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GENERAL FRÉCHET FUNCTIONAL EQUATION

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ให้ X และ Y เป็นปริภูมิเชิงเส้นเหนือฟีลด์ $\mathbb{Q}, \mathbb{R}$ หรือ $\mathbb{C}$ กำหนดให้ $a_1, a_2, \ldots, a_n$ เป็นสเกลาร์ซึ่ง $a_n \neq 0$ เราได้พิสูจน์ว่าผลเฉลย $f : X \to Y$ ทั้งหมดของสมการเชิงฟังก์ชันเฟรเชในรูปทั่วไป

$$f(x) + \sum_{k=1}^{n} a_k f(x + ky) = 0 \qquad \text{สำหรับทุก } x, y \in X$$

เป็นฟังก์ชันพหุนามวางนัยทั่วไป ซึ่งสามารถจำแนกดีกรีได้ในรูปของ $a_1, a_2, \ldots, a_n$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..........

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PATCHAREE SUMRITNORRAPONG : GENERAL FRÉCHET FUNCTIONAL EQUATION. ADVISOR : ASSOC. PROF. PATANEE UDOMKAVANICH, Ph.D., CO-ADVISOR : ASSOC. PROF. PAISAN NAKMAHACHALASINT, Ph.D., 25 pp.

Let X and Y be linear spaces over field $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Given scalars $a_1, a_2, \ldots, a_n$ such that $a_n \neq 0$, we prove that all solutions $f : X \to Y$ of the general Fréchet functional equation

$$f(x) + \sum_{k=1}^{n} a_k f(x + ky) = 0 \quad \text{for all } x, y \in X$$

are generalized polynomial functions of degrees classifiable in terms of $a_1, a_2, \ldots, a_n$.

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CONTENTS

CHAPTER I
INTRODUCTION

In this chapter, we will give some general background of functional equation and the motivation of our proposed problem.

1.1 Functional Equations

P.K. Sahoo and Pl. Kannappan [13] simply described that *"Functional equations are equations in which the unknowns are functions."* One purpose of studying a functional equation is to determine all functions satisfying the given equation. The following examples illustrate how one may determine the general solution of a given functional equation.

Example 1.1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) + 2f(y) = f(3y) + x \quad \text{for all } x, y \in \mathbb{R}. \tag{1.1}$$

Solution. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.1).
Setting $x = 0$ and $y = 0$ in (1.1), we have $f(0) = 0$.
Replacing $y = 0$ in (1.1), we obtain

$$f(x) + 2f(0) = f(0) + x \quad \text{for all } x \in \mathbb{R}.$$

Since $f(0) = 0$, we get

$$f(x) = x \quad \text{for all } x \in \mathbb{R}.$$

On the other hand, if a function f is defined by $f(x) = x$ for all $x \in \mathbb{R}$, then

$$f(x+y) + 2f(y) = (x+y) + 2y = 3y + x = f(3y) + x$$

That is, $f(x) = x$ indeed satisfies the functional equation (1.1). Therefore, the solution of (1.1) is the function f given by $f(x) = x$ for all $x \in \mathbb{R}$. $\square$

Some functional equations may not have any solutions.

Example 1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(y) + x + 1 \quad \text{for all } x, y \in \mathbb{R}. \tag{1.2}$$

Solution. Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.2).
Substituting $y = 0$ in (1.2), we have

$$f(x) = f(0) + x + 1 \quad \text{for all } x \in \mathbb{R}.$$

Setting $c = f(0) + 1$, we obtain $f(x) = x + c$ for all $x \in \mathbb{R}$.
Conversely, if a function f is given by $f(x) = x + c$ for all $x \in \mathbb{R}$, then we see that the left-hand side of (1.2) becomes

$$f(x+y) = x + y + c$$

while the right-hand side of (1.2) is

$$f(y) + x + 1 = y + c + x + 1 = x + y + c + 1.$$

Since $c \neq c+1$, the function f does not satisfy (1.2). Therefore, there is no function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1.2). □

A famous problem in functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}. \tag{1.3}$$

This additive functional equation was studied by many researchers but an important result regarding its solution was proved by A.L. Cauchy [3] in 1821. Cauchy proved that all continuous solutions $f : \mathbb{R} \to \mathbb{R}$ are of the form $f(x) = cx$ for all $x \in \mathbb{R}$, where c is a real constant. The additive functional equation is later recognized as the *Cauchy functional equation.* In general, every solution of the Cauchy functional equation is said to be an *additive function.* In 1905, G. Hamel [9] employed Hamel bases over $\mathbb{Q}$ to construct the discontinuous solution of the

Cauchy functional equation. A remarkable property of a discontinuous additive function is that the graph $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$ is dense in the plane $\mathbb{R}^2$, that is, for all $\varepsilon > 0$ and for all $(a, b) \in \mathbb{R}^2$ there exists a point $(x, f(x)) \in G(f)$ such that $(x-a)^2 + (f(x) - b)^2 < \varepsilon^2$, which indicates that the graph of f contain points scattered all the plane $\mathbb{R}^2$. This result was also proved by E. Hewitt and H.S. Zuckerman [10] without using Hamel basis in 1969.

Another functional equation that is closely related to the Cauchy functional equation is the Jensen's functional equation (for more information, please refer to [1])

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \quad \text{for all } x, y \in \mathbb{R}. \tag{1.4}$$

It can be seen that (1.4), under a suitable constant of translation, is equivalent to the Cauchy functional equation; that is, if we set $g(x) = f(x) - f(0)$, then we can show that $g(0) = 0, g(\frac{x}{2}) = \frac{1}{2}g(x)$ and $g(\frac{x+y}{2}) = \frac{1}{2}(g(x) + g(y))$ which indicate that g is an additive function. The continuous solution of the Jensen's functional equation is of the form $f(x) = c + ax$ for all $x \in \mathbb{R}$, where a and c are real constants. The general solution of the Jensen's functional equation is of the form $f(x) = c + A(x)$ where c is a real constants and A is an additive function.

If we replace y by $x + 2y$ in the Jensen's functional equation (1.4), then it becomes

$$f(x) - 2f(x+y) + f(x+2y) = 0 \quad \text{for all } x, y \in \mathbb{R}. \tag{1.5}$$

On the other hand, if we replace y by $\frac{y-x}{2}$ in (1.5), then the functional equation (1.5) becomes the Jensen's functional equation. Hence, we can take (1.5) as an equivalent form of (1.4).

In 1909, M. Fréchet [8] initiated a generalization of the Cauchy functional equation. This functional equation can be written in an explicit form as

$$\sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(x + kh) = 0 \tag{1.6}$$

where m is a nonnegative integer. The functional equation (1.6) can be written in terms of a difference operator with a span h as

$$\Delta_h^{m+1} f(x) = 0 \tag{1.7}$$

More precisely, Fréchet proved that the continuous solution of (1.6) is an ordinary of polynomial degree at most m. The functional equation (1.7) will be referred to as the *Fréchet functional equation*. In 1967, M.A. McKiernan [11] gave the general solution of the Fréchet functional equation (1.7) is of the form

$$f(x) = A^0 + A^1(x) + A^2 + \cdots + A^m(x)$$

where A^0 is a constant and $A^k(x)$ is the diagonalization of a k-additive symmetric function $A_k : \mathbb{R}^k \to \mathbb{R}$ for $k = 1, 2, \ldots, m$.

In 2003, J.K. Chung and P.K. Sahoo [4] determined the general solution of the functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4\left(f(x+y) + f(x-y) + 6f(y)\right) \tag{1.8}$$

for all $x, y \in \mathbb{R}$. The next theorem gives a general solution of the functional equation (1.8).

Theorem 1.3. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.8) if and only if f is of the form*

$$f(x) = A^4(x),$$

where $A^4(x)$ is the diagonalization of a 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$.

In 2004, P.K. Sahoo [12] solved the functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y) \tag{1.9}$$

for all $x, y \in \mathbb{R}$. The general solution of (1.9) is shown in the following theorem.

Theorem 1.4. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.9) if and only if f is of the form*

$$f(x) = A^0 + A^1(x) + A^2(x) + A^3(x)$$

where A^0 is an arbitrary constant and $A^n(x)$ is the diagonalization of an n-additive symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3$.

In 2009, M. Eshaghi Gordji and H. Khodaei [7] found the general solution of the following functional equation.

$$f(x+ky) + f(x-ky) = k^2 f(x+y) + k^2 f(x-y) + 2(1-k^2)f(x) \tag{1.10}$$

for fixed integers k with $k \neq 0, \pm 1$. The general solution of (1.10) is shown in the next theorem.

Theorem 1.5. *Let X and Y be real linear spaces. A function $f : X \to Y$ with $f(0) = 0$ satisfies (1.10) for all $x, y \in X$ if and only if for each $k = 1, 2, 3,$ there exists a k-additive symmetric function $A_k : X^k \to Y$ such that*

$$f(x) = A^1(x) + A^2(x) + A^3(x)$$

where A^k is the diagonalization of A_k.

In 2012, A. Thanyacharoen [14] solved the general solution of the functional equation

$$\begin{aligned} f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) \\ = 13\left(f(x+y) + f(x-y)\right) + 168f(y) \end{aligned} \tag{1.11}$$

for all $x, y \in \mathbb{R}$. Before he established the general solution of the functional equation (1.11), he proved an auxiliary lemma.

Lemma 1.6. *If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$f(x+4y) - 14f(x+2y) + 35f(x+y) - 35f(x) + 14f(x-y) - f(x-3y) = 0 \tag{1.12}$$

for all $x, y \in \mathbb{R}$, then f is of the form

$$f(x) = A^0 + A^1(x) + A^2(x) + A^3(x) + A^4(x)$$

where A^0 is an arbitrary constant and $A^n(x)$ is the diagonalization of an n-additive symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3, 4$.

The general solution of (1.11) is shown in the following theorem.

Theorem 1.7. *A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (1.11) if and only if it is of the form*

$$f(x) = A^4(x)$$

where $A^4(x)$ is the diagonalization of a 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$.

In 2005, J.A. Baker [2] found the general solution of a general functional equation

$$\sum_{k=0}^{m} f_k(\alpha_k x + \beta_k y) = 0$$

as shown in the following theorem.

Theorem 1.8. *Suppose that V and B are linear spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j\beta_k - \alpha_k\beta_j \neq 0$ whenever $0 \leq j < k \leq m$. If $f_k : V \to B$ for $0 \leq k \leq m$ and*

$$\sum_{k=0}^{m} f_k(\alpha_k x + \beta_k y) = 0 \quad \text{for all } x, y \in V, \tag{1.13}$$

then each f_k is a generalized polynomial function of degree at most $m - 1$.

1.2 Proposed Problem

Throughout this thesis, we let X and Y be linear spaces over $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ and $f : X \to Y$ be an arbitrary function and n be a positive integer. We will extend the Fréchet functional equation to a more general form

$$\sum_{k=0}^{n} a_k f(x + ky) = 0 \tag{1.14}$$

where $a_0, a_1, a_2, \ldots, a_n$ are scalars with $a_n \neq 0$.

We may assume that $a_0 \neq 0$. For suppose not, let m be the least positive integer m such that $a_m \neq 0$ and replace x by $x - my$ in (1.14), we obtain

$$\sum_{k=0}^{n-m} a_{k+m} f(x + ky) = 0.$$

Thus (1.14) can be written as a sum involving $n-m$, instead of n, terms.
Since $a_0 \neq 0$, we divide (1.14) by a_0. Then the coefficient of term $f(x)$ is 1. Hence, it can be assume that $a_0 = 1$.

In this thesis, we will determine the general solution of

$$\sum_{k=0}^{n} a_k f(x+ky) = 0 \qquad \text{for all } x, y \in X$$

where $a_0 = 1$ and $a_1, a_2, \ldots, a_n$ are scalars with $a_n \neq 0$. This functional equation will be referred to as the *general Fréchet functional equation of order n.*

It follows from Theorem 1.8 that the general solution of the general Fréchet functional equation of order n is a generalized polynomial function of degree at most $n-1$. There are some cases that the degree of the solution does not reach $n-1$. One of these examples was shown in Lemma 1.6 which stated that the general solution of the general Fréchet functional equation of order 7 is a generalized polynomial of degree at most 4. For this reason, it is interesting to explore much deeper on the degree of the solution.

In this thesis, we classify the degree of the general solution of the general Fréchet functional equation of order n in term of $a_1, a_2, \ldots, a_n$.

CHAPTER II
PRELIMINARIES

In this chapter, we will introduce definitions and theorems related to additive functions, multi-additive functions and the difference operators which will be used for this thesis.

First, we give the definition of an n-additive function, symmetric function and its diagonalization.

Definition 2.1. A function $A_n : X^n \to Y$ is called an *n-additive function* if for each $1 \leq i \leq n$,

$$\begin{aligned} &A_n(x_1, \ldots, x_i + y_i, \ldots, x_n) \\ &= A_n(x_1, \ldots, x_{i-1}, x_i, x_{i+1} \ldots, x_n) + A_n(x_1, \ldots, x_{i-1}, y_i, x_{i+1} \ldots, x_n), \end{aligned}$$

for all $x_1, \ldots, x_n, y_i \in X$.

A 1-additve function is said to be an additive function and a 2-additive function will be called a *bi-additive function.*

Example 2.2. A function $A_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $A_2(x, y) = xA(y)$, for all $x, y \in \mathbb{R}$, where A is an additive function, is a bi-additive function.

$\square$

Example 2.3. A function $A_3 : \mathbb{R}^3 \to \mathbb{R}$ defined by $A_3(x, y, z) = f(x)g(y)h(z)$, for all $x, y, z \in \mathbb{R}$, where f, g and h are additive functions, is a 3-additive function.

$\square$

Definition 2.4. Let $A_n : X^n \to Y$ be a function. A_n is said to be *symmetric* if $A_n(x_1, x_2, \ldots, x_n)$ is invariant under any permutation of $x_1, \ldots, x_n$, that is, for each $x_1, x_2, \dots, x_n \in X$

$$A_n(x_1, x_2, \dots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

where π is any permutation of $(1, 2, \dots, n)$.

Example 2.5. A function $A_n : \mathbb{R}^n \to \mathbb{R}$ defined by $A_n(x_1, x_2 \dots, x_n) = cx_1x_2 \cdots x_n$ for all $x_1, \dots, x_n \in \mathbb{R}$, where c is a constant, is n-additive symmetric function.

$\square$

Example 2.6. A function $A_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $A_2(x, y) = xA(y) - yA(x)$, for all $x, y \in \mathbb{R}$, where A is an additive function, is a bi-additive function but it is not symmetric.

$\square$

Definition 2.7. Let $A_n : X^n \to Y$ be a function. The function $A^n : X \to Y$ defined by

$$A^n(x) = A_n(x, x, \dots, x),$$

for all $x \in X$, is called the *diagonalization* of A_n.

Example 2.8. Define a function $A_n : \mathbb{R}^n \to \mathbb{R}$ by $A_n(x_1, x_2 \dots, x_n) = x_1x_2 \cdots x_n$ for all $x_1, \dots, x_n \in \mathbb{R}$. The diagonalization of A_n is given by

$$A^n(x) = x^n \quad \text{for all } x \in \mathbb{R}.$$

$\square$

The following lemmas give properties of an n-additive function and its diagonalization (for details, please refer to [5]).

Lemma 2.9. *Let $A_n : X^n \to Y$ be an n-additive function and let r be a rational number. Then for each $1 \le i \le n$,*

$$A_n(x_1, \dots, rx_i, \dots, x_n) = rA_n(x_1, \dots, x_i, \dots, x_n),$$

for all $x_1, \dots, x_n \in X$.

Lemma 2.10. *Let $A^n : X \to Y$ be a diagonalization of an n-additive function A_n. Then*

(i) For all $x \in X$ and for all $r \in \mathbb{Q}$,

$$A^n(rx) = r^n A^n(x).$$

(ii) For all $x, y \in X$,

$$A^n(x+y) = \sum_{i=0}^{n} \binom{n}{i} A_n(\underbrace{x, \ldots, x}_{n-i}, \underbrace{y, \ldots, y}_{i}).$$

Next, we give the definition of the difference operators and their properties.

Definition 2.11. The *difference operator* Δ_h with the span $h \in X$ is defined by

$$\Delta_h f(x) = f(x+h) - f(x),$$

for all $x \in X$.
For each $m = 0, 1, 2, \ldots$, we define the iterates Δ_h^m with the same span $h \in X$ by the recurrence relation

$$\Delta_h^0 f(x) = f(x) \text{ and } \Delta_h^{m+1} f(x) = \Delta_h \left(\Delta_h^m f(x)\right).$$

The composition of the difference operator with difference spans $h_1, h_2, \ldots, h_n \in X$ is denoted by $\Delta_{h_1, h_2, \ldots, h_n} f(x)$, that is,

$$\Delta_{h_1, h_2, \ldots, h_n} f(x) = \Delta_{h_1} \Delta_{h_2} \cdots \Delta_{h_n} f(x).$$

Example 2.12. For all $x, h \in X$,

$$\begin{aligned} \Delta_h^2 f(x) &= \Delta_h(\Delta_h f(x)) \\ &= \Delta_h(f(x+h) - f(x)) \\ &= f(x+2h) - 2f(x+h) + f(x). \end{aligned}$$

$\square$

The following lemma and theorems are properties of the difference operator (for details, please refer to [5]).

Lemma 2.13. *For any $h_1, h_2 \in X$ the difference operators $\Delta_{h_1}, \Delta_{h_2}$ commute, that is,*

$$\Delta_{h_1}\Delta_{h_2}f = \Delta_{h_2}\Delta_{h_1}f$$

Theorem 2.14. *For all $x, h_1, h_2, \ldots, h_n \in X$,*

$$\Delta_{h_1,h_2,\ldots,h_n}f(x) = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} (-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)} f(x + \varepsilon_1 h_1 + \cdots + \varepsilon_n h_n).$$

Theorem 2.15. *For all $x, h \in X$,*

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh).$$

Next theorem gives the relationship between the difference operator with the same span and the difference operator with the difference spans proved by D.Ž. Djoković [6].

Theorem 2.16. *Let $f : X \to Y$ be a function and let $h_1, \ldots, h_n \in X$ be arbitrary. For $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ define*

$$\alpha_{\varepsilon_1,\ldots,\varepsilon_n} = -\sum_{r=1}^{n} \frac{\varepsilon_r h_r}{r} \quad \text{and} \quad b_{\varepsilon_1,\ldots,\varepsilon_n} = \sum_{r=1}^{n} \varepsilon_r h_r.$$

Then for every $x \in X$,

$$\Delta_{h_1,\ldots,h_n} f(x) = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_n} \Delta^n_{\alpha_{\varepsilon_1,\ldots,\varepsilon_n}} f(x + b_{\varepsilon_1,\ldots,\varepsilon_n}).$$

Next, we give the definition of a generalized polynomial function.

Definition 2.17. [2] Let m be a nonnegative integer. A function $f : X \to Y$ is called a *generalized polynomial function of degree at most m* if for each $k = 0, 1, \ldots, m$, there exists a k-additive symmetric function $A_k : X^k \to Y$ such that

$$f(x) = A^0 + A^1(x) + \cdots + A^m(x) \quad \text{for all } x \in X,$$

where A^k is the diagonalization of A_k.

Theorem 2.18. [6] *Let m be a nonnegative integer. A function $f : X \to Y$ is a generalized polynomial function of degree at most m if and only if f satisfies $\Delta_{h_1,\ldots,h_{m+1}} f(x) = 0$ for all $x, h_1, h_2, \ldots, h_{m+1} \in X$.*

The following theorem states the general solution of Fréchet functional equation.

Theorem 2.19. [5] *Let m be a nonnegative integer. A function $f : X \to Y$ satisfies*

$$\Delta_h^{m+1} f(x) = 0 \quad \textit{for all } x, h \in X$$

if and only if f is a generalized polynomial function of degree at most m.

CHAPTER III

GENERAL JENSEN TYPE FUNCTIONAL EQUATIONS

In this chapter, we will extend the Jensen's functional equation to a more general form

$$f(x) + af(x+y) + bf(x+2y) = 0 \quad \text{for all } x, y \in X, \tag{3.1}$$

where a and b are scalars with $b \neq 0$, and will determine its general solution. The functional equation (3.1) will be called a *general Jensen type functional equation.* In order to determine the general solution of the general Jensen type functional equation. We first consider the general solution of the functional equation of the form

$$f(x) + af(x+y) = 0 \quad \text{for all } x, y \in X, \tag{3.2}$$

where a is a nonzero scalar.

Lemma 3.1. *The general solution of the functional equation (3.2) is as follows.*

1. *If $a \neq -1$, then $f(x) \equiv 0$.*

2. *If $a = -1$, then f is a constant function.*

Proof. 1. Assume that $a \neq -1$. Setting $y = 0$ in (3.2), we get $(1+a)f(x) = 0$ for all $x \in X$. Since $a \neq -1$, we get $f(x) = 0$ for all $x \in X$. On the other hand, it can be verified that $f(x) \equiv 0$ indeed satisfies (3.2).

2. Assume that $a = -1$. Then (3.2) becomes

$$f(x) - f(x+y) = 0 \quad \text{for all } x, y \in X. \tag{3.3}$$

Replacing $x = 0$ in (3.3), we get $f(y) = f(0)$ for all $y \in X$. Therefore, f is a constant function.

On the other hand, it can be verified that $f(x) = c$ where c is a constant satisfies (3.2).

$\square$

In the next theorem gives the general solution of the general Jensen type functional equation.

Theorem 3.2. *The general solutions of the general Jensen type functional equation (3.1) are the followings:*

1. *If $a + b \neq -1$, then $f(x) \equiv 0$.*

2. *Assume that $a + b = -1$.*

   (i) If $a \neq -2$, then f is a constant function.

   (ii) If $a = -2$, then $f(x) = c + A(x)$ where c is a constant and $A : X \to Y$ is an additive function.

Proof. 1. Assume that $a + b \neq -1$. Setting $y = 0$ in (3.1), we have

$$(1 + a + b)f(x) = 0 \quad \text{for all } x \in X.$$

Since $a + b \neq -1$, we get $f(x) = 0$ for all $x \in X$. The function $f(x) \equiv 0$ obviously satisfies (3.1).

2. Assume that $a + b = -1$. First, consider in the case $a = 0$. Then $b = -1$ and (3.1) becomes

$$f(x) - f(x + 2y) = 0 \quad \text{for all } x, y \in X. \tag{3.4}$$

Replacing y by $\frac{y}{2}$ in (3.4), we get $f(x) - f(x + y) = 0$. By Lemma 3.1, f is a constant function.

Assume that $a \neq 0$. Substituting $b = -1 - a$ in (3.1), we get

$$f(x) + af(x + y) + (-1 - a)f(x + 2y) = 0 \tag{3.5}$$

Replacing x by $x+y$ in (3.5) and multiplying by $-a$, we have

$$-af(x+y) - a^2 f(x+2y) - a(-1-a)f(x+3y) = 0 \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$f(x) + (-1-a-a^2)f(x+2y) - a(-1-a)f(x+3y) = 0 \tag{3.7}$$

Replacing x by $x+2y$ in (3.5) and multiplying by a^2+a+1, we have

$$(a^2+a+1)f(x+2y)+a(a^2+a+1)f(x+3y)+(-1-a)(a^2+a+1)f(x+4y) = 0 \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$f(x) + a(a^2+2a+2)f(x+3y) + (-1-a)(a^2+a+1)f(x+4y) = 0 \tag{3.9}$$

Replacing y by $2y$ in (3.5), we have

$$f(x) + af(x+2y) + (-1-a)f(x+4y) = 0 \tag{3.10}$$

Subtracting (3.10) from (3.9), then dividing the result by a, we get

$$-f(x+2y) + (a^2+2a+2)f(x+3y) + (-1-a)(a+1)f(x+4y) = 0 \tag{3.11}$$

Replacing x by $x-2y$ in (3.11) and combining (3.5), we obtain

$$(a+1)(a+2)f(x+y) - (a+1)(a+2)f(x+2y) = 0 \tag{3.12}$$

(i) Assume that $a \neq -2$.

Since it is presumed that $b \neq 0$ and $a+b = -1$, we have $a \neq -1$. Therefore, $(a+1)(a+2) \neq 0$. Setting x by $x-y$ in (3.12) and dividing by $(a+1)(a+2)$, we have $f(x) - f(x+y) = 0$. By Lemma 3.1, f is a constant function.

Substituting $f(x) = c$ in (3.1) where c is a constant, we have

$$f(x) + af(x+y) + bf(x+2y) = c + ac + bc = c(1+a+b) = 0$$

because $a+b = -1$. Hence, $f(x) = c$ satisfies (3.1).

(ii) Assume that $a = -2$. Then $b = -1 - a = 1$ and (3.1) becomes

$$f(x) - 2f(x+y) + f(x+2y) = 0$$

which is an equivalent form of the Jensen's functional equation.

Therefore, $f(x) = c + A(x)$ where c is a constant and $A : X \to Y$ is an additive function.

$\square$

Remark 3.3. *The general Jensen type functional equation (3.1) where $a = -2$ and $b = 1$ reproduces the Jensen's functional equation and the general solution is of the form $f(x) = c + A(x)$. While for all other values of a and b in (3.1), the general solution is just a constant (may be zero in certain cases) function.*

CHAPTER IV
GENERAL FRÉCHET FUNCTIONAL EQUATION

In this chapter, we will give the general solution and classify degree of the general solution of the general Fréchet functional equation of order n,

$$\sum_{k=0}^{n} a_k f(x + ky) = 0 \quad \text{for all } x, y \in X \tag{4.1}$$

where $a_0 = 1$ and $a_1, a_2, \ldots, a_n$ are scalars with $a_n \neq 0$.

4.1 General Solution

In this section, we give the general solution of the general Fréchet functional equation of order n.

We note that our work is a special case of Baker's result [2]. But our outstanding point is that we use an elementary approach.

Theorem 4.1. *If $f : X \to Y$ satisfies the general Fréchet functional equation of order n, then f is a generalized polynomial function of degree at most $n - 1$.*

Proof. Assume that f satisfies the general Fréchet functional equation of order n. For each $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0, 1\}$, we replace x with $X_{\varepsilon_1 \ldots \varepsilon_n} = x + \varepsilon_1 y_1 + \cdots + \varepsilon_n y_n$ and y with $Y_{\varepsilon_1 \ldots \varepsilon_n} = -\varepsilon_1 y_1 - \frac{1}{2}\varepsilon_2 y_2 - \cdots - \frac{1}{n}\varepsilon_n y_n$ in (4.1). We obtain

$$\sum_{k=0}^{n} a_k f(X_{\varepsilon_1 \ldots \varepsilon_n} + kY_{\varepsilon_1 \ldots \varepsilon_n}) = 0. \tag{4.2}$$

Multiplying (4.2) by $(-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)}$ and taking sum over all $\varepsilon_1, \ldots, \varepsilon_n$, to get

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}} (-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)} \sum_{k=0}^{n} a_k f(X_{\varepsilon_1 \ldots \varepsilon_n} + kY_{\varepsilon_1 \ldots \varepsilon_n}) = 0. \tag{4.3}$$

Swapping the order of the summations in (4.3), we get

$$\sum_{k=0}^{n} a_k \sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} (-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)} f(X_{\varepsilon_1\ldots\varepsilon_n} + kY_{\varepsilon_1\ldots\varepsilon_n}) = 0. \tag{4.4}$$

On the other hand, observe that

$$f(X_{\varepsilon_1\ldots\varepsilon_n} + kY_{\varepsilon_1\ldots\varepsilon_n}) = f\Big(x + \varepsilon_1(1-k)y_1 + \varepsilon_2(1-\frac{k}{2})y_2 + \cdots + \varepsilon_n(1-\frac{k}{n})y_n\Big).$$

By Theorem 2.14, we have

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} (-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)} f(X_{\varepsilon_1\ldots\varepsilon_n} + kY_{\varepsilon_1\ldots\varepsilon_n}) = \Delta_{(1-k)y_1,(1-\frac{k}{2})y_2,\ldots,(1-\frac{k}{n})y_n} f(x) \tag{4.5}$$

From (4.4) and (4.5), we obtain

$$\sum_{k=0}^{n} a_k \Delta_{(1-k)y_1,(1-\frac{k}{2})y_2,\ldots,(1-\frac{k}{n})y_n} f(x) = 0. \tag{4.6}$$

For each $k = 1, 2, \ldots, n$, let $b_k = \Delta_{(1-k)y_1,(1-\frac{k}{2})y_2,\ldots,(1-\frac{k}{n})y_n} f(x)$. We can see that

$$b_1 = \Delta_{0,(1-\frac{1}{2})y_2,\ldots,(1-\frac{1}{n})y_n} f(x),$$

$$b_n = \Delta_{(1-n)y_1,(1-\frac{n}{2})y_2,\ldots,(1-\frac{n}{n-1})y_{n-1},0} f(x)$$

and for each $k = 2, 3, \ldots, n-1$,

$$b_k = \Delta_{(1-k)y_1,(1-\frac{k}{2})y_2,\ldots,(1-\frac{k}{k-1})y_{k-1},0,(1-\frac{k}{k+1})y_{k+1},\ldots,(1-\frac{k}{n})y_n} f(x).$$

Since the difference operators commute and $\Delta_0 f(x) = 0$, we get $b_k = 0$ for all $k = 1, 2, \ldots, n$. Hence, for each $k = 1, 2, \ldots, n$,

$$\Delta_{(1-k)y_1,(1-\frac{k}{2})y_2,\ldots,(1-\frac{k}{n})y_n} f(x) = 0.$$

Therefore, the only term that survives in (4.6) is the term with $k = 0$, it follows that (4.6) becomes

$$\Delta_{y_1,y_2,\ldots,y_n} f(x) = 0.$$

Hence, f is a generalized polynomial function of degree at most $n-1$. $\square$

4.2 Classification of the Degree of the General Solution

In this section, we will classify degree of a generalized polynomial function which is the general solution of the general Fréchet functional equation of order n.

Theorem 4.2. *The general solution $f : X \to Y$ satisfying the general Fréchet functional equation of order n are the followings.*

1. *If $\displaystyle\sum_{k=0}^{n} a_k \neq 0$, then $f(x) \equiv 0$.*

2. *If $\displaystyle\sum_{k=0}^{n} a_k = 0$ and $m = \min\left\{ i \in \{1, 2, \ldots, n\} : \sum_{k=1}^{n} k^i a_k \neq 0 \right\}$, then f is a generalized polynomial function of degree at most $m - 1$.*

Proof. Define $s_0 = \displaystyle\sum_{k=0}^{n} a_k$.

1. Assume that $s_0 \neq 0$. Setting $y = 0$ in (4.1), we get $s_0 f(x) = 0$. Since $s_0 \neq 0$, it follows that $f(x) = 0$ for all $x \in X$.
   On the other hand, it can be verified that $f(x) \equiv 0$ indeed satisfies (4.1).

2. Assume that $s_0 = 0$. Since f satisfies the general Fréchet functional equation of order n, by Theorem 4.1, we have

$$f(x) = A^0 + A^1(x) + \cdots + A^{n-1}(x) \tag{4.7}$$

For each $i \in \{1, 2, \ldots, n\}$, define $s_i = \displaystyle\sum_{k=1}^{n} k^i a_k$.
If $s_i = 0$ for all $i = 1, 2, \ldots, n$, then $a_1, a_2, \ldots, a_n$ satisfy the system of linear equations

$$\begin{aligned} a_1 + 2a_2 + \cdots + na_n &= 0 \\ a_1 + 2^2 a_2 + \cdots + n^2 a_n &= 0 \\ &\vdots \\ a_1 + 2^n a_2 + \cdots + n^n a_n &= 0. \end{aligned}$$

Therefore, $a_1 = a_2 = \cdots = a_n = 0$ which is a contradiction. So, there exists $i \in \{1, 2, \ldots, n\}$ such that $s_i \neq 0$. Thus, we let

$$m = \min\{i \in \{1, 2, \ldots, n\} : s_i \neq 0\}. \tag{4.8}$$

For every $i = 0, 1, 2, \ldots, n-1$ and for all $x, y \in X$, let

$$S_i(x, y) := \sum_{k=0}^{n} a_k A^i(x + ky). \tag{4.9}$$

Substituting f from (4.7) into (4.1), we obtain

$$\sum_{i=0}^{n-1} S_i(x, y) = 0. \tag{4.10}$$

For an arbitrary $r \in \mathbb{Q}$, observe that $S_i(rx, ry) = r^i S_i(x, y)$. Replacing (x, y) with (rx, ry) in (4.10), we obtain

$$\sum_{i=0}^{n-1} S_i(x, y) r^i = 0$$

for all rational numbers r. Thus, for each $i = 0, 1, 2, \ldots, n-1$

$$S_i(x, y) = 0. \tag{4.11}$$

By Lemma 2.10, we know that

$$A^i(x + ky) = \sum_{j=0}^{i} \binom{i}{j} A_i(\underbrace{x, \ldots, x}_{i-j}, \underbrace{ky, \ldots, ky}_{j}).$$

Since $A_i(\underbrace{x, \ldots, x}_{i-j}, \underbrace{ky, \ldots, ky}_{j}) = k^j A_i(\underbrace{x, \ldots, x}_{i-j}, \underbrace{y, \ldots, y}_{j})$, we get

$$A^i(x + ky) = \sum_{j=0}^{i} \binom{i}{j} k^j A_i(\underbrace{x, \ldots, x}_{i-j}, \underbrace{y, \ldots, y}_{j}). \tag{4.12}$$

From (4.9) and (4.11), we have

$$\sum_{k=0}^{n} a_k A^i(x + ky) = 0. \tag{4.13}$$

Replacing $A^i(x+ky)$ from (4.12) in (4.13), we get

$$\sum_{j=0}^{i}\binom{i}{j}s_jA_i(\underbrace{x,\ldots,x}_{i-j},\underbrace{y,\ldots,y}_{j})=0. \tag{4.14}$$

From the definition of m in (4.8) and $s_0=0$, we know that

$$s_0=s_1=s_2=\cdots=s_{m-1}=0.$$

Therefore, for each $i=m,m+1,\ldots,n-1$,

$$\sum_{j=m}^{i}\binom{i}{j}s_jA_i(\underbrace{x,\ldots,x}_{i-j},\underbrace{y,\ldots,y}_{j})=0. \tag{4.15}$$

Let r be a rational number. Replacing y by ry in (4.15), we obtain

$$\sum_{j=m}^{i}\left(\binom{i}{j}s_jA_i(\underbrace{x,\ldots,x}_{i-j},\underbrace{y,\ldots,y}_{j})\right)r^j=0.$$

for all rational numbers r. Hence, for each $j=m,m+1,\ldots,i$

$$\binom{i}{j}s_jA_i(\underbrace{x,\ldots,x}_{i-j},\underbrace{y,\ldots,y}_{j})=0. \tag{4.16}$$

Setting $y=x$ in (4.16) and dividing by $\binom{i}{j}$, we get $s_jA^i(x)=0$.
Taking $j=m$, we have $s_mA^i(x)=0$.
Since $s_m\neq 0$, we conclude that $A^i(x)=0$ for each $i=m,m+1,\ldots,n-1$.
Therefore, $f(x)=A^0+A^1(x)+A^2(x)+\cdots+A^{m-1}(x)$.
On the other hand, replacing $f(x)=A^0+\sum_{i=1}^{m-1}A^i(x)$ in (4.1), we obtain

$$\sum_{k=0}^{n}a_kf(x+ky)=\sum_{k=0}^{n}a_kA^0+\sum_{k=0}^{n}a_k\sum_{i=1}^{m-1}A^i(x+ky). \tag{4.17}$$

We know that

$$A^i(x+ky)=A^i(x)+\sum_{j=1}^{i}k^j\binom{i}{j}A_i(\underbrace{x,\ldots,x}_{i-j},\underbrace{y,\ldots,y}_{j}). \tag{4.18}$$

Substituting $A^i(x+ky)$ from (4.18) in (4.17), we have

$$\sum_{k=0}^{n} a_k f(x+ky) = s_0 A^0 + s_0 \sum_{i=1}^{m-1} A^i(x) + \sum_{i=1}^{m-1} \sum_{j=1}^{i} s_j \binom{i}{j} A_i(\underbrace{x,\dots,x}_{i-j},\underbrace{y,\dots,y}_{j}).$$

Since $s_0 = s_1 = s_2 = \cdots = s_{m-1} = 0$, we obtain

$$\sum_{k=0}^{n} a_k f(x+ky) = 0.$$

Hence, $f(x) = A^0 + A^1(x) + A^2(x) + \cdots + A^{m-1}(x)$ indeed satisfies (4.1).

□

We apply Theorem 4.2 to find a general solution of functional equations that illustrate in the following examples.

Example 4.3. We will prove Lemma 1.6 by using Theorem 4.2.
Replacing x by $x+3y$ in (1.12), we obtain

$$f(x)-14f(x+2y)+35f(x+3y)-35f(x+4y)+14f(x+5y)-f(x+7y) = 0. \quad (4.19)$$

The functional equation (4.19) is a general Fréchet functional equation of order 7 where $a_1 = 0, a_2 = -14, a_3 = 35, a_4 = -35, a_5 = 14, a_6 = 0$ and $a_7 = -1$.
We can see that $\sum_{k=0}^{7} a_k = 0$ and for each $i = 1, 2, 3, 4$

$$\sum_{k=1}^{7} k^i a_k = 0$$

and $\sum_{k=1}^{7} k^5 a_k = -840 \neq 0$. Therefore, by Theorem 4.2, we get

$$f(x) = A^0 + A^1(x) + A^2(x) + A^3(x) + A^4(x).$$

□

Example 4.4. Find the general solution of the functional equation (1.10) in Theorem 1.5 by using Theorem 4.2.

We can assume that k is a positive integer with $k \neq 1$.

Replacing x by $x + ky$ in (1.10), we have

$$f(x) - k^2 f(x + (k-1)y) - 2(1-k^2) f(x+ky) - k^2 f(x + (k+1)y) + f(x+2ky) = 0. \tag{4.20}$$

This functional equation is a general Fréchet functional equation of order $2k$ where $a_0 = 1, a_{k-1} = -k^2, a_k = -2(1-k^2), a_{k+1} = -k^2, a_{2k} = 1$ and $a_i = 0$ in otherwise. We can compute that $\sum_{i=0}^{2k} a_i = 0$ and for each $j = 1, 2, 3$

$$\sum_{i=1}^{2k} i^j a_i = 0$$

and $\sum_{i=1}^{2k} i^4 a_i = -2k^2 + 2k^4$. Since $k \neq 1$, we get $-2k^2 + 2k^4 \neq 0$. Therefore, by Theorem 4.2, we have

$$f(x) = A^0 + A^1(x) + A^2(x) + A^3(x).$$

□

REFERENCES

[1] Aczél, J.: *Lectures on Functional Equations and Their Applications*, Academic Press, 1992.

[2] Baker, J.A.: A general functional equation and its stability, *Proc. Amer. Math. Soc.* **133**, (2005), 1657-1664.

[3] Cauchy, A.L.: *Course d' analyse de l' Ecocle Polytechnique*, l. Analyse Algébrique, V. Paris, 1821.

[4] Chung, J.K. and Sahoo, P.K.: On the general solution of a quartic functional equation, *Bull. Korean Math. Soc.* **40**, (2003), 565-576.

[5] Czerwik, S.: *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.

[6] Djoković, D.Ž.: A representation theorem for $(X_1 - 1)(X_2 - 1) \cdots (X_n - 1)$ and its applications, *Ann. Polon. Math.* **22**, (1969), 189-198.

[7] Eshaghi Gordji, M. and Khodaei, H.: Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi Banach spaces, *Nonlinear Anal.* **71**, (2009), 5629-5643.

[8] Fréchet, M.: Une definition fonctionelle des polynomes, *Nouv. Ann. de Math.* **9**, (1909), 145-162.

[9] Hamel, G.: Einer basis aller Zahlen und die unstertigen Lösungen der Funktionalgleichung $f(x+y) = f(x) + f(y)$, *Math. Ann.* **60**, (1905),

459-472.

[10] Hewitt, E. and Zukerman, H.S.: Remarks on the Functional Equation $f(x+y) = f(x) + f(y)$, *Math. Mag.* **42**, (1970), 121-123.

[11] McKiernan, M.A.: On Vanishing nth ordered differences and Hamel Bases, *Ann. Polon. Math.* **19**, (1967), 331-336.

[12] Sahoo, P.K.: On a functional equaiton characterizing polynomials of degree three, *Bull. Inst. Math. Acad. Sin.* **32**, (2004), 35-44.

[13] Sahoo, P.K. and Kannappan, Pl.: *Introduction to Functional Equations*, CRC Press, 2011.

[14] Thanyacharoen, A.: A General Solution of a Generalized Quartic Functional Equation and a Sensitivity Result, Master's Thesis, Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, 2012.

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