \widetilde{A} \cup \widetilde{\widetilde{A}} = \widetilde{A}.$$

Hence $\widetilde{A}$ is closed in (X, τ) for each $A \in \Omega$. Moreover, $\widetilde{A} \subseteq A \cup \widetilde{A} = \mathrm{cl}(A)$ for each $A \in \Omega$. $\square$

Given an essential closure, if we take out its underlying topological structure, what we get is an essential closure operator. Conversely, the following theorem shows that there is a natural way to induce an underlying topology from a given essential closure operator. However, it is not guaranteed that the induced topology is the same as the give topology. There are cases where the induced topology coincides with the given topology. A nice characterization of such cases will be derived at the end of this chapter.

Let us note that an arbitrary intersection of topologies is again a topology. So, a natural way to induce an underlying topology from a given essential closure operator is to take the (nonempty) intersection of all topologies satisfying the conditions in Lemma 2.55.

Theorem 2.56. *Let $A \mapsto \widetilde{A}$ be an essential closure operator on Ω. Define $\tau_\Omega = \bigcap \tau_\alpha$, where the nonempty intersection is taken over all topologies τ_α on X satisfying the conditions in Lemma 2.55, and let cl_Ω be the topological closure induced by τ_Ω. Then, $A \mapsto \widetilde{A} \colon \Omega \to \Omega$ satisfies the following conditions for all sets $A, B \in \Omega$:*

(i) *$\widetilde{A}$ is closed in (X, τ_Ω),*

(ii) *$\widetilde{A} \subseteq \mathrm{cl}_\Omega(A)$,*

(iii) *$\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$,*

(iv) *$\widetilde{\widetilde{A}} = \widetilde{A}$.*

In other words, $A \mapsto \widetilde{A}$ is an essential closure on (X, τ_Ω, Ω). Furthermore, τ_Ω is generated by the collection $\{(\widetilde{A})^c\}_{A \in \Omega}$.

Proof. To show that $A \mapsto \widetilde{A}$ is an essential closure, it suffices to show that $\widetilde{A}$ is closed in (X, τ_Ω) and $\widetilde{A} \subseteq \mathrm{cl}_\Omega(A)$ since it is already an essential closure operator. First of all, $\widetilde{A}$ is closed in (X, τ_Ω) since $(\widetilde{A})^c \in \tau_\alpha$ for each α. So, $(\widetilde{A})^c \in \tau_\Omega$. Moreover, $\widetilde{A} \subseteq \mathrm{cl}_\alpha(A) \subseteq \mathrm{cl}_\Omega(A)$ since $\tau_\Omega \subseteq \tau_\alpha$.

Furthermore, let τ be the topology generated by $\{(\widetilde{A})^c\}_{A \in \Omega}$. Since $\widetilde{A}$ is closed in (X, τ_Ω) for each $A \in \Omega$, we have $\tau \subseteq \tau_\Omega$. Consequently, $\mathrm{cl}_\Omega(A) \subseteq \mathrm{cl}_\tau(A)$ for each $A \in \Omega$. Therefore, $\widetilde{A} \subseteq \mathrm{cl}_\Omega(A) \subseteq \mathrm{cl}_\tau(A)$ for each $A \in \Omega$. Moreover, $\widetilde{A}$ is closed in (X, τ) for each $A \in \Omega$ since τ is generated by $\{(\widetilde{A})^c\}_{A \in \Omega}$. Hence τ is a topology satisfying the conditions in Lemma 2.55, which implies that $\tau_\Omega \subseteq \tau$. Thus the two topologies coincide. □

Remark 2.57 (Consistency)**.** *Let $A \mapsto \overline{A}$ be a topological closure operator, hence an essential closure operator. One can see that the topology induced by $A \mapsto \overline{A}$ as an essential closure operator is the same as the topology induced by $A \mapsto \overline{A}$ as a topological closure operator.*

Given an essential closure operator $\mathcal{E}$ on (X, Ω). Any topology τ containing τ_Ω with the property that $\mathcal{E}(A) \subseteq \mathrm{cl}(A)$ for all $A \in \Omega$, where cl is the topological closure induced by τ, is said to be compatible with the given essential closure operator. Notice that τ is a compatible topology if and only if $(X, \tau, \Omega, \mathcal{E})$ is an essential closure space.

There can be several compatible topologies on a given essential closure operator space. The induced topology τ_Ω in Theorem 2.56 is the smallest of such topologies and is called the canonical topology.

Example 2.58. Recall the essential closure space $(X, \tau, \Omega, \mathcal{E})$ defined in Example 2.17. According to Theorem 2.56, the canonical topology τ_Ω is generated by the collection $\{\{1\}^c, X\}$. Hence the given topology $\tau = \{\varnothing, \{1\}^c, X\}$ is canonical.

Let $\tau^* = \{\varnothing, \{1\}^c, \{2\}^c, \{1,2\}^c, X\}$, which is another topology on X. Observe that $\tau \subseteq \tau^*$ and it is straightforward to check that, for each $A \in \Omega$, $\mathcal{E}(A)$ is contained in the topological closure (with respect to τ^*) of the set A. Therefore, $(X, \tau^*, \Omega, \mathcal{E})$ is an essential closure space. Hence τ^* is a compatible topology which is not canonical.

Example 2.59. Recall the definition of the essential closure $(A \mapsto \overline{A}^e)$ on the real line defined in [13]. This is an essential closure (with respect to the standard topology τ_s) on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R})$. If we temporarily take out the standard topology and view the essential closure as an essential closure operator on $\mathfrak{L}(\mathbb{R})$, then the canonical topology (i.e., the induced topology $\tau_{\mathfrak{L}(\mathbb{R})}$ in Theorem 2.56) is indeed the given standard topology.

First of all, observe that $\overline{A}^e$ is closed with respect to the standard topology for each $A \in \mathfrak{L}(\mathbb{R})$. Hence $\tau_{\mathfrak{L}(\mathbb{R})} \subseteq \tau_s$. On the other hand, it suffices to show that each nonempty bounded open interval is of the form $(\overline{A}^e)^c$ for some $A \in \mathfrak{L}(\mathbb{R})$. Let $(a, b) \subseteq \mathbb{R}$ where $a < b$. Choose $A = (-\infty, a] \cup [b, \infty)$. Then $\overline{A}^e = (-\infty, a] \cup [b, \infty)$. Thus $(\overline{A}^e)^c = (a, b)$ as desired. Hence $\tau_s \subseteq \tau_{\mathfrak{L}(\mathbb{R})}$. Observe that $A \mapsto \overline{A}^e$ restricted to $\mathfrak{B}(\mathbb{R})$, the Borel σ-algebra over $\mathbb{R}$, is also an essential closure. Similarly, it can be shown that $\tau_{\mathfrak{B}(\mathbb{R})} = \tau_s$ as well.

We end this chapter with the following result, which gives a characterization of canonical topologies.

Theorem 2.60. *Let $\mathcal{E}$ be an essential closure on (X, τ, Ω). Then τ is the canonical topology if and only if there exists a subbase of τ whose elements are of the form $(\mathcal{E}(A))^c$ where $A \in \Omega$.*

Proof. If τ is canonical, then τ is generated by $\{(\mathcal{E}(A))^c\}_{A\in\Omega}$. Conversely, if τ is generated by a subcollection of $\{(\mathcal{E}(A))^c\}_{A\in\Omega}$, then $\tau \subseteq \tau_\Omega$. On the other hand, for each $A \in \Omega$, $\mathcal{E}(A)$ is closed in (X, τ) since $\mathcal{E}$ is an essential closure on (X, τ, Ω). Thus $\tau_\Omega \subseteq \tau$. Therefore, τ is the canonical topology. $\square$

CHAPTER III
SUBMEASURE CLOSURES

In this chapter, we construct concrete examples of essential closures via submeasures and demonstrate their applications, especially in the study of supports of measures and the notion of essential supports of functions. First, we introduce the notion of submeasures which generalizes both measures and outer measures.

Definition 3.1. Let Ω be a σ-algebra over a set X. A submeasure on Ω is a set function $\mu\colon \Omega \to [0,\infty]$ satisfying:

(i) $\mu(\varnothing) = 0$,

(ii) $\mu(A) \le \mu(B)$ for any $A, B \in \Omega$ such that $A \subseteq B$,

(iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i)$ for any $A_1, A_2, \dots \in \Omega$.

Remark 3.2. *There is a classical notion of submeasures which is different from our definition. The classical notion of submeasures were introduced in the study of one of the classical problems in measure theory known as the control measure problem. Unlike our definition of submeasures, classical submeasures are defined on algebras, finitely additive, and finite. More information on classical submeasures and the control measure problem can be found in [5, 6, 10].*

One can see that our notion of submeasures is a generalization of both the notions of measures and outer measures. However, they behave more like an outer measure than a measure. In fact, every submeasure on a σ-algebra can be extended, though not uniquely, to an outer measure. In other words, every submeasure is a restriction of some outer measure. A proof of this fact is given, in details, in the following proposition.

Proposition 3.3. *Every submeasure can be extended to an outer measure.*

Proof. Let μ be a submeasure on Ω. Define

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(B_i) \colon B_i \in \Omega \text{ and } A \subseteq \bigcup_{i=1}^{\infty} B_i\}.$$

Let $A \in \Omega$. On one hand, $\mu^*(A) \leq \mu(A)$ since $A \in \Omega$ covers itself. On the other hand, $\mu^*(A) \geq \mu(A)$ since μ is countably subadditive. Thus μ^* extends μ. Consequently, we have $\mu^*(\varnothing) = \mu(\varnothing) = 0$. In addition, if $A \subseteq B$, then a covering of B is also a covering of A. Hence $\mu^*(A) \leq \mu^*(B)$. It is left to show that μ^* is countably subadditive.

Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists a covering $\{B_{ij}\}_{i\in\mathbb{N}}$ of A_i such that $B_{ij} \in \Omega$ and

$$\sum_{j=1}^{\infty} \mu(B_{ij}) < \mu^*(A_i) + \frac{\epsilon}{2^i}.$$

Thus $\{B_{ij}\}_{i,j\in\mathbb{N}}$ covers A and

$$\mu^*(A) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(B_{ij}) < \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon.$$

Since ϵ is arbitrary, μ^* is countably subadditive. □

Definition 3.4. A submeasure is said to be trivial if the space is of submeasure zero and is said to be normalized if the space is of submeasure one.

Every nontrivial submeasure can be normalized, i.e., given any nontrivial submeasure, there is a normalized submeasure with the same collection of submeasure zero sets.

Proposition 3.5. *Every nontrivial submeasure can be normalized.*

Proof. Let μ be a nontrivial submeasure on a measurable space (X, Ω). Define μ' on (X, Ω) by $\mu'(A) = 0$ if $\mu(A) = 0$ and $\mu'(A) = 1$ otherwise. Then $\mu'(X) = 1$ since μ is nontrivial. Let $A_1, A_2, \cdots \in \Omega$.

(i) $\mu'(\varnothing) = 0$ since $\mu(\varnothing) = 0$.

(ii) Suppose $A_1 \subseteq A_2$. If $\mu(A_2) = 0$, then $\mu(A_1) = 0$. Thus $\mu'(A_1) = \mu'(A_2)$. If $\mu(A_2) > 0$, then $\mu'(A_2) = 1$. Hence $\mu'(A_1) \leq \mu'(A_2)$.

(iii) If $\mu(A_i) = 0$ for all i, then $\mu(\bigcup_{i=1}^{\infty} A_i) = 0$ by the subadditivity. Consequently,

$$\mu'(\bigcup_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} \mu'(A_i).$$

If $\mu(A_j) > 0$ for some j, then $\mu'(A_j) = 1$. Consequently,

$$\mu'(\bigcup_{i=1}^{\infty} A_i) \leq 1 \leq \sum_{i=1}^{\infty} \mu'(A_i).$$

Hence μ' is a submeasure on (X, Ω). Moreover, the collections of submeasure zero sets coincide by construction. $\square$

In practice, when we deal with a topological measurable space, we rarely encounter an open set which is not measurable. This is because open sets are considered to be well-behaved. It would be peculiar if they were not measurable. So for every topological measurable space in the sequel, the σ-algebra is assumed to contain the Borel sets, i.e., the open sets are measurable.

3.1 Definition and properties

Definition 3.6. Let $(X, \tau, \mathfrak{M}, \mu)$ be a topological submeasure space[1]. For any measurable set $A \in \mathfrak{M}$, we say that $x \in \overline{A}^{\mu}$ if $\mu(G \cap A) > 0$ for every $G \in \mathfrak{N}(x)$. The set $\overline{A}^{\mu}$ is called the μ-closure of A.

A measure ν is said to be absolutely continuous with respect to a measure μ on a common measurable space, denoted by $\nu \ll \mu$, if for every measurable set A, $\mu(A) = 0$ implies $\nu(A) = 0$.

Remark 3.7. *Suppose ν and μ are measures on a common topological measurable space with $\nu \ll \mu$. Then the ν-closure is finer than or equal to the μ-closure, i.e., $\overline{A}^{\nu} \subseteq \overline{A}^{\mu}$ for each measurable set A.*

[1] A *topological submeasure space* is a topological measurable space equipped with a submeasure.

A submeasure closure is not always an essential closure. One such example is given below.

Example 3.8. Let $X = (-\infty, 0]$. Let τ be the topology on X generated by the collection of singletons $\{x\}$ where $x \in (-\infty, 0)$. Notice that every point $x \in (-\infty, 0)$ has a neighborhood that is countable, namely, the singleton $\{x\}$. However, the only neighborhood of 0 is X, which is uncountable. Let us remark that X is, in fact, compact since every open cover of X has to contain X itself.

Consider the Borel σ-algebra $\mathfrak{B}(X)$ generated by the topology τ. Define a measure μ for each Borel measurable set A by $\mu(A) = 0$ if A is countable and $\mu(A) = \infty$ otherwise. Observe that

$$\overline{X}^{\mu} = \{0\} \text{ while } \overline{\overline{X}^{\mu}}^{\mu} = \varnothing.$$

Thus the μ-closure is not idempotent. Hence it is not an essential closure. Nevertheless, we would like to point out that every submeasure closure satisfies the other three properties in Postulate 2.1.

Note that, by inner regular measure, we mean a measure μ on a Hausdorff space equipped with a σ-algebra containing the Borel sets for which the measure of a measurable set can be approximated from within by compact subsets, i.e., for each measurable set A,

$$\mu(A) = \sup\{\mu(K) \colon K \text{ is compact and } K \subseteq A\}.$$

Two sufficient conditions which guarantee that a submeasure closure is a strong essential closure are given in the following theorem. One is a condition on the space while the other is a condition on the submeasure.

Theorem 3.9. *Assume that $(X, \tau, \mathfrak{M}, \mu)$ is either a hereditarily Lindelöf submeasure space*[2] *or an inner regular measure space*[3]*. Then the μ-closure is a strong essential closure.*

---

[2] A *hereditarily Lindelöf submeasure space* is a topological submeasure space where every subset is Lindelöf.

[3] An *inner regular measure space* is a Hausdorff measurable space equipped with an inner regular measure.

Proof. Let A and B be measurable sets.

(i) If $x \in \overline{\overline{A}^{\mu}}^{\mu}$ and $G \in \mathfrak{N}(x)$, then $G \cap \overline{A}^{\mu} \neq \varnothing$. Let $y \in G \cap \overline{A}^{\mu}$. Thus $G \in \mathfrak{N}(y)$ and $y \in \overline{A}^{\mu}$. Therefore, $\mu(G \cap A) > 0$. Hence $x \in \overline{A}^{\mu}$. Consequently, $\overline{A}^{\mu}$ is closed.

(ii) If $x \notin \overline{A}$, then there exists $G \in \mathfrak{N}(x)$ such that $G \cap A = \varnothing$. Thus we have $\mu(G \cap A) = 0$. So $x \notin \overline{A}^{\mu}$.

(iii) If $x \notin \overline{A}^{\mu}$ and $x \notin \overline{B}^{\mu}$, then there exist $G_1, G_2 \in \mathfrak{N}(x)$ with $\mu(G_1 \cap A) = 0$ and $\mu(G_2 \cap A) = 0$. Choose $G = G_1 \cap G_2 \in \mathfrak{N}(x)$. Thus

$$\mu(G \cap (A \cup B)) \leq \mu(G_1 \cap A) + \mu(G_2 \cap B) = 0.$$

So $x \notin \overline{A \cup B}^{\mu}$. Moreover, it is straightforward to show that the μ-closure is increasing with respect to set inclusion. Consequently, $\overline{A}^{\mu} \cup \overline{B}^{\mu} \subseteq \overline{A \cup B}^{\mu}$.

(iv) If $(X, \tau, \mathfrak{M}, \mu)$ is a hereditarily Lindelöf submeasure space, consider each $x \in A - \overline{A}^{\mu}$. We have $x \notin \overline{A - \overline{A}^{\mu}}^{\mu}$ otherwise $x \in \overline{A}^{\mu}$. Then there exists $\{G_x\}_{x \in A - \overline{A}^{\mu}}$ such that $G_x \in \mathfrak{N}(x)$ and $\mu(G_x \cap (A - \overline{A}^{\mu})) = 0$. Notice that $\{G_x\}_{x \in A - \overline{A}^{\mu}}$ is an open cover of $A - \overline{A}^{\mu}$, which is Lindelöf. Hence there exists a countable subcover: $\{G_1, G_2, \dots\}$. Thus

$$\mu(A - \overline{A}^{\mu}) \leq \sum_{i=1}^{\infty} \mu(G_i \cap (A - \overline{A}^{\mu})) = 0.$$

Therefore, $\overline{A - \overline{A}^{\mu}}^{\mu} = \varnothing$.

If $(X, \tau, \mathfrak{M}, \mu)$ is an inner regular measure space, consider any compact set $K \subseteq A - \overline{A}^{\mu}$. For any $x \in K$, $x \in A - \overline{A}^{\mu}$. Therefore, $x \notin \overline{A - \overline{A}^{\mu}}^{\mu}$. Then there exists $\{G_x\}_{x \in K}$ such that $G_x \in \mathfrak{N}(x)$ and

$$\mu(G_x \cap K) \leq \mu(G_x \cap (A - \overline{A}^{\mu})) = 0.$$

Notice that $\{G_x\}_{x \in K}$ is an open cover of K, which is compact. So there exists a finite subcover: $\{G_1, \dots, G_n\}$. Thus

$$\mu(K) \leq \sum_{i=1}^{n} \mu(G_i \cap K) = 0.$$

Therefore, $\mu(A - \overline{A}^{\mu}) = 0$ by the inner regularity. Thus $\overline{A - \overline{A}^{\mu}}^{\mu} = \varnothing$. $\square$

The following result gives a characterization of submeasure zero sets via the corresponding submeasure closure, which shows that a submeasure closure can be used to detect the collection of submeasure zero sets if the conditions are met.

Theorem 3.10. *Assume that $(X, \tau, \mathfrak{M}, \mu)$ is either a Lindelöf submeasure space or an inner regular measure space. Then $\overline{A}^{\mu} = \varnothing$ if and only if $\mu(A) = 0$.*

Proof. The converse follows directly from the definition of submeasure closures. So, it is left to show the implication. Let $\overline{A}^{\mu} = \varnothing$.

If $(X, \tau, \mathfrak{M}, \mu)$ is a Lindelöf submeasure space, then for each $x \in X$, there exists $G_x \in \mathfrak{N}(x)$ such that $\mu(G_x \cap A) = 0$. Thus there exists a countable subcover $\{G_1, G_2, \dots\}$ of X. Therefore,

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(G_i \cap A) = 0.$$

If $(X, \tau, \mathfrak{M}, \mu)$ is an inner regular measure space, then let K be any compact subset of A. It is straightforward from the definition of submeasure closures that $\overline{K}^{\mu} \subseteq \overline{A}^{\mu} = \varnothing$. Thus, for each $x \in X$, there exists $G_x \in \mathfrak{N}(x)$ such that $\mu(G_x \cap K) = 0$. Since K is compact and $\{G_x\}_{x \in K}$ covers K, there exists a finite subcover $\{G_1, \dots, G_n\}$ of K. Therefore,

$$\mu(K) \leq \sum_{i=1}^{n} \mu(G_i \cap K) = 0.$$

By the inner regularity, $\mu(A) = 0$. $\square$

Corollary 3.11. *Assume that $(X, \tau, \mathfrak{M}, \mu)$ is either a Lindelöf submeasure space*[4] *or an inner regular measure space. If the μ-closure is an essential closure, then it is σ-nonessential.*

[4] A *Lindelöf submeasure space* is a Lindelöf space equipped with a σ-algebra and a submeasure.

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of μ-nonessential sets. By Theorem 3.10, we have $\mu(A_i) = 0$. Consequently,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) = 0.$$

Therefore, $\bigcup_{i=1}^{\infty} A_i$ is μ-nonessential. $\square$

Corollary 3.12. *Assume that $(X, \tau, \mathfrak{M}, \mu)$ is either a Lindelöf submeasure space or an inner regular measure space. If the μ-closure is a strong essential closure, then*

$$\mu(\overline{A}^{\mu}) \geq \mu(A)$$

for each $A \in \mathfrak{M}$.

Proof. By Theorem 3.10 and the fact that the μ-closure is a strong essential closure, we have $\mu(A - \overline{A}^{\mu}) = 0$. Then

$$\begin{aligned} \mu(A) &\leq \mu(A \cap \overline{A}^{\mu}) + \mu(A - \overline{A}^{\mu}) \\ &\leq \mu(\overline{A}^{\mu}). \end{aligned}$$

This completes the proof. $\square$

On a hereditarily Lindelöf measurable space[5], Theorem 3.9 and Corollary 3.11 imply that every submeasure closure is strong and σ-nonessential. In fact, the converse also holds.

Theorem 3.13. *On a hereditarily Lindelöf measurable space, an essential closure is strong and σ-nonessential if and only if it is a submeasure closure.*

Proof. Let $(X, \tau, \mathfrak{M})$ be a hereditarily Lindelöf measurable space. The case where $\widetilde{X} = \varnothing$ is trivial. Assume that $\widetilde{X} \neq \varnothing$. Suppose $A \mapsto \widetilde{A}$ is a σ-nonessential strong essential closure on $\mathfrak{M}$. Define $\mu \colon \mathfrak{M} \to [0, \infty]$ by $\mu(A) = 0$ if $\widetilde{A} = \varnothing$, otherwise $\mu(A) = 1$. To see that μ is a submeasure on $(X, \mathfrak{M})$, let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of $\mathfrak{M}$-measurable sets.

[5]A *hereditarily Lindelöf measurable space* is a hereditarily Lindelöf space equipped with a σ-algebra.

(i) Since $\varnothing$ is nonessential, $\mu(\varnothing) = 0$.

(ii) Suppose $A_1 \subseteq A_2$. Then $\widetilde{A_1} \subseteq \widetilde{A_2}$. If A_2 is nonessential, then A_1 is nonessential, hence $\mu(A_1) = 0 = \mu(A_2)$. If A_2 is essential, then $\mu(A_1) \leq 1 = \mu(A_2)$.

(iii) If there is an essential set A_j in the collection, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq 1 \leq \sum_{i=1}^{\infty} \mu(A_i).$$

If every A_i is nonessential, then $\bigcup_{i=1}^{\infty} A_i$ is also nonessential. So, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

Therefore, μ is a submeasure on $(X, \mathfrak{M})$ and, consequently, the μ-closure defined on $(X, \tau, \mathfrak{M})$ is a σ-nonessential strong essential closure by Theorem 3.9 and Corollary 3.11. It is left to show that $A \mapsto \widetilde{A}$ and the μ-closure coincide. According to Corollary 2.12, it suffices to show that

$$\mathcal{N}_{\mathfrak{M}}(A \mapsto \widetilde{A}) = \mathcal{N}_{\mathfrak{M}}(A \mapsto \overline{A}^{\mu}).$$

By Theorem 3.10, $\overline{A}^{\mu} = \varnothing$ if and only if $\mu(A) = 0$, which is equivalent to $\widetilde{A} = \varnothing$ by construction. So the two essential closures coincide. The converse follows from Theorem 3.9 and Corollary 3.11. □

Remark 3.14. *In the proof of Theorem 3.13, the essential closure induces a normalized submeasure if the space is essential. Otherwise, it induces the trivial submeasure.*

Corollary 3.15. *Assume that $(X, \tau, \mathfrak{M})$ is a hereditarily Lindelöf measurable space. Then every σ-nonessential strong essential closure on $(X, \tau, \mathfrak{M})$ can be extended to a σ-nonessential strong essential closure on $(X, \tau, \mathcal{P}(X))$.*

Proof. Given a σ-nonessential strong essential closure on $(X, \tau, \mathfrak{M})$, by Theorem 3.13, the essential closure induces a submeasure closure, say the μ-closure. By Proposition 3.3, μ can be extended to an outer measure μ^* on $\mathcal{P}(X)$. Since μ^*

is also a submeasure, the induced μ^*-closure is a σ-nonessential strong essential closure on $(X, \tau, \mathcal{P}(X))$. Moreover, it is straightforward from the definition of submeasure closures that the μ-closure and the μ^*-closure coincide on $\mathfrak{M}$. $\square$

For set-valued functions on a common space, we may define a union of such functions in an element-wise fashion. One can verify that a finite union of two essential closures is still an essential closure. Moreover, a finite union of submeasure closures is still a submeasure closure.

Proposition 3.16. *Given two submeasure closures, μ_1-closure and μ_2-closure, on a common topological measurable space, the set function*

$$A \mapsto \overline{A}^{\mu_1} \cup \overline{A}^{\mu_2}$$

is, in fact, the $(\mu_1 + \mu_2)$-closure. Moreover, if both the μ_1-closure and the μ_2-closure are essential closures, then so is the $(\mu_1 + \mu_2)$-closure.

Proof. Let $\mu = \mu_1 + \mu_2$, which is still a submeasure. If $x \notin \overline{A}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap A) = 0$. Thus $\mu_1(G \cap A) = 0 = \mu_2(G \cap A)$. Therefore, $x \notin \overline{A}^{\mu_1} \cup \overline{A}^{\mu_2}$. Conversely, if $x \notin \overline{A}^{\mu_1} \cup \overline{A}^{\mu_2}$, then there exist $G_1, G_2 \in \mathfrak{N}(x)$ such that $\mu_1(G_1 \cap A) = 0$ and $\mu_2(G_2 \cap A) = 0$. Choose $G = G_1 \cap G_2 \in \mathfrak{N}(x)$. Then $\mu_1(G \cap A) = 0 = \mu_2(G \cap A)$. Consequently, $\mu(G \cap A) = 0$. Hence $x \notin \overline{A}^{\mu}$. $\square$

Example 3.17. We know that $\overline{A - \overline{A}^{\mu}}^{\mu} = \varnothing$, which means $A - \overline{A}^{\mu}$ is a nonessential set, as long as the μ-closure is a strong essential closure. But $\overline{A}^{\mu} - A$ can be an essential set, meaning it can be large with respect to the submeasure μ.

To see this, take $\mu = \lambda_1$, the 1-dimensional Lebesgue measure on $[0, 1]$, and take the set $A = [0, 1] - C$, where C is a positive Lebesgue measure Cantor set on $[0, 1]$. (For more information on positive Lebesgue measure Cantor sets, see [1].) Then, for each $x \in [0, 1]$ and $G \in \mathfrak{N}(x)$, $G \cap A$ contains a nonempty open interval. Hence $\lambda_1(G \cap A) > 0$. Therefore, $\overline{A}^{\lambda_1} = [0, 1]$. Thus

$$\overline{A}^{\lambda_1} - A = [0, 1] - A = C,$$

which is of positive Lebesgue measure. By Theorem 3.10, $\overline{A}^{\lambda_1} - A$ is essential with respect to λ_1.

3.2 Applications

In this section, we demonstrate some of the applications of submeasure closures, especially the study of supports of measures and the notion of essential supports of functions.

3.2.1 Supports of measures

The support of a submeasure can be defined analogous to the definition of the support of a measure.

Definition 3.18. Let $(X, \tau, \mathfrak{M}, \mu)$ be a topological submeasure space. Then the support of μ, denoted by $\operatorname{supp} \mu$, is defined to be the complement of the union of all open sets G such that $\mu(G) = 0$.

Example 3.19. Consider the real line $\mathbb{R}$ equipped with the discrete topology τ and the σ-algebra $\mathcal{P}(\mathbb{R})$. Define a measure μ on $(\mathbb{R}, \tau, \mathcal{P}(\mathbb{R}))$ by $\mu(A) = 0$ if A is countable and $\mu(A) = \infty$ otherwise. Observe that each singleton is an open set of μ-measure zero. Hence $\operatorname{supp} \mu = \varnothing$. Consequently, $\mu((\operatorname{supp} \mu)^c) = \infty$. So, there is a measure whose complement of the support is of positive measure.

Proposition 3.20. *Assume that* $(X, \tau, \mathfrak{M}, \mu)$ *is either a hereditarily Lindelöf submeasure space or an inner regular measure space. Then*

$$\mu((\operatorname{supp} \mu)^c) = 0.$$

Proof. If the space is a hereditarily Lindelöf submeasure space, then for each $x \in (\operatorname{supp} \mu)^c$, there exists $G_x \in \mathfrak{N}(x)$ such that $\mu(G_x) = 0$. Therefore, $\{G_x\}_{x \in (\operatorname{supp} \mu)^c}$ is an open cover of $(\operatorname{supp} \mu)^c$, which is Lindelöf. Thus there is a countable subcover: $\{G_1, G_2, \dots\}$. Then

$$\mu((\operatorname{supp} \mu)^c) \leq \sum_{i=1}^{\infty} \mu(G_i) = 0.$$

If the space is an inner regular space. Consider a compact set $K \subseteq (\operatorname{supp} \mu)^c$. For each $x \in K$, since $x \notin \operatorname{supp} \mu$, there exists $G_x \in \mathfrak{N}(x)$ such that $\mu(G_x) = 0$.

Therefore, $\{G_x\}_{x\in K}$ is an open cover of K, which is compact. Thus there is a finite subcover: $\{G_1, \ldots, G_n\}$. Then

$$\mu(K) \leq \sum_{i=1}^{n} \mu(G_i) = 0.$$

By the inner regularity, $\mu((\operatorname{supp}\mu)^c) = 0$. $\square$

An essential closure can be viewed as a tool to eliminate the nonessential part and collect the essential part of a set. In the case of submeasure closures, one can expect that eliminating the nonessential part of the space should give the support of that submeasure.

Theorem 3.21. *Let $(X, \tau, \mathfrak{M}, \mu)$ be a topological submeasure space. Then we have*

$$\operatorname{supp}\mu = \overline{B}^{\mu}$$

for any $\mathfrak{M}$-measurable set B such that $\mu(B^c) = 0$. In particular, if the μ-closure is an essential closure, then $\operatorname{supp}\mu$ *is μ-essentially closed.*

Proof. If $x \notin \operatorname{supp}\mu$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G) = 0$. Then $\mu(G \cap B) = 0$ for any $\mathfrak{M}$-measurable set B. Hence $x \notin \overline{B}^{\mu}$. Conversely, if $x \notin \overline{B}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap B) = 0$. Since $\mu(B^c) = 0$, we have $\mu(G) \leq \mu(G \cap B) + \mu(G \cap B^c) = 0$. Therefore, $x \notin \operatorname{supp}\mu$. $\square$

The following result gives a characterization of μ-essentially closed sets on a hereditarily Lindelöf measure space. Notice the repeated use of Proposition 3.20 in the proof.

Theorem 3.22. *Suppose $(X, \tau, \mathfrak{M}, \mu)$ is a hereditarily Lindelöf measure space. Then a set $A \in \mathfrak{M}$ is μ-essentially closed if and only if there is a measure ν such that $\nu \ll \mu$ and* $\operatorname{supp}\nu = A$.

Proof. Let A be a μ-essentially closed set. For each $\mathfrak{M}$-measurable set B, define $\nu(B) = \mu(A \cap B)$. Observe that $\nu \ll \mu$. It is left to show that $\operatorname{supp}\nu = A$. On one hand, observe that A is closed and $\nu(A^c) = \mu(A \cap A^c) = 0$. Hence $\operatorname{supp}\nu \subseteq A$. On the other hand, observe that

$$\mu((\operatorname{supp}\nu)^c \cap A) = \nu((\operatorname{supp}\nu)^c) = 0.$$

Suppose $(\operatorname{supp}\nu)^c \cap A \neq \varnothing$. Let $x \in (\operatorname{supp}\nu)^c \cap A$. Then $(\operatorname{supp}\nu)^c \in \mathfrak{N}(x)$ and $x \in A = \overline{A}^{\mu}$. Therefore, $\mu((\operatorname{supp}\nu)^c \cap A) > 0$, a contradiction. Hence $(\operatorname{supp}\nu)^c \cap A = \varnothing$. In other words, $A \subseteq \operatorname{supp}\nu$.

Conversely, it suffices to show that $\operatorname{supp}\nu \subseteq \overline{\operatorname{supp}\nu}^{\mu}$. If $x \notin \overline{\operatorname{supp}\nu}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$, $\mu(G \cap \operatorname{supp}\nu) = 0$. By absolute continuity, we have

$$\nu(G) \leq \nu(G \cap \operatorname{supp}\nu) + \nu((\operatorname{supp}\nu)^c) = 0.$$

So $x \notin \operatorname{supp}\nu$. Hence $\operatorname{supp}\nu$ is μ-essentially closed. $\square$

If we instead assume that the space is an inner regular measure space, we obtain a result similar to Theorem 3.22. But first, we need the following lemma.

Lemma 3.23. *Let μ be an inner regular measure on a Hausdorff measurable space and let A be measurable. Define, for each measurable set B,*

$$\mu_A(B) = \mu(A \cap B).$$

Then μ_A is inner regular and $\mu_A \ll \mu$.

Proof. Let B be a measurable set. Observe that

$$\begin{aligned} \mu_A(B) &= \mu(A \cap B) \\ &\geq \sup\{\mu(A \cap K) \colon K \text{ is compact and } K \subseteq B\} \\ &= \sup\{\mu_A(K) \colon K \text{ is compact and } K \subseteq B\}. \end{aligned}$$

On the other hand, if K is a compact set such that $K \subseteq A \cap B$, then $K \subseteq B$ and $\mu(A \cap K) = \mu(K)$. Hence

$$\begin{aligned} \mu_A(B) &= \mu(A \cap B) \\ &= \sup\{\mu(K) \colon K \text{ is compact and } K \subseteq A \cap B\} \\ &\leq \sup\{\mu(A \cap K) \colon K \text{ is compact and } K \subseteq B\} \\ &= \sup\{\mu_A(K) \colon K \text{ is compact and } K \subseteq B\}. \end{aligned}$$

Therefore, μ_A is inner regular. Moreover, it is clear that $\mu_A \ll \mu$. $\square$

The above result is probably known to experts. But to the best of our knowledge, no proof has been given. So we give one for completeness of this thesis.

The following corollary can be proved in the same manner as the proof of Theorem 3.22. Notice the difference in the inner regularity of the measures μ and ν compared to the ones in Theorem 3.22.

Corollary 3.24. *Suppose $(X, \tau, \mathfrak{M}, \mu)$ is an inner regular measure space. Then a set $A \in \mathfrak{M}$ is μ-essentially closed if and only if there is an inner regular measure ν such that $\nu \ll \mu$ and $\operatorname{supp} \nu = A$.*

Proof. For each $\mathfrak{M}$-measurable set B, define $\nu(B) = \mu(A \cap B)$. By Lemma 3.23, ν is inner regular and $\nu \ll \mu$. It is left to show that $\operatorname{supp} \nu = A$. On one hand, observe that A is closed and $\nu(A^c) = \mu(A \cap A^c) = 0$. Hence $\operatorname{supp} \nu \subseteq A$. On the other hand, observe that

$$\mu((\operatorname{supp} \nu)^c \cap A) = \nu((\operatorname{supp} \nu)^c) = 0.$$

Suppose $(\operatorname{supp} \nu)^c \cap A \neq \varnothing$. Let $x \in (\operatorname{supp} \nu)^c \cap A$. Then $(\operatorname{supp} \nu)^c \in \mathfrak{N}(x)$ and $x \in A = \overline{A}^{\mu}$. Therefore, $\mu((\operatorname{supp} \nu)^c \cap A) > 0$, a contradiction. Hence $(\operatorname{supp} \nu)^c \cap A = \varnothing$. In other words, $A \subseteq \operatorname{supp} \nu$.

Conversely, it suffices to show that $\operatorname{supp} \nu \subseteq \overline{\operatorname{supp} \nu}^{\mu}$. If $x \notin \overline{\operatorname{supp} \nu}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$, $\mu(G \cap \operatorname{supp} \nu) = 0$. By absolute continuity,

$$\nu(G) \leq \nu(G \cap \operatorname{supp} \nu) + \nu((\operatorname{supp} \nu)^c) = 0.$$

So $x \notin \operatorname{supp} \nu$. Hence $\operatorname{supp} \nu$ is μ-essentially closed. $\square$

Measures ν and μ on a common measurable space are said to be <u>singular</u>, denoted by $\nu \perp \mu$, if there is a measurable partition $\{A, B\}$ of the space such that $\mu(A) = 0$ while $\nu(B) = 0$. Recall that a σ-finite measure can be uniquely decomposed, with respect to another σ-finite measure on a common measurable space, into two parts: absolutely continuous part and singular part. This result is known as Lebesgue's decomposition theorem (see [20, page 278]).

Theorem 3.25 (Lebesgue's decomposition theorem)**.** *For any σ-finite measures μ and ν on a common measurable space, there exist unique σ-finite measures ν_a and ν_s such that*

(i) $\nu = \nu_a + \nu_s$,

(ii) $\nu_a \ll \mu$,

(iii) $\nu_s \perp \mu$.

Consider the Lebesgue decomposition of a σ-finite measure with respect to an underlying σ-finite measure on a hereditarily Lindelöf measurable space. If the support of the singular part is negligible, then the support of the absolutely continuous part can be determined via the submeasure closure induced by the underlying measure.

Theorem 3.26. *Assume that μ and η are σ-finite measures on a hereditarily Lindelöf measurable space with the Lebesgue decomposition $\eta = \eta_a + \eta_s$ with respect to μ. If $\mu(\operatorname{supp}\eta_s) = 0$, then $\operatorname{supp}\eta_a = \overline{\operatorname{supp}\eta}^{\mu}$.*

Proof. First of all, we claim that $\operatorname{supp}\eta = \operatorname{supp}\eta_a \cup \operatorname{supp}\eta_s$. Note that this holds in general, not just for the Lebesgue decomposition. Since $\eta((\operatorname{supp}\eta)^c) = 0$, we have $\eta_a((\operatorname{supp}\eta)^c) = 0 = \eta_s((\operatorname{supp}\eta)^c)$. Hence

$$(\operatorname{supp}\eta)^c \subseteq (\operatorname{supp}\eta_a)^c \cap (\operatorname{supp}\eta_s)^c.$$

On the other hand, if $x \notin \operatorname{supp}\eta_a \cup \operatorname{supp}\eta_s$, then there exist $G_1, G_2 \in \mathfrak{N}(x)$ such that $\eta_a(G_1) = 0 = \eta_s(G_2)$. Choose $G = G_1 \cap G_2 \in \mathfrak{N}(x)$. Then $\eta_a(G) = 0 = \eta_s(G)$. Thus $\eta(G) = 0$. Therefore, $x \notin \operatorname{supp}\eta$. We conclude that

$$\operatorname{supp}\eta = \operatorname{supp}\eta_a \cup \operatorname{supp}\eta_s.$$

By Theorem 3.10, since $\mu(\operatorname{supp}\eta_s) = 0$, we have $\overline{\operatorname{supp}\eta_s}^{\mu} = \varnothing$. Therefore,

$$\overline{\operatorname{supp}\eta}^{\mu} = \overline{\operatorname{supp}\eta_a}^{\mu} \cup \overline{\operatorname{supp}\eta_s}^{\mu} = \overline{\operatorname{supp}\eta_a}^{\mu}.$$

Thus $\overline{\operatorname{supp}\eta}^{\mu} = \operatorname{supp}\eta_a$ since $\operatorname{supp}\eta_a$ is μ-essentially closed by Theorem 3.22 $\square$

Example 3.27. Suppose μ and η are σ-finite measures on a hereditarily Lindelöf measurable space with the Lebesgue decomposition $\eta = \eta_a + \eta_s$ with respect to μ. If $\mu(\operatorname{supp}\eta) = 0$, then

$$\mu(\operatorname{supp}\eta_s) \leq \mu(\operatorname{supp}\eta) = 0$$

since $\operatorname{supp}\eta_s \subseteq \operatorname{supp}\eta$. By Theorems 3.10 and 3.26, $\operatorname{supp}\eta_a = \overline{\operatorname{supp}\eta}^{\mu} = \varnothing$. Thus $\eta_a \equiv 0$, which implies that η is singular with respect to μ.

3.2.2 Essential supports of functions

In this section, we introduce the notion of essential supports of functions, which is partly motivated by the study of supports of Radon-Nikodym derivatives. We are particularly interested in the study of Radon-Nikodym derivatives via techniques from geometric measure theory.

For any pair of Radon measures[6] ν and μ on a Euclidean space $\mathbb{R}^n$ (equipped with a σ-algebra containing the Borel sets) such that $\nu \ll \mu$, it was shown in [17, Theorem 2.12] that the function

$$D_{\nu,\mu}(x) = \lim_{\epsilon \to 0^+} \frac{\nu(B(x,\epsilon))}{\mu(B(x,\epsilon))}$$

is defined for μ-almost everywhere on $\mathbb{R}^n$ and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Similarly, for any locally finite measure ν defined on the Borel σ-algebra over $\mathbb{R}^n$ such that $\nu \ll \lambda_n$, it was shown in [3, Theorem 2.3.8] that the function

$$D_{\nu,\lambda_n}(x) = \lim_{\epsilon \to 0^+} \frac{\nu(B(x,\epsilon))}{\lambda_n(B(x,\epsilon))}$$

is defined for Lebesgue almost everywhere on $\mathbb{R}^n$ and coincides Lebesgue almost everywhere with the Radon-Nikodym derivative of ν with respect to λ_n.

From the two examples above, we propose a more general definition for a σ-finite measure to be differentiable with respect to another σ-finite measure as follows.

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[6] See Definition 1.5 and Corollary 1.11 in Mattila's book [17].

Definition 3.28. Let ν and μ be σ-finite measures on a metric measurable space[7] $(X, d, \mathfrak{M})$. We say that ν is differentiable with respect to μ if $\nu \ll \mu$ and

$$D_{\nu,\mu}(x) = \lim_{\epsilon \to 0^+} \frac{\nu(B(x,\epsilon))}{\mu(B(x,\epsilon))}$$

is defined for μ-almost everywhere and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Proposition 3.29. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then*

$$\operatorname{supp} D_{\nu,\mu} = \operatorname{supp} \nu.$$

Proof. If $x \notin \operatorname{supp} \nu$, then there exists $\epsilon > 0$ such that $\nu(B(x,\epsilon)) = 0$. Hence $D_{\nu,\mu}(x) = 0$. So we have $\{x \colon D_{\nu,\mu}(x) \neq 0\} \subseteq \operatorname{supp} \nu$. Therefore,

$$\operatorname{supp} D_{\nu,\mu} \subseteq \operatorname{supp} \nu.$$

On the other hand, if $x \notin \operatorname{supp} D_{\nu,\mu}$, then there is $G \in \mathfrak{N}(x)$ such that $D_{\nu,\mu} \equiv 0$ on G. Observe that $\nu(G) = \displaystyle\int_G D_{\nu,\mu}\, d\mu = 0$. Thus $x \notin \operatorname{supp} \nu$. Hence

$$\operatorname{supp} \nu \subseteq \operatorname{supp} D_{\nu,\mu}.$$

Therefore, $\operatorname{supp} D_{\nu,\mu} = \operatorname{supp} \nu$. □

Radon-Nikodym derivatives are unique up to sets of measure zero. As a result, the notion of topological supports fails to detect essential parts of such functions. We demonstrate an extreme case in the following example.

Example 3.30. Consider the trivial measure $\nu \equiv 0$ on the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R})$, which is absolutely continuous with respect to the 1-dimensional Lebesgue measure. Observe that both $f \equiv 0$ and $g = \chi_{\mathbb{Q}}$ are the Radon-Nikodym derivative of ν with respect to λ_1. However, $\operatorname{supp} f = \varnothing$ while $\operatorname{supp} g = \mathbb{R}$.

In the above example, observe that even though $\mathbb{Q}$ is negligible since it has Lebesgue measure zero, it is dense in $\mathbb{R}$. This is the cause of the problem.

[7] A *metric measurable space* is a metric space equipped with a σ-algebra containing the Borel sets.

Definition 3.31. Let f be an extended real-valued $\mathfrak{M}$-measurable function on a topological submeasure space $(X, \tau, \mathfrak{M}, \mu)$. Define the essential support of f with respect to μ by

$$\operatorname{ess\,supp}_\mu f = \overline{\{x \in X \colon f(x) \neq 0\}}^{\mu}.$$

From the definition, one can see that the essential support of a function is always contained in the topological support of that function. An extreme case is presented in the following example.

Similar to the notion of almost everywhere for measures, for the case of submeasures, we say that a property holds almost everywhere if the set of elements for which the property does not hold is a submeasure zero set.

Proposition 3.32. *Let f and g be extended real-valued $\mathfrak{M}$-measurable functions on a topological submeasure space $(X, \tau, \mathfrak{M}, \mu)$. If f and g are equal μ-almost everywhere, then*

$$\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu g.$$

Proof. Since $f = g$ μ-almost everywhere, we have

$$\mu(\{x \in X \colon g(x) \neq 0\}) = \mu(\{x \in X \colon f(x) \neq 0\}),$$

which implies that the essential supports of f and g coincide. $\square$

Theorem 3.33. *Assume that $(X, \tau, \mathfrak{M}, \mu)$ is either a hereditarily Lindelöf submeasure space or an inner regular measure space. For each extended real-valued $\mathfrak{M}$-measurable function f, let $[f]_\mu$ denote the class of extended real-valued $\mathfrak{M}$-measurable functions on X which are equal to f μ-almost everywhere. Then there exists $f_0 \in [f]_\mu$ such that*

$$\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f$$

which is μ-essentially closed.

Proof. Define f_0 to be the function which coincides with f on $\operatorname{ess\,supp}_\mu f$ and vanishes everywhere else. Since the μ-closure is a strong essential closure,

$$\begin{aligned}\{x \in X : f(x) \neq f_0(x)\} &= \{x \in X : f(x) \neq 0\} - \operatorname{ess\,supp}_\mu f \\ &= \{x \in X : f(x) \neq 0\} - \overline{\{x \in X : f(x) \neq 0\}}^{\mu}\end{aligned}$$

is μ-nonessential. Hence $\mu(\{x \in X : f(x) \neq f_0(x)\}) = 0$ by Theorem 3.10. Thus f and f_0 are equal μ-almost everywhere, i.e., $f_0 \in [f]_\mu$. By Proposition 3.32, we have $\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0$.

If $f_0(x) \neq 0$, then $x \in \operatorname{ess\,supp}_\mu f$ by construction. Therefore, we have that $\{x \in X \colon f_0(x) \neq 0\} \subseteq \operatorname{ess\,supp}_\mu f$. Hence

$$\operatorname{supp} f_0 \subseteq \operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0 \subseteq \operatorname{supp} f_0.$$

Thus $\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f_0 = \operatorname{ess\,supp}_\mu f$. In particular, $\operatorname{supp} f_0$ is μ-essentially closed. □

Proposition 3.34. *Let ν and μ be σ-finite measures on $(X, \tau, \mathfrak{M})$ with $\nu \ll \mu$ and let $\frac{d\nu}{d\mu}$ denote the Radon-Nikodym derivative. Then*

$$\operatorname{ess\,supp}_\mu \tfrac{d\nu}{d\mu} = \operatorname{supp} \nu.$$

Proof. Let f denote $\frac{d\nu}{d\mu}$. If $x \notin \operatorname{supp} \nu$, then there exists $G \in \mathfrak{N}(x)$ such that $\nu(G) = 0$. Thus $f = 0$ μ-almost everywhere on G. Therefore, we have that $\mu(G \cap \{x \in X \colon f(x) \neq 0\}) = 0$. Hence $x \notin \operatorname{ess\,supp}_\mu f$. Conversely, if $x \notin \operatorname{ess\,supp}_\mu f$, then $x \notin \overline{\{x \in X \colon f(x) \neq 0\}}^{\mu}$. Therefore, there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \{x \in X \colon f(x) \neq 0\}) = 0$. Thus $f = 0$ μ-almost everywhere on G. Hence $\nu(G) = 0$. So $x \notin \operatorname{supp} \nu$. □

Corollary 3.35. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then*

$$\operatorname{ess\,supp}_\mu D_{\nu,\mu} = \operatorname{supp} D_{\nu,\mu}.$$

Proof. This follows directly from Proposition 3.29 and Proposition 3.34. □

Example 3.36. There exists an absolutely continuous measure $\nu \ll \mu$ with full support such that μ is not absolutely continuous with respect to ν. To see this, let μ be the 1-dimensional Lebesgue measure on $[0,1]$ and let ν be a measure on $[0,1]$ defined, for each Lebesgue measurable set $B \subseteq [0,1]$, by $\nu(B) = \lambda_1(B \cap A^c)$ where A is a positive Lebesgue measure Cantor set on $[0,1]$. Obviously, we have $\nu \ll \lambda_1$ by construction. Moreover, by Proposition 3.34,

$$\operatorname{supp} \nu = \text{ess supp}_{\lambda_1} \chi_{A^c} = \overline{A^c}^{\lambda_1} = [0,1].$$

Therefore, ν has full support. However, $\nu(A) = \lambda_1(A \cap A^c) = 0$ while $\lambda_1(A) > 0$. So λ_1 is not absolutely continuous with respect to ν.

Example 3.37. Let $(X, \tau, \mathfrak{M}, \mu)$ be a hereditarily Lindelöf measure space and let f be an extended real-valued $\mathfrak{M}$-measurable function. We already know that

$$\int_X f \, d\mu = \int_{\operatorname{supp} f} f \, d\mu.$$

For each $x \notin \text{ess supp}_\mu f$, there is $G_x \in \mathfrak{N}(x)$ with $\mu(G_x \cap \{f \neq 0\}) = 0$. Then the collection $\{G_x\}_{x \in (\text{ess supp}_\mu f)^c}$ is an open cover of $(\text{ess supp}_\mu f)^c$. Let $\{G_1, G_2, \dots\}$ be a countable subcover. By the countable subadditivity of measures, we can show that $\mu((\text{ess supp}_\mu f)^c \cap \{f \neq 0\}) = 0$. Thus

$$\int_X f \, d\mu = \int_{\text{ess supp}_\mu f} f \, d\mu = \int_{\operatorname{supp} f_0} f_0 \, d\mu$$

where f_0 is a representative of the class $[f]_\mu$ in Theorem 3.33. Also note that

$$\operatorname{supp} f_0 = \text{ess supp}_\mu f \subseteq \operatorname{supp} f.$$

In this case, we see that f_0 is indeed a good representative of the class $[f]_\mu$.

3.3 Existing and related notions

There are various notions related to the notion of essential closures. In this section, we pick a few of them to discuss in details, most of which are related to submeasure closures.

3.3.1 Lebesgue closures

Recall that the essential closure introduced in [13], which we call the Lebesgue closure to avoid confusion, of a Lebesgue measurable set $A \subseteq \mathbb{R}$ is defined by

$$\overline{A}^e = \{x \in \mathbb{R} \colon \text{for all } \epsilon > 0, \lambda_1((x-\epsilon, x+\epsilon) \cap A) > 0\}.$$

It is easy to see that the Lebesgue closure coincides with the λ_1-closure on the Lebesgue σ-algebra. Also recall in [13] the definition of the Lebesgue closure defined on S^1, the unit circle centered at the origin in $\mathbb{R}^2$.

According to Theorem $265E$ in Fremlin's book [12], we have that the pushforward of the 1-dimensional Lebesgue measure on S^1 (with respect to the canonical map, $\theta \mapsto e^{i\theta}$) coincides with the 1-dimensional Hausdorff measure on S^1. As a consequence, the Lebesgue closure coincides with the $\mathcal{H}^1$-closure on the induced σ-algebra on S^1.

3.3.2 Lebesgue density closures

To avoid confusion, the essential closures cl^* in Buczolich and Pfeffer's work [4] and in Fremlin's book [11], defined for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$ by

$$\mathrm{cl}^* A = \{x \in \mathbb{R}^n \colon \limsup_{\epsilon \to 0^+} \frac{\lambda_n(B(x,\epsilon) \cap A)}{\lambda_n(B(x,\epsilon))} > 0\},$$

will be called Lebesgue density closures. Note that Lebesgue density closures fail to satisfy at least the first property of essential closures.

For each λ_n-density closure cl^* on the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^n)$, define the modified λ_n-density closure of $A \in \mathcal{L}(\mathbb{R}^n)$ by $\widetilde{A} = \overline{\mathrm{cl}^* A}$. As a consequence of taking the topological closure of $\mathrm{cl}^* A$, the modified λ_n-density closure is forced to satisfy the first property of essential closures. The question is whether the modified λ_n-density closure satisfies the other three properties, which makes it an essential closure. Surprisingly, it can be shown that the modified λ_n-density closure is, in fact, the λ_n-closure defined on $\mathcal{L}(\mathbb{R}^n)$.

First, we show that the modified λ_n-density closure and the λ_n-closure coincide on the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Borel measurable. For each Borel measurable set $B \subseteq \mathbb{R}^n$, define $\lambda_A(B) = \lambda_n(B \cap A)$. It is clear that λ_A is σ-finite and $\lambda_A \ll \lambda_n$ on the Borel σ-algebra. According to Theorem 2.3.8 in Ash's book [3], λ_A is differentiable with respect to λ_n. As a result,

$$D_{\lambda_A,\lambda_n}(x) = \lim_{\epsilon\to 0^+} \frac{\lambda_A(B(x,\epsilon))}{\lambda_n(B(x,\epsilon))} = \limsup_{\epsilon\to 0^+} \frac{\lambda_n(B(x,\epsilon)\cap A)}{\lambda_n(B(x,\epsilon))}$$

defines the Radon-Nikodym derivative of λ_A with respect to λ_n. By Theorem 3.21 and Proposition 3.29, we have

$$\widetilde{A} = \overline{\mathrm{cl}^* A} = \operatorname{supp} D_{\lambda_A,\lambda_n} = \operatorname{supp} \lambda_A = \overline{A}^{\lambda_A}.$$

Moreover, it is straightforward to verify that $\overline{A}^{\lambda_A} = \overline{A}^{\lambda_n}$. Hence $\widetilde{A} = \overline{A}^{\lambda_n}$ for each Borel measurable set $A \subseteq \mathbb{R}^n$.

Finally, we extend the result to the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Lebesgue measurable. There is a Borel measurable set $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\lambda_n(B - A) = 0$. According to Lemma 475C in Fremlin's book [11], cl^* is distributive over finite unions and $\mathrm{cl}^*(A) = \varnothing$ if $\lambda_n(A) = 0$. As a result,

$$\mathrm{cl}^*(A) = \mathrm{cl}^*(A) \cup \mathrm{cl}^*(B - A) = \mathrm{cl}^*(B).$$

Similarly, $\overline{A}^{\lambda_n} = \overline{B}^{\lambda_n}$. Therefore, $\widetilde{A} = \overline{\mathrm{cl}^*(A)} = \overline{\mathrm{cl}^*(B)} = \widetilde{B} = \overline{B}^{\lambda_n} = \overline{A}^{\lambda_n}$ for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$.

3.3.3 Essential range

Recall the definition of the essential range of a complex-valued $\mathfrak{M}$-measurable function $f\colon (X, \mathfrak{M}, \mu) \to \mathbb{C}$, which is defined to be the set

$$S = \{z \in \mathbb{C}\colon \mu(\{x \in X\colon |f(x) - z| < \epsilon\}) > 0 \text{ for all } \epsilon > 0\}.$$

Let $\mu_f\colon \mathfrak{B}_{\mathbb{C}} \to [0, \infty]$ be the pushforward of μ, i.e., $\mu_f(B) = \mu(f^{-1}(B))$ for each Borel measurable set B. Then

$$S = \{z \in \mathbb{C}\colon \mu_f(B(z,\epsilon)) > 0 \text{ for all } \epsilon > 0\} = \overline{\mathbb{C}}^{\mu_f}.$$

Consequently, by Theorem 3.21, the essential range of f is the support of μ_f.

3.3.4 Prevalence

Prevalence is a measure-theoretic approach to define what it means for a statement to hold almost everywhere in a possibly infinite-dimensional complete metric vector space. It was proved by Hunt et al. in [15] that the notion of prevalence extends that of Lebesgue almost everywhere in finite-dimensional Euclidean spaces.

It is well-known that there is no nontrivial translation-invariant measure in infinite-dimensional spaces. So the question is whether we can find something weaker, e.g., a nontrivial translation-invariant submeasure whose submeasure zero sets are exactly the shy sets, i.e., the complements of the prevalent sets. Via the notion of prevalence and the theory of essential closures, such a submeasure can be constructed. Let us recall some basic properties of shy sets derived in [15]. Let $A, A_1, A_2, \ldots$ be shy sets and let v be a vector. Then the following hold:

(i) $A + v$ is shy,

(ii) $B \subseteq A$ implies B is shy,

(iii) $\bigcup_{n=1}^{\infty} A_n$ is shy.

Observe that, with a suitable underlying σ-algebra, the collection of shy sets satisfies the conditions in Definition 2.20. In the sequel, let V be a hereditarily Lindelöf complete metric vector space.

Theorem 3.38. *There exists a finite nontrivial translation-invariant submeasure on V whose submeasure zero sets are exactly the shy sets. Moreover, the induced submeasure closure commutes with the translations.*

Proof. We call the σ-algebra generated by the open subsets and the shy subsets of V the prevalence σ-algebra, denoted by $\mathcal{L}(V)$. According to Hunt et al., the collection of shy sets on V satisfies the conditions in Definition 2.20 with respect to $\mathcal{L}(V)$. By Theorem 2.22, there exists a unique σ-nonessential strong essential closure whose collection of nonessential sets is exactly the collection of shy sets. We call the induced essential closure the prevalence closure.

By Theorem 3.13, the prevalence closure induces a submeasure on $\mathcal{L}(V)$. Note that an induced submeasure is not unique. We call such a submeasure a prevalence submeasure. Moreover, by Theorem 3.10, the collection of nonessential sets, which is the collection of shy sets, is exactly the collection of prevalence submeasure zero sets. In addition, it is worth mentioning that the vector space V is essential and essentially closed with respect to the prevalence closure. This is due to the fact that nonempty open sets are not shy, hence are of positive prevalence submeasure.

To conclude, we have a prevalence submeasure on $\mathcal{L}(V)$ whose prevalence submeasure zero sets are exactly the shy sets on V. Moreover, it is straightforward to verify that the prevalence closure commutes with the translations. However, a prevalence submeasure is generally not translation-invariant. Nevertheless, there is a special prevalence submeasure which is translation-invariant.

Recall the proof of Theorem 3.13, we call the normalized submeasure obtained from the prevalence closure the normalized prevalence submeasure, denoted by μ_p. For each vector $v \in V$, $\mu_p(A+v) = 0$ if and only if $\mu_p(A)$=0 by the definition of μ_p. Since μ_p assumes the value of either 0 or 1, μ_p is translation-invariant. $\square$

CHAPTER IV

STOCHASTIC CLOSURES

We begin this chapter with a short introduction containing a handful of notions and results with which mathematicians outside the field are probably unfamiliar. Detailed explanations shall be given along the way as we proceed.

It is known that the class of doubly stochastic measures on $[0,1]^2$ fully describes the class of joint distributions of two random variables uniformly distributed on $[0,1]$ (see [19]). For decades, the supports of doubly stochastic measures have been extensively studied by many mathematicians because the support of a doubly stochastic measure tells us where the mass of the measure is concentrated. A handful of necessary conditions and sufficient conditions for a set to be the support of a doubly stochastic measure have been obtained (see, for example, [14, 24, 25]).

Analogously, it is also known that the class of n-stochastic measures on $[0,1]^n$ fully describes the class of joint distributions of n random variables uniformly distributed on $[0,1]$. In this chapter, we study the supports of multivariate copulas, equivalently the supports of multivariate stochastic measures, from a different approach. We introduce the notion of stochastic closures, which are submeasure closures on a hereditarily Lindelöf space $[0,1]^n$, hence strong and σ-nonessential. We obtain geometric necessary conditions via the notion of essential closedness. Moreover, in some special cases, it turns out that an explicit formula of the support can be derived in terms of stochastic closures. One such case is the case of doubly stochastic measures whose underlying continuous random variables are mutually completely dependent, i.e., each one of them is a Borel measurable function of the other almost surely[1]. In that case, if we further assume that the underlying random variables are uniformly distributed on $[0,1]$, then there is a

[1]A property is said to hold *almost surely* if the set on which the property holds has probability one.

measure-preserving bijective Borel measurable function connecting the two random variables. It has been observed that the graph of such function and the support of the corresponding doubly stochastic measure are closely related. For instance, it has been shown in [2] that the mass of a doubly stochastic measure ν, which is a Borel probability measure, is concentrated on the graph of a corresponding function, i.e., $\nu(\operatorname{gr} f) = 1$. In addition to deriving the geometric necessary conditions, we also introduce a notion of complete dependence in higher dimensions and study the supports of multivariate stochastic measures whose underlying continuous random variables are completely dependent.

To study multivariate stochastic measures, it is more convenient to use the notion of multivariate copulas. It is well-known that there is a one-to-one correspondence between the collection of n-stochastic measures and the collection of n-copulas. Among them, doubly stochastic measures and copulas (i.e., 2-copulas) are most studied. More information on copulas and multivariate copulas can be found in Nelsen's book [19].

Definition 4.1. For an integer $n \geq 2$, an <u>n-copula</u> is a function $C\colon [0,1]^n \to [0,1]$ satisfying

(i) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$,

(ii) $C(1, \ldots, 1, u, 1, \ldots, 1) = u$, and

(iii) C is n-increasing, i.e., for each hyperrectangle $B = \times_{i=1}^n [x_i, y_i] \subseteq [0,1]^n$,

$$V_C(B) = \sum_{z \in \times_{i=1}^n \{x_i, y_i\}} (-1)^{N(z)} C(z) \geq 0,$$

where $N(z)$ denotes the size of the set $\{k\colon z_k = x_k\}$.

Example 4.2. Given a copula C, we have

$$V_C([x_1, x_2] \times [y_1, y_2]) = C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0$$

for each $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$.

The set function V_C is a Borel probability measure on $[0,1]^n$ and is often called C-volume. In fact, V_C is an n-stochastic measure, i.e., it pushes forward to the 1-dimensional Lebesgue measure on each axis: for each Borel measurable set $A \subseteq [0,1]$ and for each $k = 0, 1, \ldots, n-1$,

$$V_C([0,1]^k \times A \times [0,1]^{n-k-1}) = \lambda_1(A).$$

Moreover, the support of C is defined to be the support of the measure V_C.

Remark 4.3. *An n-copula C induces an n-stochastic measure on $[0,1]^n$ via the C-volume. Conversely, given an n-stochastic measure ν on $[0,1]^n$, the function $C\colon [0,1]^n \to [0,1]$ defined by*

$$C(x_1, \ldots, x_n) = \nu([0, x_1] \times \cdots \times [0, x_n])$$

is an n-copula.

The support of a copula C can be used to compute values of the copula at some, if not all, points $(x, y) \in [0,1]^2$. We demonstrate such a technique in the following example.

Example 4.4 ([21], Example 1.5)**.** Let C be a copula whose support is shown in the figure below.

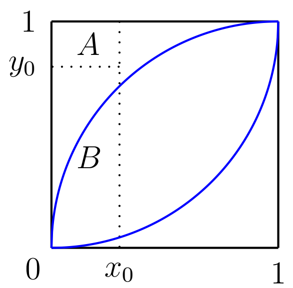

Figure 4.1: the support of C

For any point (x_0, y_0) in the upper left area, let A denote the rectangle whose vertices are $(0, y_0), (x_0, y_0), (x_0, 1)$ and $(0, 1)$ and let B denote the rectangle whose vertices are $(0, 0), (x_0, 0), (x_0, y_0)$ and $(0, y_0)$. Then $V_C(A) = 0$ since it does not intersect the support of C. Moreover,

$$V_C(A \cup B) = C(x_0, 1) - C(x_0, 0) - C(0, 1) + C(0, 0) = x_0.$$

Then, $V_C(B) = V_C(A \cup B) - V_C(A) = x_0$. Hence,

$$C(x_0, y_0) = V_C(B) + C(0, y_0) + C(x_0, 0) - C(0, 0) = x_0.$$

The values of C at the points in the lower right area can be computed similarly.

A multivariate copula can be viewed as a joint distribution of uniform $[0, 1]$ random variables. This fact is shown in one of the most important theorems in copula theory.

Theorem 4.5 (Sklar's theorem)**.** *Let $X_1, \ldots, X_n$ be random variables on a common probability space. Let H be the joint distribution and F_i be the marginal distribution of X_i. Then there is an n-copula C such that*

$$H(x_1, \ldots x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Moreover, if $X_1, \ldots, X_n$ are continuous, C is unique and is denoted by $C_{X_1,\ldots,X_n}$.

In theoretical practices, the most important copulas are the Fréchet-Hoeffding upper and lower bounds and the independence copula. Their formulas are given, respectively, by

$$\begin{aligned} M_2(u, v) &= \min(u, v), \\ W_2(u, v) &= \max(u + v - 1, 0), \\ \Pi_2(u, v) &= uv. \end{aligned}$$

These copulas represent comonotonicity, countermonotonicity and independence, respectively, between the two random variables. The support of M_2 is the set $\{(x, x) \colon x \in [0, 1]\}$, the support of W_2 is the set $\{(x, 1-x) \colon x \in [0, 1]\}$, and Π_2 has full support. In higher dimensions, these three formulas are defined analogously:

$$\begin{aligned} M_n(u_1, \ldots, u_n) &= \min(u_1, \ldots, u_n), \\ W_n(u_1, \ldots, u_n) &= \max(u_1 + \cdots + u_n - n + 1, 0), \\ \Pi_n(u_1, \ldots, u_n) &= u_1 \ldots u_n. \end{aligned}$$

Note that W_n is no longer an n-copula unless $n = 2$ while M_n and Π_n are n-copulas for each integer $n \geq 2$.

Another important class of copulas is known as the class of shuffles of M_2. A shuffle of M_2 can be viewed as a special pushforward of the doubly stochastic measure induced by the copula M_2. This way of defining shuffles of M_2 involves measure-theoretic techniques, which makes it complicated. So, in the following definition, we present a simpler way of defining shuffles of M_2. More information on shuffles of copulas can be found in, e.g., Durante et al.'s work [7] and the first author et al.'s work [23].

Definition 4.6 ([21], Definition 1.8)**.** A copula C is a shuffle of M_2 if there exist a positive integer n, partitions

$$0 = s_0 < s_1 < \cdots < s_n = 1 \text{ and } 0 = t_0 < t_1 < \cdots < t_n = 1$$

of $[0, 1]$, and a permutation σ on the set $\{1, 2, \ldots, n\}$ such that each rectangle $(s_{i-1}, s_i) \times (t_{\sigma(i)-1}, t_{\sigma(i)})$ is a square of C-volume $s_i - s_{i-1}$ and its intersection with the support of C is one of the diagonals of the square.

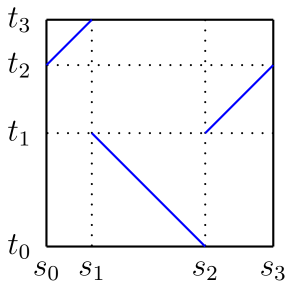

Figure 4.2: the support of a shuffle of M_2 where $\sigma = (1\ 3\ 2)$

Remark 4.7. *Because of a special characteristic of the supports of shuffles of M_2, each of them is uniquely determined by its support via the technique demonstrated in Example 4.4.*

4.1 Multivariate complete dependence

Multivariate dependence structures are much more complex than 2-dimensional dependence structures. In this section, we introduce a special kind of complete dependence in higher dimensions where one set of random variables is completely dependent on another set of random variables. This kind of dependence structure often occurs in practice, such as the case where a set of random variables is used to predict another set of random variables.

Definition 4.8. Given two nonempty finite sets $\mathbf{A}$ and $\mathbf{B}$ of random variables on a common probability space, we say that $\mathbf{A}$ is completely dependent on $\mathbf{B}$ (viewed as a random vector) if, for every $X \in \mathbf{A}$, there exists a Borel measurable function f such that $X = f(\mathbf{B})$ almost surely.

We would like to mention that there is a similar notion of multivariate complete dependence introduced by Tasena and Dhompongsa in [26, Definition 2.2]. In their work, a random vector $(X_1, \ldots, X_n)$ is said to be completely dependent on the i-th coordinate if each X_j is a Borel measurable function of X_i for each $j \neq i$. Compared to our definition of multivariate complete dependence, their definition means exactly that $\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}$ is completely dependent on X_i. So, our notion of multivariate complete dependence is more general.

Definition 4.9. A bipartite dependence n-copula C is the n-copula of continuous random variables $X_1, \ldots, X_n$ where the continuous random variables can be partitioned into two sets so that one set is completely dependent on the other.

In two dimensions, a bipartite dependence copula is simply a copula whose underlying random variables are completely dependent, i.e., one random variable is a Borel measurable function of the other almost surely. We observe that the support of such copula and the graph of a corresponding Borel measurable function are closely related. The following example is an attempt to compute the support of a complete dependence copula from the graph of a Borel measurable function connecting the underlying random variables.

Example 4.10. Let $f\colon [0,1] \to [0,1]$ be such that $f(x) = x$ if x is irrational and $f(x) = 1 - x$ otherwise. Let U be a uniform $[0,1]$ random variable and $V = f(U)$ almost surely. So V is also a uniform $[0,1]$ random variable since f is measure-preserving. Notice that the topological closure of the graph of f, shown in Figure 4.3, is the union of the sets $\{(x,x)\colon x \in [0,1]\}$ and $\{(x, 1-x)\colon x \in [0,1]\}$. But since $V = f(U) = U$ almost surely, the copula of the random vector (U, V) is the 2-dimensional Fréchet-Hoeffding upper bound M_2 whose support, shown in Figure 4.3, is just the set $\{(x,x)\colon x \in [0,1]\}$. Therefore, the topological closure is not a suitable type of essential closure to use in this case. The cause of this is the thin part in the graph of f, namely, the set $\{(x, 1-x)\colon x \in \mathbb{Q}\}$. Even though $\{(x, 1-x)\colon x \in \mathbb{Q}\}$ is negligible since $\mathbb{Q}$ has Lebesgue measure zero, it is dense in $\{(x, 1-x)\colon x \in [0,1]\}$. This is a hindrance to the effectiveness of applying the topological closure to the graph of f.

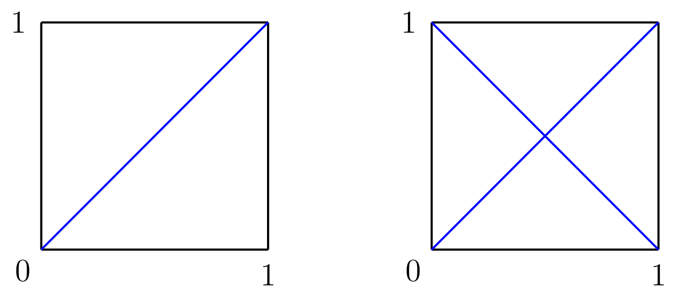

Figure 4.3: the support of $C_{U,V}$ and the topological closure of the graph of f

4.2 Supports of multivariate copulas

We know very little about supports of multivariate copulas compared to what we know about supports of bivariate copulas. One reason is the lack of tools to study them. A suitable tool to study supports of multivariate copulas (or equivalently, supports of multivariate stochastic measures) should be constructed according to the nature of multivariate stochastic measures, i.e., the nature of pushing forward to 1-dimensional Lebesgue measure on each axis.

Definition 4.11. For each integers $1 \leq d \leq n$, define an outer measure $\mathcal{S}_d$ on $\mathcal{P}([0,1]^n)$ as follows: for each $A \subseteq [0,1]^n$,

$$\mathcal{S}_d(A) = \sum_W \lambda_d^*(\pi_W(A)),$$

where the sum is taken over all d-dimensional standard subspaces[2] W of $\mathbb{R}^n$. Define the <u>d-stochastic closure</u> to be the submeasure closure on $\mathcal{P}([0,1]^n)$ induced by $\mathcal{S}_d$.

Remark 4.12. *Equivalently, $x \in \overline{A}^{\mathcal{S}_d}$ if and only if, for each $G \in \mathfrak{N}(x)$, there is a d-dimensional standard subspace W such that $\lambda_d^*(\pi_W(G \cap A)) > 0$. Moreover, since each stochastic closure is a submeasure closure on a Euclidean space, it is both strong and σ-nonessential.*

Remark 4.13. *For each integers $1 \leq e \leq d$, the d-stochastic closure is finer than or equal to the e-stochastic closure, i.e., $\overline{A}^{\mathcal{S}_d} \subseteq \overline{A}^{\mathcal{S}_e}$ for each set A.*

Theorem 4.14. *For every n-copula C, $\operatorname{supp} C$ is 1-stochastic essentially closed.*

Proof. For any Borel set $A \subseteq [0,1]^n$, write

$$V_C(A) = V_C(A \cap \operatorname{supp} C) + V_C(A \cap (\operatorname{supp} C)^c).$$

Observe that $V_C(A \cap (\operatorname{supp} C)^c) \leq V_C((\operatorname{supp} C)^c) = 0$. Consequently, we have $V_C(A) = V_C(A \cap \operatorname{supp} C)$.

Since $\operatorname{supp} C$ is closed, it follows that $\overline{\operatorname{supp} C}^{\mathcal{S}_1} \subseteq \overline{\operatorname{supp} C} = \operatorname{supp} C$. It is left to show that $\operatorname{supp} C \subseteq \overline{\operatorname{supp} C}^{\mathcal{S}_1}$. By the definition of stochastic closures, if $x \notin \overline{\operatorname{supp} C}^{\mathcal{S}_1}$, then there exists $G \in \mathfrak{N}(x)$ such that $\lambda_1(\pi_1(G \cap \operatorname{supp} C)) = 0$. Since V_C is n-stochastic,

$$\begin{aligned} V_C(G) &= V_C(G \cap \operatorname{supp} C) \\ &\leq V_C(\pi_1(G \cap \operatorname{supp} C) \times [0,1]^{n-1}) \\ &= \lambda_1(\pi_1(G \cap \operatorname{supp} C)) = 0. \end{aligned}$$

So $x \notin \operatorname{supp} C$. $\square$

[2]A *standard subspace* is a subspace of a Euclidean space spanned by a set of standard basis elements.

Proposition 4.15. *Let C be an n-copula. Then, for any open set G intersecting the support of C, the intersection cannot be a subset of an $(n-1)$-dimensional hyperplane perpendicular to an axis.*

Proof. Let $W \subseteq \mathbb{R}^n$ be an $(n-1)$-dimensional hyperplane perpendicular to an axis. It suffices to show that $W \cap [0,1]^n$ has V_C-measure zero. Suppose W is perpendicular to the i-th axis at a point $x \in [0,1]$. Since V_C is n-stochastic,

$$V_C(W \cap [0,1]^n) = \lambda_1(\{x\}) = 0.$$

Therefore, any Borel subset of W inside $[0,1]^n$ also has V_C-measure zero. □

Theorem 4.14 and Proposition 4.15 give geometric necessary conditions for a set to be the support of a multivariate copula. However, these necessary conditions are not sufficient. For example, a hairpin-like set is 1-stochastic essentially closed but not always the support of a copula as mentioned in [25].

Example 4.16. As a consequence of Proposition 4.15, the set shown in Figure 4.4 cannot be the support of a copula because it contains a line segment perpendicular to an axis.

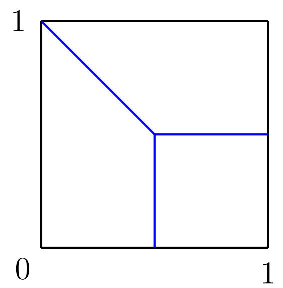

Figure 4.4: a set that is not the support of a copula

In the next section, we explore a special case in which it is possible to explicitly determine the supports via stochastic closures.

4.3 Supports of bipartite dependence multivariate copulas

Definition 4.17. Let $A \subseteq [0,1]^n$ and σ be a permutation on $\{1,2,\ldots,n\}$. Define the coordinate permutation of A with respect to σ by

$$A_\sigma = \{(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})\colon (x_1, x_2, \ldots, x_n) \in A\}.$$

Proposition 4.18. *Let σ be a permutation on $\{1,2,\ldots,n\}$. Let C be the n-copula of uniform $[0,1]$ random variables $X_1,\ldots,X_n$ and let C_σ be the n-copula of uniform $[0,1]$ random variables $X_{\sigma(1)},\ldots,X_{\sigma(n)}$. Then $\operatorname{supp} C_\sigma = (\operatorname{supp} C)_\sigma$.*

Proof. Observe that, for each open set $G \subseteq [0,1]^n$,

$$\begin{aligned}
V_C(G) &= P((X_1,\ldots,X_n) \in G)\\
&= P((X_{\sigma(1)},\ldots,X_{\sigma(n)}) \in G_\sigma)\\
&= V_{C_\sigma}(G_\sigma).
\end{aligned}$$

As a result, $x \in \operatorname{supp} C$ if and only if $x_\sigma \in \operatorname{supp} C_\sigma$. □

According to Proposition 4.18, in the case of bipartite dependence n-copulas, we may rearrange the random variables so that, for some k, each random variable X_j, $j \in \{k+1,\ldots,n\}$, is completely dependent on the random vector $(X_1,\ldots,X_k)$.

Definition 4.19. A function $\mathcal{F}\colon [0,1]^n \to [0,1]^m$ with Borel coordinate functions is said to have a Borel essential refinement if there is $\mathcal{F}^*\colon [0,1]^n \to [0,1]^m$ with Borel coordinate functions such that each corresponding pair of coordinate functions of $\mathcal{F}^*$ and $\mathcal{F}$ are equal Lebesgue almost everywhere and, for any open set $G \subseteq \mathbb{R}^{n+m}$, the following holds:

$$\lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) = 0 \text{ implies } \lambda_1(\pi_j(G \cap \operatorname{gr} \mathcal{F}^*)) = 0 \text{ for all } j > n$$

where W_0 is the subspace spanned by the first n standard basis elements.

Remark 4.20. *A function with Borel coordinate functions is Borel measurable. Hence its graph is a Borel measurable set.*

The notion of Borel essential refinements is introduced in order to deal with certain wild functions. An example of such functions is as follows.

Example 4.21. Let A denote the Cantor ternary set on $[0,1]$ and $f\colon [0,1] \to [0,1]$ denote the Cantor function. Define $g\colon [0,1] \to [0,1]$ by

$$g = f - f \cdot \chi_{A^c}.$$

Observe that g is Borel measurable and $g \equiv 0$ Lebesgue almost everywhere but the range of g is of Lebesgue measure one. This is because f is constant on each of the open intervals contained in A^c. Roughly speaking, this is an example of a function whose graph has a portion with negligible projection image on the domain but non-negligible projection image on the codomain.

This type of wild function is a hindrance to the effectiveness of applying stochastic closures to their graphs. Fortunately, the following theorem guarantees that a function with Borel coordinate functions can always be Borel essentially refined.

Theorem 4.22. *Every function $\mathcal{F}\colon [0,1]^n \to [0,1]^m$ with Borel coordinate functions has a Borel essential refinement.*

Proof. The idea is to redefine $\mathcal{F}$ on a set of Borel measure zero. Let W_0 denote the subspace spanned by the first n standard basis elements of $\mathbb{R}^{n+m}$. Let

$$V = \bigcup_{\alpha \in \Lambda} V_\alpha$$

where the union is taken over all open sets V_α such that $\lambda_n(\pi_{W_0}(V_\alpha \cap \operatorname{gr} \mathcal{F})) = 0$ while $\lambda_1(\pi_j(V_\alpha \cap \operatorname{gr} \mathcal{F})) > 0$ for some $j > n$. By the Lindelöf property of Euclidean spaces, there exists a countable subcollection with the same union: $\{V_1, V_2, \dots\}$. Then we have

$$\lambda_n(\pi_{W_0}(V \cap \operatorname{gr} \mathcal{F})) \leq \sum_{i=1}^{\infty} \lambda_n(\pi_{W_0}(V_i \cap \operatorname{gr} \mathcal{F})) = 0.$$

Thus there exists a Borel measure zero set $B \subseteq \mathbb{R}^n$ with $\pi_{W_0}(V \cap \operatorname{gr} \mathcal{F}) \subseteq B$. Define $\mathcal{F}^* = \mathcal{F} - \mathcal{F} \cdot \chi_B$, i.e., $\mathcal{F}$ is redefined on B to be identically zero. Consequently, the coordinate functions of $\mathcal{F}^*$ are Borel measurable and each corresponding pair of coordinate functions of $\mathcal{F}^*$ and $\mathcal{F}$ are equal almost everywhere.

Suppose there is an open set $G \in \mathbb{R}^{n+m}$ with $\lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) = 0$ while $\lambda_1(\pi_j(G \cap \operatorname{gr} \mathcal{F}^*)) > 0$ for some $j > n$. Observe that the projection images $\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F})$ and $\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)$ differ by a subset of B, which is of Borel measure zero. Moreover, $\pi_j(G \cap \operatorname{gr} \mathcal{F})$ contains $\pi_j(G \cap \operatorname{gr} \mathcal{F}^*) - \{0\}$ due to the redefining. Thus

$$\lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F})) = 0 \text{ while } \lambda_1(\pi_j(G \cap \operatorname{gr} \mathcal{F})) > 0.$$

Hence $G \subseteq V$. This means that the points inside $G \cap \operatorname{gr} \mathcal{F}$ have been redefined. As a consequence, $\lambda_1(\pi_j(G \cap \operatorname{gr} \mathcal{F}^*)) = \lambda_1(\{0\}) = 0$, which is a contradiction. Therefore, $\mathcal{F}^*$ is a Borel essential refinement of $\mathcal{F}$. □

Remark 4.23. *Our results often assume that the random variables are uniformly distributed on* $[0, 1]$*. This is by no means restrictive since, for given continuous random variables* $X_1, \ldots, X_n$*, each* $U_i = F_i(X_i)$ *is uniform on* $[0, 1]$ *for each* i*. Moreover,*

$$C_{X_1,\ldots,X_n} = C_{U_1,\ldots,U_n}.$$

So, it suffices to study only uniform $[0, 1]$ *random variables.*

In the following theorem, we derive an explicit formula of the support of a bipartite dependence multivariate copula in terms of a stochastic closure.

Theorem 4.24. *Let* $U_1, U_2, \ldots, U_{n+m}$ *be uniform* $[0, 1]$ *random variables and* C *be their multivariate copula. Let* $\mathbf{U}$ *denote the random vector* $(U_1, U_2, \ldots, U_n)$*. Suppose that* $\lambda_n \ll V_{C_{\mathbf{U}}} \ll \lambda_n$ *if* $n \geq 2$*. If, for each* $i \in \{1, 2, \ldots, m\}$*,* U_{n+i} *is completely dependent on the random vector* $\mathbf{U}$*, i.e., there exist Borel measurable functions* $f_i \colon [0, 1]^n \to [0, 1]$ *such that* $U_{n+i} = f_i(\mathbf{U})$ *almost surely, then*

$$\operatorname{supp} C = \overline{\operatorname{gr} \mathcal{F}^*}^{\,\mathcal{S}_n}$$

where $\mathcal{F}^* = (f_1^*, \ldots, f_m^*)$ *is a Borel essential refinement of* $\mathcal{F} = (f_1, \ldots, f_m)$*.*

Proof. First of all, since $f_i^* = f_i$ Lebesgue almost everywhere, $f_i^*(\mathbf{U}) = f_i(\mathbf{U})$ almost surely. Hence $C = C_{\mathbf{U},\mathcal{F}^*(\mathbf{U})}$. Let W_0 be the subspace spanned by the first n standard basis elements of $\mathbb{R}^{n+m}$. For a given open set $G \subseteq \mathbb{R}^{n+m}$, it is straightforward to show that

$$\begin{aligned} V_C(G) &= P((\mathbf{U}, \mathcal{F}^*(\mathbf{U})) \in G) \\ &= P((\mathbf{U}, \mathcal{F}^*(\mathbf{U})) \in G \cap \operatorname{gr} \mathcal{F}^*) \\ &= P(\mathbf{U} \in \pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) \end{aligned}$$

since $(\mathbf{U}, \mathcal{F}^*(\mathbf{U})) \in \operatorname{gr} \mathcal{F}^*$ almost surely. Moreover, since $\lambda_n \ll V_{C_\mathbf{U}} \ll \lambda_n$ if $n \geq 2$, we have

$$P(\mathbf{U} \in \pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) > 0 \text{ if and only if } \lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) > 0.$$

Thus $\operatorname{supp} C \subseteq \overline{\operatorname{gr} \mathcal{F}^*}^{\mathcal{S}_n}$.

Conversely, let $G \subseteq \mathbb{R}^{n+m}$ be an open set such that $\lambda_n(\pi_W(G \cap \operatorname{gr} \mathcal{F}^*)) > 0$ for some n-dimensional standard subspace W. Then there exists $j > n$ such that $\lambda_1(\pi_j(G \cap \operatorname{gr} \mathcal{F}^*)) > 0$. Consequently, $\lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) > 0$ since $\mathcal{F}^*$ is a Borel essential refinement. We have previously shown in the proof that

$$V_C(G) > 0 \text{ if and only if } \lambda_n(\pi_{W_0}(G \cap \operatorname{gr} \mathcal{F}^*)) > 0.$$

Thus $\overline{\operatorname{gr} \mathcal{F}^*}^{\mathcal{S}_n} \subseteq \operatorname{supp} C$. $\square$

Remark 4.25. *In Theorem 4.24, if $U_1, \ldots, U_n$ are independent, then $C_\mathbf{U}$ is the independence n-copula Π_n. Thus, for Borel measurable sets $A_1, \ldots, A_n \subseteq [0,1]$,*

$$\begin{aligned} V_{C_\mathbf{U}}(A_1 \times \cdots \times A_n) &= P(\mathbf{U} \in A_1 \times \cdots \times A_n) \\ &= P(U_1 \in A_1) \ldots P(U_n \in A_n) \\ &= \lambda_1(A_1) \ldots \lambda_1(A_n) \\ &= \lambda_n(A_1 \times \cdots \times A_n) \end{aligned}$$

which implies that $V_{C_\mathbf{U}} = \lambda_n$, i.e., $\lambda_n \ll V_{C_\mathbf{U}} \ll \lambda_n$.

Example 4.26. This example demonstrates a way to extract a transformation, connecting the two uniform $[0,1]$ random variables, from the support of a shuffle of M_2.

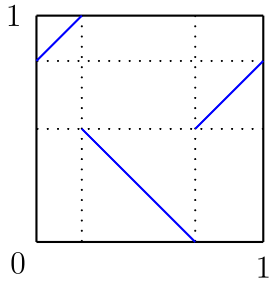

Figure 4.5: the support of a shuffle of M_2

Observe that the support of a shuffle of M_2 looks like the graph of a function. In fact, removing a few points from the support of a shuffle of M_2 gives us the graph of some function. This can be done in many ways. One way to do it is to simply remove the rightmost point from each linear piece except the last piece. One can see that the remaining set is the graph of a function whose explicit formula can be derived. Then, by Theorem 4.24 and the fact that shuffles of M_2 are uniquely determined by their supports, it ensures that we have the right function since the 1-stochastic closure of the graph of that function is equal to the support we started with.

Example 4.27. Write $M_3 = C_{U,U,U}$ for some uniform $[0,1]$ random variable U. In this case, choose $\mathcal{F} = (id_{[0,1]}, id_{[0,1]})$. Thus $M_3 = C_{U,\mathcal{F}(U)}$. Moreover, $\operatorname{gr}\mathcal{F}$ is 1-stochastic essentially closed. Therefore,

$$\operatorname{supp} M_3 = \operatorname{gr}\mathcal{F} = \{(x,x,x) \colon x \in [0,1]\},$$

which is the main diagonal of the unit cube $[0,1]^3$. Similarly, $\operatorname{supp} M_n$ is the main diagonal of the hypercube $[0,1]^n$. Let us remark that even though this result is intuitively known to experts, to the best of our knowledge, no rigorous proof has ever been given. This example is probably the first.

Notice that we need to carefully choose a bipartition in order to apply Theorem 4.24. For example, we can view the last random variable as being completely dependent on the first two. In this case, there are many functions which connect them, e.g., $f_1(x,y) = \frac{x+y}{2}$, $f_2(x,y) = \sqrt{xy}$, etc. Certainly, we cannot apply the theorem and one reason is because the copula of the first two random variables is M_2 whose induced measure is not absolutely continuous with respect to λ_2.

4.4 Local Hausdorff dimension

We end this chapter with the notion of local Hausdorff dimension and the notion of Hausdorff closures, which is yet another type of essential closure suitable for the study of supports of multivariate copulas.

The Hausdorff dimension is a generalization of the notion of dimension of vector spaces, i.e., the Hausdorff dimension of an n-dimensional inner product space is equal to n. The Hausdorff dimension is defined, for all metric spaces, via a class of outer measures called Hausdorff measures. Since Hausdorff measures are outer measures, hence submeasures, they induce submeasure closures which we call Hausdorff closures. In addition to the basic properties of submeasure closures, we derive a connection between Hausdorff closures and the local Hausdorff dimension. More details on Hausdorff measures and the Hausdorff dimension can be found in Falconer's book [8] and Fremlin's book [12].

Definition 4.28. Let (X, d) be a metric space and let $\dim_{\mathrm{H}}$ denote the Hausdorff dimension. Then $A \subseteq X$ is said to have local Hausdorff dimension at least s if, for every open set G intersecting A, $\dim_{\mathrm{H}}(G \cap A) \geq s$.

Let us remark that, in our research, it is not necessary to know the exact value of the local Hausdorff dimension of a set. Knowing a lower bound of the local Hausdorff dimension of the set is sufficient. This is why we use the above definition instead of the one that gives the exact value of the local Hausdorff dimension.

Definition 4.29. Let (X, d) be a metric space and let τ_d denote the topology induced by the metric d. The s-Hausdorff closure is defined to be the submeasure closure on $(X, \tau_d, \mathcal{P}(X))$ induced by $\mathcal{H}^{\mathrm{s}}$.

Basic properties and most applications of Hausdorff closures are analogous to the basic properties and applications of submeasure closures. Moreover, the aforementioned connection between Hausdorff closures and the local Hausdorff dimension is demonstrated in the following result.

Lemma 4.30. *If A is s-Hausdorff essentially closed, then A has local Hausdorff dimension at least s.*

Proof. Suppose there exist $x \in A$ and $G \in \mathfrak{N}(x)$ such that $\dim_{\mathrm{H}}(G \cap A) < s$. Then $\mathcal{H}^{s}(G \cap A) = 0$, contradicting the fact that $x \in A = \overline{A}^{\mathcal{H}^s}$. □

Theorem 4.31. *Let ν be an n-stochastic measure. Then* $\operatorname{supp}\nu$ *is* 1*-Hausdorff essentially closed. In particular,* $\operatorname{supp}\nu$ *has local Hausdorff dimension at least one.*

Proof. It suffices to show that $\operatorname{supp}\nu \subseteq \overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$. If $x \notin \overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$, then there exists $G \in \mathfrak{N}(x)$ such that $\mathcal{H}^1(G \cap \operatorname{supp}\nu) = 0$. Note that $\nu(G) = \nu(G \cap \operatorname{supp}\nu)$. If $\nu(G \cap \operatorname{supp}\nu) > 0$, then $\mathcal{H}^1(\pi_1(G \cap \operatorname{supp}\nu)) = \lambda_1(\pi_1(G \cap \operatorname{supp}\nu)) > 0$. Thus $\mathcal{H}^1(G \cap \operatorname{supp}\nu) > 0$. So $\nu(G) = \nu(G \cap \operatorname{supp}\nu) = 0$. Hence $x \notin \operatorname{supp}\nu$. Therefore, $\operatorname{supp}\nu = \overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$. Hence $\operatorname{supp}\nu$ is 1-Hausdorff essentially closed. Consequently, by Lemma 4.30, $\operatorname{supp}\nu$ has local Hausdorff dimension at least one. □

Example 4.32. In [9, Theorem 1], Fredricks et al. show that for each value $s \in (1, 2)$, there is a copula with fractal support of Hausdorff dimension s. In fact, there are copulas with supports of Hausdorff dimension one and two, which are the 2-dimensional Fréchet-Hoeffding upper bound and the independence copula, respectively. Moreover, Theorem 4.31 implies, in particular, that the support of a copula has Hausdorff dimension at least one. Together with the result of Fredricks et al., we can conclude that supports of copulas have Hausdorff dimension at least one and for each possible value $s \in [1, 2]$, there is a copula whose support is of Hausdorff dimension s.

Lemma 4.33. *Let A be a subset of a Euclidean space. Then*

$$\dim_{\mathrm{H}}(A) \geq \dim_{\mathrm{H}}(\pi_W(A))$$

for any orthogonal projection π_W.

Proof. It suffices to show that for any δ-cover $\{C_\alpha\}_{\alpha\in\Lambda}$ of the set A, there exists a δ-cover $\{D_\alpha\}_{\alpha\in\Lambda}$ of $\pi_W(A)$ such that $\operatorname{diam}(D_\alpha) \le \operatorname{diam}(C_\alpha)$, which implies that $\mathcal{H}^{\mathrm{s}}(\pi_W(A)) \le \mathcal{H}^{\mathrm{s}}(A)$ for each s.

Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be a δ-cover of A. Choose $D_\alpha = \pi_W(C_\alpha)$. It is clear that $\{D_\alpha\}_{\alpha\in\Lambda}$ covers $\pi_W(A)$. Moreover,

$$\operatorname{diam}(D_\alpha) \le \operatorname{diam}(C_\alpha) \le \delta$$

since π_W is an orthogonal projection. Hence $\{D_\alpha\}_{\alpha\in\Lambda}$ is a δ-cover of $\pi_W(A)$ $\square$

As a consequence of Theorem 4.24, the support of a bipartite dependence multivariate copula is essentially closed with respect to the associated stochastic closure. The following result gives a geometric interpretation derived directly from the essential closedness of the support of the bipartite dependence multivariate copula.

Theorem 4.34. *Let C be the bipartite dependence multivariate copula defined in Theorem 4.24. Then $\operatorname{supp} C$ is an n-stochastic essentially closed set. In particular, $\operatorname{supp} C$ has local Hausdorff dimension at least n.*

Proof. Since $\operatorname{supp} C$ can be written as an n-stochastic closure of some set, it is n-stochastic essentially closed. Suppose there exists an open set $G \subseteq \mathbb{R}^{n+m}$ such that $G \cap \operatorname{supp} C \neq \varnothing$ and $\dim_{\mathrm{H}}(G \cap \operatorname{supp} C) < n$. By Lemma 4.33, for each n-dimensional standard subspace W, $\dim_{\mathrm{H}}(\pi_W(G \cap \operatorname{supp} C)) < n$ which implies $\lambda_n(\pi_W(G \cap \operatorname{supp} C)) = 0$. Consequently, $\overline{G \cap \operatorname{supp} C}^{\mathcal{S}_n} = \varnothing$, contradicting with Lemma 2.15. $\square$

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