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UPPER CRITICAL FIELD AND CRITICAL TEMPERATURE IN SUPERCONDUCTOR/FERROMAGNET SANDWICHES

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ทฤษฎีของระบบประกบตัวนำยิ่งยวด-แม่เหล็กเฟร์โรได้รับการพัฒนาโดยได้แสดงความสมมูลระหว่างสมการอนุพันธ์เชิงขนส่งแบบอูสาเดลและทฤษฎีของทากาฮาชิ–ทาชิกิที่มีพื้นฐานมาจากวิธีฟังก์ชันสหสมพันธ์ของเดอจอง นอกจากนี้ได้อนุพันธ์สมการอูสาเดลแบบทั่วไปที่รวมผลของการทำลายคู่อิเล็กตรอนได้แก่แม่เหล็กไดอะของวงโคจร สภาวะสปินแม่เหล็กพาราของพอลลี การกระเจิงเชิงสปินวงโคจร และการกระเจิงของสารเจือแม่เหล็กเป็นครั้งแรก ได้เสนอวิธีการแก้ปัญหาแบบหลายโมด อุณหภูมิวิกฤตและสนามแม่เหล็กวิกฤตตั้งฉากบนของระบบประกบสามชั้นแม่เหล็กเฟร์โร-ตัวนำยิ่งยวด-แม่เหล็กเฟร์โรถูกคำนวณเป็นฟังก์ชั่นของพารามิเตอร์วัสดุเช่นความหนาชั้นวัสดุ ระยะอาพันธ์และสภาพการนำไฟฟ้าในสถานะปกติ และการจัดเรียงตัวของโมเมนต์แม่เหล็กเฟร์โร ในที่นี้ได้คำนึงถึงความโปร่งใสของวัสดุที่รอยต่อระหว่างผิวและพลังงานแลกเปลี่ยนของแม่เหล็กเฟร์โรที่มีค่าใดๆทั้งแบบอ่อนและแบบแรง พบว่าการขึ้นกับความหนาของชั้นแม่เหล็กเฟร์โรที่มีต่ออุณหภูมิวิกฤตแสดงพฤติกรรมหลายแบบดังนี้ 1.การสลายตัวแบบเชิงเดี่ยวสภาพนำยิ่งยวดหายไปอย่างสมบูรณ์ที่ค่าเฉพาะหนึ่งของความหนาชั้นแม่เหล็ก 2.การสลายตัวแบบไม่เป็นเชิงเดี่ยวสภาพนำยิ่งยวดมีค่าตลอดความหนาชั้นแม่เหล็กโดยที่อุณหภูมิวิกฤตมีค่าต่ำสุดที่ค่าเฉพาะหนึ่งของความหนาชั้นแม่เหล็ก 3.สภาพนำยิ่งยวดแบบเกิดใหม่อุณหภูมิวิกฤตตกลงเป็นศูนย์ที่ค่าเฉพาะค่าหนึ่งของชั้นความหนาแม่เหล็กและคืนค่าอีกครั้งโดยเกิดเฉพาะในการจัดเรียงตัวของโมเมนต์แม่เหล็กเฟร์โรแบบขนาน ผลเชิงตัวเลขแสดงให้เห็นว่าสัณฐานแบบขนานสวนเสริมค่าสภาพนำยิ่งยวดมากกว่าสัณฐานแบบขนาน ผลการศึกษาสนามแม่เหล็กวิกฤตตั้งฉากบนแสดงให้เห็นว่าค่าความต้านทานต่ำที่รอยต่อระหว่างผิวมีผลเฉพาะสัณฐานแบบขนานสวนเนื่องจากลักษณะการกดข่มอย่างอ่อนของสนามแลกเปลี่ยนแบบเฟร์โรแมกนิติก

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BOONLIT KRUNAVAKARN:UPPER CRITICAL FIELD AND CRITICAL TEMPERATURE IN SUPERCONDUCTOR/FERROMAGNET SANDWICHES. THESIS ADVISOR :ASSOCIATE PROFESSOR DR. WICHIT SRITRAKOOL, THESIS COADVISOR : PROFESSOR DR. SUTHAT YOKSAN, 111 pp.ISBN 974-17 -4082 -4.

A theory of proximity effect for superconductor/ferromagnet sandwiches is developed. We show the equivalence between the Usadel transport-like differential equations and the Takahashi-Tachiki theory, which is based on the de Gennes correlation function method. Moreover the generalized Usadel equations, including the pair breaking effects such as orbital diamagnetism, Pauli spin paramagnetism, spin-orbit scattering and magnetic impurity scattering, are obtained for the first time. The method for solving the problem in the multimode method is proposed. The critical temperature and the perpendicular upper critical magnetic field of ferromagnet/ superconductor/ ferromagnet trilayers are calculated as a function of material parameters, such as the layer thickness, the coherence length and the normal state conductivity, as well as the mutual orientation of ferromagnetic moments. Therefore we treat the role of finite transparency at boundary interfaces and the arbitrary exchange energy, which account for both weak and strong ferromagnets. It is found that the dependence of the critical temperature, T_c , over the ferromagnetic layer thickness, d_f , exhibits various types of $T_c(d_f)$ behavior; (i) the monotonic decay of T_c, superconductivity completely vanishes at a particular d_f, (ii) the nonmonotonic decay of T_c, superconductivity existing throughout the ferromagnet but T_c has a minimum value at a particular d_f, and (iii) the reentrant superconductivity occurs only in the parallel phase, T_c drops to zero at finite d_f and it restores again. Numerical results show that the antiparallel configuration enhances superconductivity than the parallel one. Investigations of the perpendicular upper critical magnetic field reveal that the low interface boundary resistance has influence only in the antiparallel phase due to the weak suppression character of the ferromagnetic exchange fields.

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Superconductors in a Magnetic Field

1.1 Introduction

The microscopic theory of superconductivity was established in 1957 by Bardeen, Cooper and Schrieffer (BCS). The condensed state occurs when electrons of opposite spin and momentum form singlet pairs, so called the Cooper pair. The phonon mediated interaction is responsible for attractive electrons. The crucial concept of the BCS theory is the presence of an instability of the normal state toward the formation of the Cooper pair. An infinite conductivity, perfect diamagnetism and existence of the energy gap are features of superconductivity. The Cooper pair is always assumed to be uniform in a bulk superconductor. Therefore the critical magnetic field is purely the response of the orbital electron, the effect of electron spins is completely neglected. This superconductor is named as type I. For the type II superconductor, the transition to the normal state with increasing field is of the second order phase transition which is characterized by the superconducting order parameter vanishing continuously at a field H_{c2} greater than the bulk thermodynamic critical field H_c.

The effect of a uniform static magnetic field on a superconductor particularly for the type II superconductor has attracted much attention recently. The coupling of electron spins with the Zeeman magnetic field can make a significant contribution to the upper critical field H_{c2} in a high field superconductor.

Furthermore, the possibility of inhomogeneous superconductivity with spatial oscillation of the order parameter was investigated by Fulde and Ferrell (1964) and Larkin and Ovchinnikov (1965), FFLO, who demonstrated that the pairing electrons with nonzero total momentum can occur when the conduction electrons have the spin energy splitting the Fermi surface due to the action of the Zeeman energy.

The external applied magnetic field can be idealized as an internal exchange field acting on the electron spins. The ferromagnetic alignment of paramagnetic impurities may be regarded as an external molecular field which produces the spin exchange field. The FFLO state provides the coexistence of superconductivity and magnetism. Nevertheless, the FFLO state was never observed in bulk materials due to the Clogston 's criterion (Clogston, 1962) which stated that the superconducting state at zero temperature is completely destroyed when the paramagnetic field $H_p(0)$ exceeds $\triangle(0)/\sqrt{2}\mu$, where $\triangle(0)$ is the zero temperature superconducting energy gap and μ the Bohr magneton. Although the FFLO state is not favorable in bulk superconductors, the ferromagnetic/superconductor proximity effect may be a good candidate because the Cooper pairs in a ferromagnetic layer behave like a damped oscillation which reveals the inhomogeneous superconducting state.

Chapter 1 is organized as follows, in §1.2 we review the superconductor in the presence of a magnetic field. It is argued that both the gap and the thermodynamic properties change drastically due to the magnetic field. §1.3 explains the inhomogeneous superconducting state where the possibility of the FFLO phase is discussed.

1.2 Paramagnetic superconductors

In this section we will examine the effect of spin paramagnetism on a superconducting state as well as the thermodynamic behavior by assuming that the magnetic field is uniform. The model Hamiltonian describes the conduction electrons interact with other electrons by means of the attractive interaction in which the uniform magnetic field acting on the electron spin.

$$H = \sum_{\vec{k}\sigma} \xi_{\vec{k}\sigma} a^{\dagger}_{\vec{k}\sigma} a_{\vec{k}\sigma} - V \sum_{\vec{k}\vec{k}'} a^{\dagger}_{\vec{k}\uparrow} a^{\dagger}_{-\vec{k}'\downarrow} a_{\vec{k}'\uparrow} a_{-\vec{k}\downarrow} \tag{1.1}$$

where $\xi_{\vec{k}\sigma} = \xi_{\vec{k}} - \sigma h$, $\xi_{\vec{k}}$ is the one-electron energy measured from the Fermi surface, $h = \mu_0 \mathbf{H}$ is the Zeeman energy represents the electron magnetic moment μ_0 coupled to the magnetic field $\mathbf{H}$. $a^{\dagger}_{\vec{k}\sigma}$ and $a_{\vec{k}\sigma}$ are the creation and the destruction operators of an electron state $(\vec{k}, \sigma)$. The interaction range is confined in the stripe of thickness $2\omega_D$ around the Fermi surface, ω_D is the maximum phonon (Debye) frequency. This model Hamiltonian neglects the vector potential involving the Meissner effect, this enables us to generalize the BCS treatment by taking into account only the effect of electron spins. As a result

$$H_{eff} = \sum_{\vec{k}\sigma} \xi_{\vec{k}\sigma} a^{\dagger}_{\vec{k}\sigma} a_{\vec{k}\sigma} + \sum_{\vec{k}} (\triangle a^{\dagger}_{\vec{k}\uparrow} a^{\dagger}_{-\vec{k}\downarrow} + h.c.) \tag{1.2}$$

with the order parameter

$$\triangle = -V \sum_{\vec{k}} \langle a_{-\vec{k}\downarrow} a_{\vec{k}\uparrow} \rangle \tag{1.3}$$

where $< .... >$ means the grand canonical average of any operator and $h.c.$ refers to the hermitian conjugate. The self-consistency condition for the order parameter $\triangle(T, h)$ is obtained by virtue of the Bogoliubov transformation and one has the result

$$\triangle = V \sum_{\vec{k}} \triangle \frac{1 - f(E_{\vec{k}\uparrow}) - f(E_{\vec{k}\downarrow})}{2E_{\vec{k}}} \tag{1.4}$$

with $E_{\vec{k}\sigma} = E_{\vec{k}} - \sigma h$, $E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + \triangle^2(T,h)}$ is the finite temperature superconducting order parameter in a magnetic field, and the Fermi distribution function $f(E) = 1/(1 + \exp E/T)$.

We pass from the sum over momentum space to the integral over energy and introduce the density of states at the Fermi level $N(0)$, (1.4) becomes

$$1 = N(0)V \int_0^{\omega_D} \frac{d\xi}{2\sqrt{\xi^2 + \triangle^2(T,h)}} [\tanh(\frac{\sqrt{\xi^2 + \triangle^2(T,h)} - h}{2T}) + \tanh(\frac{\sqrt{\xi^2 + \triangle^2(T,h)} + h}{2T})]. \quad (1.5)$$

We must solve (1.5) self-consistency to obtain the order parameter $\triangle(T,h)$. If we taking $h = 0$ in (1.5), the ordinary BCS gap equation is obtained as expected, further by observing that the replacement $h \to -h$ leaves $\triangle(T,h)$ invariant then we can write down (1.5) as

$$1 = N(0)V \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \triangle^2(T,h)}} \tanh(\frac{\sqrt{\xi^2 + \triangle^2(T,h)} - h}{2T}). \quad (1.6)$$

At $T = 0$ this equation reduces to

$$1 = N(0)V \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \triangle^2(0,h)}} \theta(\sqrt{\xi^2 + \triangle^2(0,h)} - h), \quad (1.7)$$

where the step function $\theta(x) = 1$ when $x > 1$ and $\theta(x) = 0$ when $x < 1$. The argument of the step function must satisfies the inequality

$$\xi > \sqrt{h^2 - \triangle^2(0,h).} \quad (1.8)$$

This inequality shows that the conduction electron energy is bounded only when $h > \triangle(0,h)$ while in the opposite case i.e., when $h < \triangle(0,h)$, (1.7) has no solution so the possible one is $\triangle(0,h) = \triangle(0,0)$. We now determine the solution of (1.7) in the case $h > \triangle(0,h)$ by using (1.8)

$$\frac{1}{N(0)V} = \int_{\sqrt{h^2-\triangle^2(0,h)}}^{\omega_D} \frac{d\xi}{\sqrt{\xi^2+\triangle^2(0,h)}}. \tag{1.9}$$

Performing the integral and eliminating the coupling constant we obtain

$$\sinh^{-1}\frac{\omega_D}{\triangle(0,0)} = \sinh^{-1}\frac{\omega_D}{\triangle(0,h)} - \cosh^{-1}\frac{h}{\triangle(0,h)}. \tag{1.10}$$

Eq.(1.10) in the weak-coupling approximation yields the zero temperature solution

$$\triangle(0,h) = \triangle(0,0)\sqrt{\frac{2h}{\triangle(0,0)} - 1}, \tag{1.11}$$

which is the result first obtained by Sarma (1963), the range of the Sarma's gap energy lie in the intervals $2h/\triangle(0,0) - 1 > 0$ and $h > \triangle(0,h)$ together with the fact that the Zeeman energy tends to break electron pairs i.e., $\triangle(0,h) < \triangle(0,0)$. Combining these inequalities we thus get the confined region of (2.11)

$$\frac{1}{2}\triangle(0,0) < h < \triangle(0,0). \tag{1.12}$$

To investigate the stability of the superconducting state we employ the Abrikosov's formula (1988) of the free energy difference between the superconductive and the normal phases

$$F_s(h) - F_n(h) = \int_0^{\triangle(0,h)} \triangle^2 d\frac{1}{V}. \tag{1.13}$$

For the BCS superconductor we have the gap equation in the weak-coupling limit

$$1 = N(0)V\ln\frac{2\omega_D}{\triangle(0,0)},$$

with $\triangle(0,h) = \triangle(0,0)$, after inserting the BCS gap equation into the equation (1.13), the calculated result is

$$F_s(h) - F_n(h) = -\frac{1}{2}N(0)\triangle^2(0,0) = F_s(0) - F_n(0). \tag{1.14}$$

This result reveals that the BCS free energy is independent of the magnetic field, however, the energy of the normal state paramagnetism is given by

$$F_n(h) - F_n(0) = -N(0)h^2. \tag{1.15}$$

The free energy difference should be rewritten as

$$F_s(h) - F_n(h) = [F_s(h) - F_s(0)] + [F_s(0) - F_n(0)] + [F_n(0) - F_n(h)]. \tag{1.16}$$

Combining (1.14)-(1.16) together with $F_s(h) = F_s(0)$ for the superconducting BCS state (the perfect Meissner effect) we get

$$F_s(h) - F_n(h) = -N(0)(\frac{1}{2}\triangle^2(0,0) - h^2). \tag{1.17}$$

Because of the condition $h < \triangle(0,0)$ for the BCS superconductor this obtained result indicates that $F_s(h)$ is smaller than $F_n(h)$ at low h. As increasing h the free energy difference, (1.17), increases gradually. There will be the critical field $h = h_c$ in which $F_s(h) = F_n(h)$ and for the field below the critical value $h < h_c$ there is a stable superconducting state . At the critical field h_c, (1.17) implies

$$h_c = \frac{1}{\sqrt{2}}\triangle(0,0), \tag{1.18}$$

which is the result first obtained by Clogton(1962).

For the spin polarized superconductor we have the weak-coupling solution of (1.9)

$$1 = N(0)V[\ln\frac{2\omega_D}{\triangle(0,h)} - \cosh^{-1}\frac{h}{\triangle(0,h)}],$$

we calculate (1.13) by using this expression the result is

$$F_s(h) - F_n(h) = N(0)[-\frac{1}{2}\triangle^2(0,h) + h(h - \sqrt{h^2 - \triangle^2(0,h)})],$$

By substituting the paramagnetic superconducting gap solution of Sarma,(1.11), into the above equation, we obtain

$$F_s(h) - F_n(h) = \frac{1}{2}N(0)(\triangle(0,0) - 2h)^2, \tag{1.19}$$

in the interval $1/2 < h/\triangle(0,0) < 1$, this result indicates that $F_s(h)$ is greater than $F_n(h)$. As h increases from the lower limit the free energy difference being initially zero and then increases continuously until they reach the upper field limit. So when comparing their energies,(1.14) and (1.15), we find the spin polarized superconductive state places above the BCS state. Thus we conclude that the formation of the spin polarized state is unstable towards the BCS state.

We consider now the situation when $T \neq 0$ the finite temperature gap equation (1.6) can be rewritten in a more convenient form as

$$\ln\frac{\triangle(T,h)}{\triangle(0,0)} = \int_0^{\omega_D}\frac{d\xi}{\sqrt{\xi^2 + \triangle^2(T,h)}}[\tanh\frac{\sqrt{\xi^2 + \triangle^2(T,h)} - h}{2T} - 1]. \tag{1.20}$$

If we take $h = 0$, this equation reduces to the usual BCS expression of the BCS theory. In the low temperature region, $T \ll T_c$, (1.20) is expressed as

$$\ln\frac{\triangle(T,h)}{\triangle(0,0)} = \int_0^{\omega_D}\frac{d\xi}{\sqrt{\xi^2 + \triangle^2(T,h)}}[\theta(\sqrt{\xi^2 + \triangle^2(T,h)} - h) - 1$$

$$+2\sum_{n=1}^{\infty}(-1)^n \exp\frac{n(h - \sqrt{\xi^2 + \triangle^2(T,h)})}{T}. \tag{1.21}$$

It follows from the argument of the step function that for $h < \triangle(T,h)$ we obtain the solution to the BCS superconducting state

$$\ln\frac{\triangle(T,h)}{\triangle(0,0)} = 2\sum_{n=1}^{\infty}(-1)^n \exp\frac{nh}{T}K_0(\frac{n\triangle(T,h)}{T}), \tag{1.22}$$

where $K_0(x)$ is the zeroth order Bessel function, while in the case $h > \triangle(T,h)$ we have the gap in the spin polarized superconducting state

$$\ln\frac{\triangle(T,h)}{\triangle(0,0)} = -\cosh^{-1}\frac{h}{\triangle(T,h)} + 2\sum_{n=1}^{\infty}\exp\frac{nh}{T}K_0(\frac{n\triangle(T,h)}{T}). \tag{1.23}$$

Both solutions reduce to the zero temperature gap energy when $T = 0$.

In the vicinity of the critical temperature $T < T_c$ we express (1.6) in an alternative form by using the series representation of the hyperbolic tangent

$$\tanh\frac{x}{2T} = 2T\sum_{n=-\infty}^{\infty}\frac{1}{x \pm i\omega_n},$$

with $\omega_n = (2n+1)\pi T$, then (1.6) becomes

$$\frac{1}{N(0)V} = 4T\sum_{n=0}^{\infty}\int_0^{\omega_D}\frac{d\xi}{\xi^2 + \triangle^2(T,h) + (\omega_n + ih)^2}. \tag{1.24}$$

We do further by expanding the denominator of the integrand in power series of $\triangle(T,h)$ and integrating over the energy in the weak-coupling limit $\omega_D/T_c \to \infty$ to obtain

$$\ln\frac{T}{T_{c0}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + i\rho) - \frac{1}{2}f_1(\rho)(\frac{\triangle(T,h)}{2\pi T})^2 + \frac{3}{8}f_2(\rho)(\frac{\triangle(T,h)}{2\pi T})^4 + ... \tag{1.25}$$

where T_{c0} is the transition temperature at $h = 0, \rho = h/2\pi T, \psi(z)$ is the digamma function

$$\psi(z) = \sum_{n=0}^{\infty}(\frac{1}{n+1} - \frac{1}{n+z}) - 0.5772..$$

and

$$f_1(z) = Re\sum_{n=0}^{\infty}\frac{1}{(n + \frac{1}{2} + iz)^3},$$

and

$$f_2(z) = Re\sum_{n=0}^{\infty}\frac{1}{(n + \frac{1}{2} + iz)^5}.$$

In (1.25) the small $\triangle$ has been assumed while the parameter ρ is arbitrary. We consider the two limiting cases: the high field limit $\rho \gg 1$ using the asymptotic representation of the digamma function,$|z| \gg 1$,

$$\psi(z) = \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + ..., \tag{1.26}$$

and taking $\triangle = 0$ in (1.25), the critical temperature can be approximated as

$$T_c^2 = \frac{2h^2}{\pi^2} \ln \frac{2h}{\triangle(0,0)}, \tag{1.27}$$

where the BCS universal ratio $\triangle(0,0) = 1.65T_{c0}$ has been used. In the case of low field limit $\rho \ll 1$, (1.25) with $\triangle = 0$ implies

$$\frac{T_c}{T_{c0}} = 1 - 7\zeta(3)(\frac{h}{2\pi T_{c0}})^2, \tag{1.28}$$

in which $\zeta(z)$ is the Riemann-zeta function.

We will calculate the free energy difference between the superconducting and normal phases according to the formula (Maki and Tsuneto, 1964)

$$F_s(T,h) - F_n(T,h) = \int_0^{\triangle(T,h)} \triangle^2 d(\frac{1}{V}). \tag{1.29}$$

Near the critical temperature we have

$$d(\frac{1}{V}) = -N(0)[f_1(\rho)\frac{\triangle}{(2\pi T)^2} - \frac{3}{2}f_2(\rho)\frac{\triangle^3}{(2\pi T)^4}]d\triangle, \tag{1.30}$$

and for the free energy

$$F_s(T,h) - F_n(T,h) = -\frac{1}{4}N(0)\triangle^2[f_1(\rho)(\frac{\triangle}{2\pi T})^2 - f_2(\rho)(\frac{\triangle}{2\pi T})^4]. \tag{1.31}$$

This result shows that $F_s(T,h)$ is smaller than $F_n(T,h)$. As h and T are increased to the critical value $h = h_c$ and $T = T_c$, we have the condition $F_s(T,h) = F_n(T,h)$, which form the critical line. When $\triangle = 0$, the transition is of the the second order while $\triangle \neq 0$ indicates the first order phase transition and is determined by

$$\triangle^2 = \frac{f_1(\rho)}{f_2(\rho)}(2\pi T)^2. \tag{1.32}$$

The critical point in which the type of phase transition changes is characterized by the root of the equation

$$f_1(\rho_0) = 0, \tag{1.33}$$

which gives $\rho_0 = 0.308$ and corresponds to $t = T_c/T_{c0} = 0.566$, the function $f_1(\rho)$ is a monotonously decreasing function of ρ, then for $\rho < \rho_0$ or $t > 0.566$, $f_1(\rho)$ is positive and the phase transition is of the second order while for $\rho > \rho_0$ or $t < 0.566$, $f_1(\rho)$ is negative and the phase transition is of the first order.

1.3 The Fulde-Ferrell-Larkin-Ovchinnikov state

In the previous section we have considered the superconductor under the action of a magnetic field within the assumption that the pair electrons have the net zero momentum, however there is some possibility of the oscillatory pair wave function in which the pair momentum is non-zero, this state has been studied by Fulde and Ferrell (1964) and by Larkin and Ovchinnikov (1965). We follow the work of Takada and Izuyama (1969). The effective BCS Hamiltonian of the electron in the presence of a magnetic field is

$$H_{eff} = \sum_{\vec{k}\sigma} \xi_\sigma(\vec{k}) a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} + \sum_{\vec{k}} [\triangle_q a^\dagger_{\vec{k}+\vec{q}/2,\uparrow} a^\dagger_{-\vec{k}+\vec{q}/2,\downarrow} + h.c.], \tag{1.34}$$

with

$$\xi_\sigma(\vec{k}) = \xi(\vec{k}) - \sigma h, \tag{1.35}$$

where $\xi(\vec{k})$ is the conduction electron energy measured from the Fermi surface and $\sigma = 1(-1)$ for up(down) electron spins. The magnetic energy h acting on the electron spins can be regarded as a ferromagnetic impurity which produces the molecular field in the superconducting state which is the coexistence problem of superconductivity and ferromagnetism, in this circumstance, h is modelled as

$$h = n_i J < S_z >,$$

where n_i is the number of impurity atoms per unit volume, J is the strength of impurity spin S and $< S_z >$ denotes the thermal average of the impurity spin along the z-axis. The inhomogeneous superconducting order parameter is defined as

$$\triangle_q = -V \sum_{\vec{k}} < a_{-\vec{k}+\vec{q}/2,\downarrow} a_{\vec{k}+\vec{q}/2,\uparrow} >, \tag{1.36}$$

when $q = 0$ is taken, (1.36) reduces to the BCS order parameter with zero pair momentum. The superconducting state can be described by the Gorkov's equations

$$[i\omega_n - \xi_\sigma(\sigma\vec{k} + \vec{q}/2)]G_\sigma(\vec{k}, i\omega_n) + \triangle_q F_\sigma^\dagger(\vec{k}, i\omega_n) = 1,$$
$$[i\omega_n - \xi_{-\sigma}(-\sigma\vec{k} + \vec{q}/2)]F_\sigma^\dagger(\vec{k}, i\omega_n) + \triangle_q^* G_\sigma(\vec{k}, i\omega_n) = 0. \tag{1.37}$$

We will assume the wave vector $|\vec{q}|$ is small when compared with the Fermi wave vector k_F since $|\vec{q}|$ represents the displaced Fermi surfaces of up-and -down spin electrons due to the Zeeman energy h according to the relation

$$k_{F\uparrow} - k_{F\downarrow} = \frac{h}{\mu} k_F, \tag{1.38}$$

here μ is the chemical potential, so we define the dimensionless quantity, $\bar{q}$,

$$q = \bar{q}(k_{F\uparrow} - k_{F\downarrow}). \tag{1.39}$$

Combining both equations we have $\bar{q} = v_F q/2h$. We linearize the electron energy near the Fermi surface and neglected the q^2 term

$$\xi_\sigma(\sigma\vec{k} + \vec{q}/2) = \xi(\vec{k}) + \sigma(\bar{q}x - 1)h, \tag{1.40}$$

where $x = \cos\theta$ is the cosine of the angle between vector $\vec{k}$ and $\vec{q}$. Solving the Gorkov's equations,(1.37), we obtain

$$G_\sigma(\vec{k}, i\omega_n) = \frac{i\omega_n + \xi(\vec{k}) - \sigma(\bar{q}x - 1)h}{(i\omega_n - E_\sigma(\vec{k}))(i\omega_n + E_{-\sigma}(\vec{k}))},$$

$$F^\dagger_\sigma(\vec{k}, i\omega_n) = -\frac{\triangle^*_q}{(i\omega_n - E_\sigma(\vec{k}))(i\omega_n + E_{-\sigma}(\vec{k}))}, \tag{1.41}$$

where the excitation spectrum of the superconductive FFLO state is defined by

$$E_\sigma(\vec{k}) = \sigma(\bar{q}x - 1)h + \sqrt{\xi^2(\vec{k}) + \triangle^2_q}. \tag{1.42}$$

The FFLO gap equation (1.36) can be expressed in terms of the anomalous Green's function as

$$\triangle^*_q = VT \sum_{\vec{k},\omega_n} F^\dagger_\uparrow(\vec{k}, i\omega_n). \tag{1.43}$$

Inserting (1.41) into (1.43) and performing the sum over the fermionic Matsubara's frequency by means of the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} F(i\omega_n) = \frac{1}{2T} \oint_C \frac{d\omega}{2\pi i} F(\omega) \tanh\frac{\omega}{2T},$$

where the contour C does not enclose any singularities in the imaginary axis of the complex $\omega-$ plane, as the result one arrives at the result

$$1 = V \sum_{\vec{k}} \frac{\tanh(\frac{E_\uparrow(\vec{k})}{2T}) + \tanh(\frac{E_\downarrow(\vec{k})}{2T})}{4\sqrt{\xi^2(\vec{k}) + \triangle^2_q}}. \tag{1.44}$$

In the absence of the Zeeman energy $h = 0$, (1.44) becomes the self-consistent equation of the BCS superconductor. From the excitation energy in the superconducting state (1.42), we see that

$$E_\uparrow(\vec{k}) + E_\downarrow(\vec{k}) = 2\sqrt{\xi^2(\vec{k}) + \triangle^2_q} > 0.$$

However, we may have $E_\sigma(\vec{k}) < 0$ for some value of $\vec{k}$ which is another type of the superconducting state, such a state is called the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state. For the BCS superconducting state, both $E_\uparrow(\vec{k})$ and $E_\downarrow(\vec{k})$ are greater than zero, this means that the electron state $(\vec{k}+\vec{q}/2, \uparrow)$ and $(-\vec{k}+\vec{q}/2, \downarrow)$ are occupied in pairs. For the FFLO superconducting state, there is some value of $\vec{k}$ which satisfies the condition which is the depairing zone, i.e, the Cooper pairs are broken. So the electron state $(\sigma\vec{k} + \vec{q}/2, \sigma)$ is blocked while $(-\sigma\vec{k} + \vec{q}/2, -\sigma)$ a vacant one. This means that the unoccupied electron states must obey the inequality $\xi_{-\sigma}(-\sigma\vec{k} + \vec{q}/2) > \triangle_q$ at the Fermi surface e.g., we have

$$-\sigma(\bar{q}x - 1)h > \triangle_q. \tag{1.45}$$

The FFLO state will be analyzed for two values of $\bar{q}$, i.e., $0 < \bar{q} < 1$, and $\bar{q} > 1$.

In the first case $0 < \bar{q} < 1$, $E_\downarrow(\vec{k})$ is positive definite while $E_\uparrow(\vec{k})$ may be less than zero for some value of $\vec{k}$. The inequality (1.45) of the down spin electron, $\sigma = -1$, implies the condition

$$h > \frac{\triangle_q}{1 - \bar{q}}. \tag{1.46}$$

The blocking region for $E_\uparrow(\vec{k}) < 0$ is determined to be

$$\begin{aligned} -1 \le x \le \phi^-(\xi), \\ |\xi| \le (1 - \bar{q})hx_2, \end{aligned} \tag{1.47}$$

where

$$x_{1,2} = \sqrt{1 - [\frac{\triangle_q}{(\bar{q} \pm 1)h}]^2},$$

$$\phi^\pm(\xi) = \frac{1}{\bar{q}h}(h \pm \sqrt{\xi^2(\vec{k}) + \triangle_q^2}). \tag{1.48}$$

We note that the condition (1.46) and the blocked region (1.47) differ from the results obtained by Takada and Izuyama. They have used only the condition of the unoccupied up - spin electron state, $h > \triangle_q/(1+\bar{q})$, which is incorrect.

In the latter case: $\bar{q} > 1$ both $E_\uparrow(\vec{k})$ and $E_\downarrow(\vec{k})$ are less than zero for some values of $\vec{k}$. The blocking region of the up-spin electron is found to be

$$h > \frac{\triangle_q}{\bar{q}+1},$$

$$-1 \le x \le \phi^-(\xi),$$

$$|\xi(\vec{k})| \le (\bar{q}+1)hx_1, \tag{1.49}$$

and for the down- spin electron

$$h > \frac{\triangle_q}{\bar{q}-1},$$

$$\phi^+(\xi) \le x \le 1,$$

$$|\xi(\vec{k})| \le (\bar{q}-1)hx_2. \tag{1.50}$$

The parameter $x_{1,2}$,(1.48) measuring the blocking and the ranges are taken between zero and unity, when $x_1 = x_2 = 0$ we get the BCS state while $x_1 = x_2 = 1$ the normal state is recovered.

Let us calculate the gap energy at $T = 0$, we have from (1.44)

$$1 = V(\sum_{\vec{k}} - \sum_{E_\sigma(\vec{k})<0})\frac{1}{2\sqrt{\xi^2(\vec{k}) + \triangle_q^2}}, \tag{1.51}$$

here $\triangle_q = \triangle_q(T, h)$. The effect of blocking is expressed in the second term of the right side of (1.51) in which the unpaired electron will reduce the energy gap. In the weak coupling limit, (1.51) is transformed to be

$$\ln\frac{\triangle_q}{\triangle_0} = -\frac{1}{2N(0)}\sum_{E_\sigma(\vec{k})<0}\frac{1}{\sqrt{\xi^2(\vec{k}) + \triangle_q^2}}, \tag{1.52}$$

where $\triangle_0 = \triangle_0(0,0)$ is the BCS energy gap

$$\triangle_0 = 2\omega_D \exp(-\frac{1}{N(0)V}),$$

$N(0)$ is the density of states at the Fermi level. Performing the right side summation of (1.52) and using the blocked regions, (1.49) and (1.50), the results are

$$\ln\frac{\triangle_q}{\triangle_0} = \frac{1+\bar{q}}{4\bar{q}}[\ln\frac{1-x_2}{1+x_2} + 2\frac{1-\bar{q}}{1+\bar{q}}x_2], \quad (1.53)$$

for $0 < \bar{q} < 1$, and

$$\ln\frac{\triangle_q}{\triangle_0} = \frac{\bar{q}+1}{4\bar{q}}[\ln\frac{1-x_1}{1+x_1} + 2x_1] + \frac{\bar{q}-1}{4\bar{q}}[\ln\frac{1-x_2}{1+x_2} + 2x_2], \quad (1.54)$$

for $\bar{q} > 1$. We note that Takada and Izuyama have obtained only (1.54) for all values of $\bar{q}$. The equations (1.53)and (1.54), are complicated equations because $x_{1,2}$, (1.48), also contains $\triangle_q$.

It can be shown later that the FFLO state is stable only when $\bar{q} > 1$. We will assume $\bar{q} > 1$ and consider the two limiting cases, the BCS limit (no blocking) and the normal limit (perfect blocking).

BCS limit: In this case $x_1 = x_2 = 0$, (1.54) implies $\triangle_q = \triangle_0$ for $h \leq h_B$ where the critical field defined by

$$\frac{h_B}{\triangle_0} = \frac{1}{\bar{q}+1}, \quad (1.55)$$

when $h > h_B$ is the necessary condition for the FFLO state.

Normal limit: We expand $x_{1,2}$ in powers of $(\triangle_q/h)^2$, retaining to the term $(\triangle_q/h)^4$ and neglected the higher order terms, substituting these expansions into (1.54), we obtain

$$\triangle_q^2 = 4(\bar{q}^2-1)h^2 f(\bar{q},h), \quad (1.56)$$

with

$$f(\bar{q},h) = 1 - \ln\frac{2h}{\triangle_0} - \frac{1}{2}\ln(\bar{q}^2-1) - \frac{1}{2\bar{q}}\ln\frac{\bar{q}+1}{\bar{q}-1}. \tag{1.57}$$

In this limit there has the critical field h_N defined by

$$\triangle_q = \triangle_q\theta(h_N - h), \tag{1.58}$$

with fixed value of $\bar{q}$. Then (1.56) gives

$$f(\bar{q},h_N) = 0.$$

Solving this equation we get the normal critical field

$$\frac{h_N}{\triangle_0} = \frac{e}{2(\bar{q}+1)}(\frac{\bar{q}+1}{\bar{q}-1})^{\frac{\bar{q}-1}{2\bar{q}}}. \tag{1.59}$$

The FFLO state is confined in the range $h_B < h < h_N$ for fixed value of $\bar{q}$. The second order phase transition from the normal to FFLO states can occur at the critical value of $\bar{q} = \bar{q}_c$ at which the normal critical field is maximized, $h_c = h_N(\bar{q}_c)$. We determine $\bar{q}_c$ by differentiating (1.59)

$$\frac{d}{d\bar{q}}\ln h_N(\bar{q}_c) = 0,$$

which gives

$$2\bar{q}_c = \ln\frac{\bar{q}_c+1}{\bar{q}_c-1}.$$

This equation gives the numerical value $\bar{q}_c = 1.2$, and consequently, we get $h_c = 0.75\triangle_0$.

In order to calculate the ground state energy of the FFLO phase we denote $\triangle E =$ (Energy of FFLO state) -(Energy of the normal state), by means of the variation of the thermodynamic potential, we have

$$\triangle E = \int_0^{\triangle_q} \triangle_q^2 d\frac{1}{V},$$

by using (1.54)

$$d(\frac{1}{V}) = -N(0)[\frac{1}{\triangle_q} + \frac{\bar{q}+1}{2\bar{q}}\frac{x_1^2}{1-x_1^2}\frac{dx_1}{d\triangle_q} + \frac{\bar{q}-1}{2\bar{q}}\frac{x_2^2}{1-x_2^2}\frac{dx_2}{d\triangle_q}]d\triangle_q.$$

The straightforward calculation yields

$$\frac{\triangle E}{N(0)} = h^2 + (\frac{h^2\bar{q}^2}{3} - \frac{1}{2}\triangle_q^2) - \frac{h^2}{6\bar{q}}[(\bar{q}+1)^3 x_1^3 + (\bar{q}-1)^3 x_2^3]. \quad (1.60)$$

Let us examine (1.60) in some limiting cases.

For the perfectly paired state we put $x_1 = x_2 = 0$ with $\triangle_q = \triangle_0$ then

$$\triangle E = N(0)[h^2 + \frac{1}{3}(h\bar{q})^2 - \frac{1}{2}\triangle_0^2], \quad (1.61)$$

where the first and second and third terms represent the Zeeman energy of the normal state, the kinetic energy of Cooper pairs and the formation energy of the superconductive state, respectively.

For the normal limit, since $\triangle_q/h \ll 1$ in expanding x_1 and x_2, we are retaining to the term $(\triangle_q/h)^4$. Upon substituting these expansions into (1.60), we obtain

$$\triangle E = -\frac{N(0)h^2}{8(\bar{q}^2-1)}(\frac{\triangle_q}{h})^4. \quad (1.62)$$

This formula clearly indicates that only for $\bar{q} > 1$ is necessary to establish the superconductive state.

Near the critical point it is worth determining the transition temperature and the free energy difference between the FF state and the normal one, the gap equation (1.43) can be written as

$$1 = VT\sum_{\vec{k},\omega_n}\frac{1}{[\omega_n + i(\bar{q}x-1)h]^2 + \xi^2(\vec{k}) + \triangle_q^2}, \quad (1.63)$$

where $\omega_n = (2n+1)\pi T$ and the sum is taken for all value of n. Near the critical temperature, (1.63) is approximated as

$$1 = VT \sum_{\vec{k},\omega_n} [\frac{1}{[\omega_n + i(\bar{q}x-1)h]^2 + \xi^2(\vec{k})} - \frac{\triangle_q^2}{([\omega_n + i(\bar{q}x-1)h]^2 + \xi^2(\vec{k}))^2}]. \quad (1.64)$$

Performing the calculation yields

$$\ln\frac{T}{T_{c0}} = g_0(\bar{q},h,T) + g_1(\bar{q},h,T)(\frac{\triangle_q}{2\pi T})^2, \quad (1.65)$$

where

$$g_0(\bar{q},h,T) = -\frac{1}{2}\int_{-1}^{1} dx Re[\psi(\frac{1}{2}+i\rho(x)) - \psi(\frac{1}{2})],$$
$$g_0(\bar{q},h,T) = \psi(\frac{1}{2}) - \frac{T\pi}{h\bar{q}} Im[\ln(\frac{1}{2}+i\rho(1)) - \ln(\frac{1}{2}+i\rho(-1))], \quad (1.66)$$

and

$$g_1(\bar{q},h,T) = -\frac{1}{2}\int_{-1}^{1} dx Re[\psi(\frac{1}{2}+i\rho(x)) - \psi(\frac{1}{2})],$$
$$g_1(\bar{q},h,T) = \frac{T\pi}{4h\bar{q}} Im[\psi'(\frac{1}{2}+i\rho(1)) - \psi'(\frac{1}{2}+i\rho(-1))]. \quad (1.67)$$

Therefore $\psi(z)$ is the digamma function, and the parameter

$$\rho(x) = \frac{h(\bar{q}x-1)}{2\pi T}, \quad (1.68)$$

and T_{c0} is the BCS transition temperature. By taking $\triangle_q = 0$ in (1.65) we get the transition temperature equation

$$\ln\frac{T}{T_{c0}} = g_0(\bar{q},h,T). \quad (1.69)$$

We are interested in the high field limit $h \gg T$, the asymptotic form of $g_0(\bar{q},h,T)$ is represented as

$$g_0(\bar{q},h,T) = \ln\frac{T}{T_{c0}} + f(\bar{q},h) - \frac{1}{8(\bar{q}^2-1)}(\frac{2\pi T}{h})^2, \quad (1.70)$$

where $f(\bar{q}, h)$ has been already stated in (1.57). Thus in this limit we arrive at the relation

$$T^2 = \frac{2}{\pi^2}(\bar{q}^2 - 1)h^2 f(\bar{q}, h). \tag{1.71}$$

Next let us evaluate the free energy difference near T_c. The usual expression for the free energy difference between the FF and normal state is given by

$$\triangle F = \int_0^{\triangle_q} \triangle_q^2 d(\frac{1}{V}),$$

with the derivation of (1.65)

$$d(\frac{1}{V}) = \frac{2N(0)}{(2\pi T)^2} g_1(\bar{q}, h, T)\triangle_q d\triangle_q.$$

The result is thus

$$\triangle F = \frac{N(0)}{2(2\pi T)^2}\triangle_q^4. \tag{1.72}$$

In the high field limit $h \gg T$, employing the asymptotic behavior of $g_1(\bar{q}, h, T)$, we have

$$g_1(\bar{q}, h, T) = -\frac{4}{(\bar{q}^2 - 1)}(\frac{\pi T}{h})^2. \tag{1.73}$$

The expression (1.72) becomes

$$\triangle F = -\frac{N(0)h^2}{2(\bar{q}^2 - 1)}(\frac{\triangle_q}{h})^4. \tag{1.74}$$

This expression shows that we cannot obtain the stable FFLO state with $\bar{q} < 1$.

In this section, the theoretical investigation of the FFLO state is presented. The properties such as the energy gap and the thermodynamic potential at zero temperature as well as near the critical temperature T_c are shown. Although, there is a possibility in the theoretical model but it has never been detected yet.

Chapter 2
Proximity Effect Theories

2.1 Introduction

The proximity effect was proposed by de Gennes(1964) and applied to the layered structures of superconducting (S) and normal-metal (N) materials. In his work the S/N sandwiches in which the normal metals having a smaller critical temperature were considered. The boundary conditions were derived and the critical temperature in the Cooper-de Gennes limit for thin layers was determined.

The de Gennes theory was later developed by Takahashi and Tachiki (1986) for multilayers systems. In their formalism the method of the eigenfunction expansion and the de Gennes boundary conditions were used to calculate the upper critical fields both in the perpendicular and parallel directions. Auvil and Ketterson (1988) used this theory to show that the Cooper-de Gennes limit for the critical temperature in the limit of small layer thickness corresponds to the diagonal approximation method, the critical temperature was calculated exactly by Jin and Ketterson (1989) whereas Takanaka (1991) numerically calculated the transition temperature and the upper critical fields for the S/N superlattice and found the qualitative agreement between the exact result and the approximated one of Takahashi and Tachiki. Auvil, Ketterson and Song (1989) have generalized the Takahashi and Tachiki theory of dirty coupled superconductors to include the effects of orbital diamagnetism, Pauli spin paramagnetism, spin-orbit scattering and magnetic impurity scattering. Koperdraad and Lodder (1995, 1996) attempted to analyze the experimental data of the upper critical fields for the metallic multilayers system within the concept of the unrealistic magnetic coherence length scaling factor and fitting to the theory of Takahashi and Tachiki. Kuboya and Takanaka (1998) studied theoretically the temperature dependence of the upper critical fields, perpendicular and parallel to the layers for dirty superconducting (S)/ferromagnetic (F) superlattices. An interesting feature of the S/F system is the existence of the π -phase state in which the phase shift of the superconducting order parameter between adjacent layers is π. The superconducting transition temperature of the superlattice as a function of the ferromagnetic layers thickness shows an oscillatory behavior under appropriate conditions which is due to the π-phase state.

Another theory of proximity effect is the Usadel's dirty-limit version of the quasiclassical theory of superconductivity (Usadel, 1970). This theory requires a zero critical temperature for the non-superconducting material and only the perpendicular upper critical field was calculated. The Usadel equations were solved exactly by Radovic et al. (1991 a, b, c) to calculate the transition temperature and the perpendicular upper critical field of S/N and of S/F superlattices. The equivalence between the Takahashi and Tachiki theory and the Usadel's quasiclassical theory has been shown by Lodder and Koperdraad (1993) to be valid only for the S/N system. The Usadel's equations including spin-orbit scattering terms were solved for S/F bi-layers by Demler, Arnold and Beasley (1997) and for S/F superlattices by Oh, Kim, Youm and Beasley (2000). The results indicate that the oscillatory behavior of the transition temperature is reduced by the spin-orbit scattering and strongly depends on the properties of materials. Khusianov and Proshin (1997) and Tagirov (1998) have developed the Usadel's theory of proximity effect in S/F bi-layers by taking into account the finite transparency of the F/S interface. They have shown that the oscillatory dependance of the critical temperature on the F-layer thickness is due to a periodic modulation of the F/S boundary transparency by the pair amplitude within the F layer.

Motivated by studying the interaction of superconductivity and magnetism, we review the proximity effect theories in this chapter. §2.2-2.7 is devoted to the de Gennes-Takahashi-Tachiki theory in the context of the correlation function method while the rest describes the Usadel equations and its applicability in the S/F problem.

2.2 Bogoliubov-de Gennes's equations

It is well known that the microscopic theory of superconductivity was established by BCS, the trial wave function method is used to determine the motion of electrons self-consistently, however, this method is applicable only for the homogeneous system.

For the non-uniform system such as in the presence of impurity atom or in the vicinity of the external field the BCS theory must be treated appropriately. One approach in this section is the Bogoliubov-de Gennes self-consistent field method which we follow de Gennes (1966). The Hamiltonian in the second quantized form reads as

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{2.1}$$

where the circumflex denotes the second quantized operator, $\hat{H}_0$ is the one electron Hamiltonian including the one point interaction term, $U_{\alpha\beta}(\vec{r})$,

$$\hat{H}_0 = \int d^3r \hat{\psi}^\dagger_\alpha(\vec{r})[-\frac{1}{2m}(\nabla - i\frac{e}{c}\vec{A})^2\delta_{\alpha\beta} + U_{\alpha\beta}(\vec{r})]\hat{\psi}_\beta(\vec{r}), \tag{2.2}$$

and

$$\hat{V} = \int d^3r d^3r' \hat{\psi}^\dagger_\delta(\vec{r})\hat{\psi}^\dagger_\gamma(\vec{r'})V_{\delta\gamma,\alpha\beta}(\vec{r},\vec{r'})\hat{\psi}_\alpha(\vec{r'})\hat{\psi}_\beta(\vec{r}), \tag{2.3}$$

is the instantaneous electron-electron interaction. The lower indices refer to their spins and the sum over repeated indices is understood. For simplicity the spin independent interactions are assumed

$$U_{\alpha\beta}(\vec{r}) = U(\vec{r})\delta_{\alpha\beta},$$

$$V_{\delta\gamma,\alpha\beta}(\vec{r},\vec{r'}) = V(\vec{r})\delta_{\alpha\gamma}\delta_{\delta\beta}\delta(\vec{r}-\vec{r'}), \tag{2.4}$$

in the last line the delta function-like form is made for the two body interaction as well as the inhomogeneity of the system. $\hat{\psi}^{\dagger}_{\alpha}(\vec{r})(\hat{\psi}_{\alpha}(\vec{r}))$ is the field operator which creates (destroys) a particle of spin α at the point $\vec{r}$ and obeys the anti-commutation relations

$$\hat{\psi}^{\dagger}_{\alpha}(\vec{r})\hat{\psi}_{\beta}(\vec{r'}) + \hat{\psi}_{\beta}(\vec{r'})\hat{\psi}^{\dagger}_{\alpha}(\vec{r}) = \delta_{\alpha\beta}\delta(\vec{r}-\vec{r'}),$$

$$\hat{\psi}_{\alpha}(\vec{r})\hat{\psi}_{\beta}(\vec{r'}) + \hat{\psi}_{\beta}(\vec{r'})\hat{\psi}_{\alpha}(\vec{r}) = 0. \tag{2.5}$$

The number operator

$$\hat{N} = \int d^3r \hat{\psi}^{\dagger}_{\alpha}(\vec{r})\hat{\psi}_{\alpha}(\vec{r}), \tag{2.6}$$

will be added to the one electron Hamiltonian when the electron energy is measured with respect to the Fermi energy, μ, i.e.,

$$\hat{H}_0 \longrightarrow \hat{H}_0 - \mu\hat{N}.$$

In the many-body problem the two-particle interaction is factorized to be an effective one-electron Hamiltonian and the result is of the form

$$\hat{H}_{eff} = \int d^3r[\hat{\psi}^{\dagger}_{\alpha}(\vec{r})\mathcal{H}\hat{\psi}_{\alpha}(\vec{r}) + (\triangle(\vec{r})\hat{\psi}^{\dagger}_{\uparrow}(\vec{r})\hat{\psi}^{\dagger}_{\downarrow}(\vec{r}) + h.c.)] \tag{2.7}$$

where

$$\mathcal{H} = -\frac{1}{2m}(\nabla - \frac{ie}{c}\vec{A})^2 + U(\vec{r}) - \mu. \tag{2.8}$$

The self-consistent field $\triangle(\vec{r})$ is called the pair potential because it changes the number of particles by two.

Having obtained the effective Hamiltonian, we employ the canonical transformation to calculate the trace of the effective Hamiltonian as follows

$$\hat{\psi}_{\uparrow}(\vec{r}) = \sum_n [u_n(\vec{r})\hat{\gamma}_{n\uparrow} - v_n^*(\vec{r})\hat{\gamma}_{n\downarrow}^{\dagger}]$$

and

$$\hat{\psi}_{\downarrow}(\vec{r}) = \sum_n [u_n(\vec{r})\hat{\gamma}_{n\downarrow} + v_n^*(\vec{r})\hat{\gamma}_{n\uparrow}^{\dagger}]. \tag{2.9}$$

where $u_n(\vec{r})$ and $v_n(\vec{r})$ are the eigenfunctions and in order to preserve the anti-commutation relation,(2.5), between the old and the new bases, the eigenfunctions $u_n(\vec{r})$ and $v_n(\vec{r})$ must satisfy the normalization and completeness conditions

$$\int d^3r[u_n^*(\vec{r})u_m(\vec{r}) + v_n^*(\vec{r})v_m(\vec{r})] = \delta_{n,m}$$

and

$$\sum_n [u_n^*(\vec{r})u_n(\vec{r'}) + v_n^*(\vec{r})v_n(\vec{r'})] = \delta(\vec{r} - \vec{r'}). \tag{2.10}$$

by means of equations (2.9)and (2.10) the diagonalized Hamiltonian has the simple form

$$\hat{H}_{eff} = E_g + \sum_{n,\alpha} \epsilon_n \hat{\gamma}_{n,\alpha}^{\dagger}\hat{\gamma}_{n,\alpha}, \tag{2.11}$$

where E_g is the ground-state energy and ϵ_n is the excitation energy in the state n. The equations of motion are governed by

$$[\hat{H}_{eff}, \hat{\gamma}_{n,\alpha}] = -\epsilon_n \hat{\gamma}_{n,\alpha},$$

$$[\hat{H}_{eff}, \hat{\gamma}_{n,\alpha}^{\dagger}] = \epsilon_n \hat{\gamma}_{n,\alpha}^{\dagger}. \tag{2.12}$$

These equations impose $u_n(\vec{r})$ and $v_n(\vec{r})$, we then obtain the set of equations for $u_n(\vec{r})$ and $v_n(\vec{r})$ known as the Bogoliubov-de Gennes equations

$$\epsilon_n u_n(\vec{r}) = \mathcal{H}u_n(\vec{r}) + \triangle(\vec{r})v_n(\vec{r}),$$

$$\epsilon_n v_n(\vec{r}) = -\mathcal{H}^* v_n(\vec{r}) + \triangle^*(\vec{r}) u_n(\vec{r}). \tag{2.13}$$

The pair of coupled equations for eigenfunctions $u_n(\vec{r})$ and $v_n(\vec{r})$ describe the superconducting state.

Upon factorizing the effective Hamiltonian, (2.7), we have the pair potential

$$\triangle(\vec{r}) = -V(\vec{r}) < \hat{\psi}_\downarrow(\vec{r})\hat{\psi}_\uparrow(\vec{r}) >, \tag{2.14}$$

where the angular brackets represent the statistical average of the field operator over the Gibbs ensemble. Using the canonical transformation of field operator,(2.9), as well as the thermal average of the products of operators $\hat{\gamma}_{n\alpha}$

$$< \hat{\gamma}^\dagger_{n\alpha}\hat{\gamma}_{m\beta} >= \delta_{\alpha\beta}\delta_{nm} f_n,$$

$$< \hat{\gamma}_{n\alpha}\hat{\gamma}_{m\beta} >= 0, \tag{2.15}$$

where

$$f_n = \frac{1}{\exp(\epsilon_n/T) + 1} \tag{2.16}$$

is the Fermi distribution function at temperature T in the eigenstate n. We obtain

$$\triangle(\vec{r}) = V(\vec{r}) \sum_n u_n(\vec{r}) v_n^*(\vec{r})(1 - 2f_n). \tag{2.17}$$

The Bogoliubov-de Gennes approach covers the inhomogeneous system. If $\triangle$ is constant, the system becomes the uniform superconductor.

2.3 Linearized pair potential

Near the transition point between the superconducting and the normal phases, the pair potential $\triangle(\vec{r})$ in the linearized form may be treated as a perturbation. The eigenfunctions $u_n(\vec{r})$and $v_n(\vec{r})$ of the Bogoliubov-de Gennes equation,(2.13), coincide with the normal state eigenfunctions, say $\phi_n(\vec{r})$, at the transition point. To verify them, we introduce the eigenvalue equation in the normal state

$$\mathcal{H}\phi_n(\vec{r}) = \xi_n\phi_n(\vec{r}),$$

or in the explicit form

$$[-\frac{1}{2m}(\nabla - \frac{ie}{c}\vec{A})^2 + U(\vec{r}) - \mu]\phi_n(\vec{r}) = \xi_n\phi_n(\vec{r}). \tag{2.18}$$

The solutions to the Bogoliubov-de Gennes equations can be derived in an approximate way by expanding the eigenfunctions $u_n(\vec{r})$ and $v_n(\vec{r})$ to first order

$$u_n = u_n^{(0)} + u_n^{(1)} + ...,$$

$$v_n = v_n^{(0)} + v_n^{(1)} + ...,$$

and also

$$\epsilon_n = \epsilon_n^{(0)} + \epsilon_n^{(1)} + ... \tag{2.19}$$

Substituting (2.19) into the Bogoliubov-de Gennes equations, (2.13), and arranging them along the order of expansions, we obtain

$$\epsilon_n^{(0)}u_n^{(0)} = \mathcal{H}u_n^{(0)},$$

$$\epsilon_n^{(0)}v_n^{(0)} = -\mathcal{H}^*v_n^{(0)}. \tag{2.20}$$

in the zeroth order. Note that the pair potential $\triangle$ is of the first order manner. The solutions to (2.20) are

$$u_n^{(0)} = \phi_n, v_n^{(0)} = 0, \epsilon_n^{(0)} = \xi_n,$$

and

$$u_n^{(0)} = 0, v_n^{(0)} = \phi^*, \epsilon_n^{(0)} = -\xi_n,$$

respectively, we rewrite these solutions in a more compact form as follows

$$u_n^{(0)} = \phi_n \theta(\xi_n),$$

$$v_n^{(0)} = \phi_n^* \theta(-\xi_n), \tag{2.21}$$

where $\theta(x)$ is the step function. The product $u_n^{(0)} v_n^{(0)} = 0$, this gives, when insert it into the self-consistent equation, (2.17), the pair potential $\triangle$ identically vanishes in the zeroth order. To the first order correction of the Bogoliubov-de Gennes equations, we have

$$\epsilon_n^{(0)} u_n^{(1)} + \epsilon_n^{(1)} u_n^{(0)} = \mathcal{H} u_n^{(1)} + \triangle v_n^{(0)},$$

$$\epsilon_n^{(0)} v_n^{(1)} + \epsilon_n^{(1)} v_n^{(0)} = -\mathcal{H}^* v_n^{(1)} + \triangle^* u_n^{(0)}, \tag{2.22}$$

Using the eigenfunction ϕ_n as a basis set, we expand

$$u_n^{(1)} = \sum_m a_{nm} \phi_m,$$

$$v_n^{(1)} = \sum_m b_{nm} \phi_m^*. \tag{2.23}$$

We now determine the coefficients a_{nm} and b_{nm} by inserting (2.23) into (2.22), multiplying by ϕ_n^* and ϕ_n respectively, and performing the integration over d^3r, we arrive at the following relation

$$(|\xi_n| - \xi_m) a_{nm} + \epsilon_n^{(1)} \theta(\xi_n) \delta_{nm} = \theta(-\xi_n) \int d^3r \triangle(\vec{r}) \phi_n^*(\vec{r}) \phi_m^*(\vec{r}),$$

$$(|\xi_n| + \xi_m) b_{nm} + \epsilon_n^{(1)} \theta(-\xi_n) \delta_{nm} = \theta(\xi_n) \int d^3r \triangle^*(\vec{r}) \phi_n(\vec{r}) \phi_m(\vec{r}). \tag{2.24}$$

For $n \neq m$ we have

$$a_{nm} = \frac{\theta(-\xi_n)}{|\xi_n| - \xi_m} \int d^3r \triangle(\vec{r}) \phi_n^*(\vec{r}) \phi_m^*(\vec{r}),$$

$$b_{nm} = \frac{\theta(\xi_n)}{|\xi_n| + \xi_m} \int d^3r \triangle^*(\vec{r}) \phi_n(\vec{r}) \phi_m(\vec{r}) \tag{2.25}$$

while for $n = m$, the coefficients a_{nm} and b_{nm} are set to be zero in order to preserve the normalized wave functions $u_n^{(1)}$ and $v_n^{(1)}$.

The self-consistent equation for the pair potential,(2.17), to the first order correction, can be written as

$$\triangle(\vec{r}) = V(\vec{r})\sum_n [u_n^{(0)}(\vec{r})v_n^{(0)*}(\vec{r}) + (u_n^{(0)}(\vec{r})v_n^{(1)*}(\vec{r}) + u_n^{(1)}(\vec{r})v_n^{(0)*}(\vec{r}))](1 - 2f_n(\epsilon^{(0)})). \tag{2.26}$$

Since the product of the zeroth order eigenfunction $u_n^{(0)}v_n^{(0)*} = 0$, this equation becomes

$$\triangle(\vec{r}) = V(\vec{r})\sum_{n,m}[\theta(\xi_n)b_{nm}^* + \theta(-\xi_n)a_{nm}]\phi_n(\vec{r})\phi_m(\vec{r})(1 - 2f(|\xi_n|)). \tag{2.27}$$

Inserting the coefficients a_{nm} and b_{nm} in (2.25) we then obtain the spatial variation of the pair potential in an integral form

$$\triangle(\vec{r}) = \int K(\vec{r},\vec{r'})\triangle(\vec{r'})d^3r', \tag{2.28}$$

where the kernel $K(\vec{r},\vec{r'})$ is defined as

$$K(\vec{r},\vec{r'}) = V(\vec{r})\sum_{nm}[1 - 2f(|\xi_n|)](\frac{\theta(\xi_n)}{|\xi_n| + \xi_m} + \frac{\theta(-\xi_n)}{|\xi_n| - \xi_m})\phi_n^*(\vec{r'})\phi_m^*(\vec{r'})\phi_n(\vec{r})\phi_m(\vec{r}). \tag{2.29}$$

If we use the properties of the step function

$$\frac{\theta(-\xi_n)}{|\xi_n| - \xi_m} = -\frac{\theta(-\xi_n)}{\xi_n + \xi_m},$$

$$\theta(\xi_n) + \theta(-\xi_n) = 1,$$

together with the identity $1 - 2f(|\xi_n|) = \tanh(\xi_n/2T)$, the kernel may be rewritten as

$$K(\vec{r},\vec{r'}) = \frac{1}{2}V(\vec{r})\sum_{n,m}\frac{\tanh(\xi_n/2T) + \tanh(\xi_m/2T)}{\xi_n + \xi_m}\phi_n^*(\vec{r'})\phi_m^*(\vec{r'})\phi_n(\vec{r})\phi_m(\vec{r}). \tag{2.30}$$

Note that the hyperbolic tangent function has an odd symmetry and the kernel has the symmetry when the indices n and m are interchanged. The kernel may be expressed in an alternative form by making use of the analytic continuation of the function tanh x in the complex plane which implies

$$\tanh(x) = \sum_{\nu} \frac{1}{x \pm (\nu + \frac{1}{2}) i\pi},$$

or in our desired formula

$$\tanh(\frac{\xi}{2T}) = 2T \sum_{\omega_\nu} \frac{1}{\xi \pm i\omega_\nu},$$

where $\omega_\nu = (2\nu + 1)\pi T$, the sum is taken over positive and negative integers ν. and either $\pm$ in the denominator of (2.30) both contribute. Then

$$\frac{\tanh(\xi_n/2T) + \tanh(\xi_m/2T)}{\xi_n + \xi_m} = 2T \sum_{\omega_\nu} \frac{1}{(\xi_n - i\omega_\nu)(\xi_m + i\omega_\nu)}.$$

Thus we obtain the kernel in another form

$$K(\vec{r}, \vec{r'}) = V(\vec{r}) T \sum_{\omega_\nu} \sum_{n,m} \frac{\phi_n^*(\vec{r'}) \phi_m^*(\vec{r'}) \phi_n(\vec{r}) \phi_m(\vec{r})}{(\xi_n - i\omega_\nu)(\xi_m + i\omega_\nu)}. \tag{2.31}$$

2.4 Correlation functions

Although the kernel (2.31) is obtained, it remains to calculate the matrix element between states $\phi_m^*(\vec{r})$ and $\phi_n^*(\vec{r})$. In practice this is an unusual work. However, with the aid of the definition of the complex conjugation operator C, we can write the products like $\phi_m^*(\vec{r})\phi_n^*(\vec{r})$ in a suitable form. The complex conjugation operator C having the property

$$C\phi_m = \phi_m^*,$$

then permits us to write

$$\phi_n^*(\vec{r})\phi_m^*(\vec{r}) = \phi_n^*(\vec{r}) C \phi_m(\vec{r}).$$

In doing so, let us insert the completeness relations of the position states $\vec{r}_1$ and $\vec{r}_2$ into the above equation and performing the integration over d^3r_1 and d^3r_2 yields

$$\phi_n^*(\vec{r})\phi_m^*(\vec{r}) =< n|\delta(\vec{R}-\vec{r})C|m >,$$

$$\phi_n(\vec{r})\phi_m(\vec{r}) =< n|C^\dagger\delta(\vec{R}-\vec{r})|m > . \tag{2.32}$$

where $\vec{R}$ is the position operator. The energy denominator of (2.31) shall be written as

$$\frac{1}{(\xi_n - i\omega_\nu)(\xi + i\omega_\nu)} = \int d\xi d\xi' \frac{\delta(\xi-\xi_n)\delta(\xi'-\xi_m)}{(\xi - i\omega_\nu)(\xi + i\omega_\nu)},$$

$$= \int d\xi d\xi' \frac{\delta(\xi-\xi_n)\delta(\xi_m-\xi_n-\Omega)}{(\xi - i\omega_\nu)(\xi' + i\omega_\nu)}, \tag{2.33}$$

with $\Omega = \xi' - \xi$. We introduce the correlation function $g_\xi(\vec{r},\vec{r'},\Omega)$ of a one-electron in the normal state having the energy ξ located at a point $\vec{r}$ move to another point $\vec{r'}$ at the energy state ξ'

$$g_\xi(\vec{r},\vec{r'},\Omega) = \sum_{n,m} < n|\delta(\vec{R}-\vec{r})C|m >< m|C^\dagger\delta(\vec{R}-\vec{r'})|n > \delta(\xi-\xi_n)\delta(\xi_m-\xi_n-\Omega). \tag{2.34}$$

Then the kernel can be expressed in terms of the correlation function as

$$K(\vec{r},\vec{r'}) = V(\vec{r})T\sum_{\omega_\nu}\int d\xi d\xi' \frac{g_\xi(\vec{r},\vec{r'},\Omega)}{(\xi - i\omega_\nu)(\xi' + i\omega_\nu)}. \tag{2.35}$$

Since the dominant contribution of energy states lies on the Fermi level then we shall approximate $\xi \approx 0$.

The Fourier's transform of the correlation function is

$$g_{\xi\approx 0}(\vec{r},\vec{r'},\Omega) = \int \frac{dt}{2\pi} e^{i\Omega t} g_{\xi\approx 0}(\vec{r},\vec{r'},t), \tag{2.36}$$

and its inverse transform

$$g_{\xi\approx 0}(\vec{r},\vec{r'},t) = \int d\Omega e^{-i\Omega t} g_{\xi\approx 0}(\vec{r},\vec{r'},\Omega). \tag{2.37}$$

The kernel (2.35) is simply written as

$$K(\vec{r},\vec{r'}) = V(\vec{r})T\sum_{\omega_\nu}\int\frac{dt}{2\pi}\int d\xi d\xi'\frac{e^{i\Omega t}g_{\xi\approx 0}(\vec{r},\vec{r'},t)}{(\xi - i\omega_\nu)(\xi' + i\omega_\nu)}. \tag{2.38}$$

Performing the energy integrals of (2.38) using the residue theorem, there are four cases: $t < 0, \omega_\nu < 0; t > 0, \omega_\nu > 0; t < 0, \omega_\nu > 0; t > 0, \omega_\nu < 0$, the nonvanishing integrals are

$$\int d\xi d\xi'\frac{e^{i\Omega t}}{(\xi - i\omega_\nu)(\xi + \omega_\nu)} = (2\pi)^2[\theta(t)\theta(-\omega_\nu)e^{-2|\omega_\nu|t} + \theta(-t)\theta(\omega_\nu)e^{-2\omega_\nu t}].$$

We insert the above integral into (2.38) and arrange the time interval to obtain

$$K(\vec{r},\vec{r'}) = 2\pi V(\vec{r})T\sum_{\omega_\nu}\int_0^\infty dte^{-2|\omega_\nu|t}g_{\xi\approx 0}(\vec{r},\vec{r'},t). \tag{2.39}$$

We proceed further to calculate the correlation function, the time derivative of the complex conjugation operator C is governed by the Heisenberg equation of motion

$$\frac{dC}{dt} = i[\mathcal{H}, C] = -\frac{e}{mc}(\nabla.\vec{A} + \vec{A}.\nabla)C,$$

we work in the gauge $\nabla.\vec{A} = 0$, this provides

$$C(t) = e^{-i\phi(t)}C(0), \tag{2.40}$$

where

$$\phi(t) = \frac{2e}{c}\int_{\vec{r}(0)}^{\vec{r}(t)}\vec{A}(\vec{l}).d\vec{l}. \tag{2.41}$$

let us calculate (2.37) by using (2.34) then we arrive at the following result

$$g_{\xi\approx 0}(\vec{r},\vec{r'};t) = \sum_{n,m}e^{-i(\xi_m - \xi_n)t} < n|\delta(\vec{R} - \vec{r})C|m >< m|C^\dagger\delta(\vec{R} - \vec{r'})|n > \delta(\xi - \xi_n). \tag{2.42}$$

We note that the delta function contains Ω in $g_{\xi\approx 0}(\Omega)$. When it is integrated, an exponential factor $\exp[-i(\xi_m - \xi_n)t]$ in (2.42) is obtained. This factor and the first matrix element in (2.42) is recasted as

$$e^{-i(\xi_m-\xi_n)t} < n|\delta(\vec{R}-\vec{r})C|m >=< n|\delta(\vec{R}-\vec{r})C(t)|m >, \qquad (2.43)$$

where the Heisenberg operators are defined as

$$\delta(\vec{R}(t)-\vec{r}) = e^{i\mathcal{H}t}\delta(\vec{R}-\vec{r})e^{-i\mathcal{H}t}, \qquad (2.44)$$

and

$$C(t) = e^{-i\mathcal{H}t}Ce^{-i\mathcal{H}t}. \qquad (2.45)$$

Substituting (2.43) into (2.42) and using the closer relation $\sum_m |m >< m| = 1$, we have

$$g_{\xi\approx 0}(\vec{r},\vec{r'},t) = \sum_n < n|\delta(\vec{R}(t)-\vec{r})C(t)C^\dagger(0)\delta(\vec{R}(0)-\vec{r})|n > \delta(\xi-\xi_n). \qquad (2.46)$$

2.5 Spin-dependent correlation functions

When the potentials are spin dependent, the generalized Bogoliubov-de Gennes equation is required, the main goal is to obtain the kernel of the linearized pair potential. We begin with the eigenvalue equation

$$[\frac{1}{2m}(\vec{p}-\frac{e}{c}\vec{A})^2-\mu]\phi_N(\vec{r},\alpha) + \sum_\beta U_{\alpha\beta}(\vec{r})\phi_N(\vec{r},\alpha) = \xi_N\phi_N(\vec{r},\alpha), \qquad (2.47)$$

where the subscript N includes both the translational and the spin quantum numbers while the indices α or β refer to the spin components of a wavefunction. Instead of (2.31), the kernel is generalized to include spins as

$$K(\vec{r},\vec{r'}) = \frac{1}{2}V(\vec{r})T\sum_{\omega_\nu}\sum_{M,N}\sum_{\alpha\beta\gamma\delta}\frac{\phi^*_N(\vec{r'},\alpha)\rho_{\alpha\beta}\phi^*_M(\vec{r'},\beta)\phi_M(\vec{r},\gamma)\rho_{\gamma\delta}\phi_N(\vec{r},\delta)}{(\xi_N-i\omega_\nu)(\xi_M+i\omega_\nu)}, \qquad (2.48)$$

here the factor 1/2 taking into account the sum over spin conmponents, the operator $\rho = i\sigma_y$ where σ_y is a Pauli spin matrix. Next we introduce the Wigner time reversal operator

$$\mathcal{K} = \rho C, \tag{2.49}$$

with the usual complex conjugate operator C, the product of the wavefunctions can be written formally as

$$\sum_{\alpha\beta} \phi_N^*(\vec{r},\alpha)\rho_{\alpha\beta}\phi_M^*(\vec{r},\beta) =< N|\delta(\vec{R}-\vec{r})\mathcal{K}|M >,$$

and

$$\sum_{\alpha\beta} \phi_M(\vec{r},\alpha)\rho_{\alpha\beta}\phi_N(\vec{r},\beta) = -< M|\mathcal{K}^\dagger\delta(\vec{R}-\vec{r})|N > . \tag{2.50}$$

The correlation function analogous to (2.34) is

$$g_{\xi\approx 0}(\vec{r},\vec{r'},\Omega) = -\frac{1}{2}\sum_{NM} < N|\delta(\vec{R}-\vec{r})\mathcal{K}|M >< M|\mathcal{K}^\dagger\delta(\vec{R}-\vec{r'})|N >$$
$$\delta(\xi-\xi_N)\delta(\xi_M-\xi_N-\Omega), \tag{2.51}$$

with $\Omega = \xi - \xi'$.

The Fourier transform of this equation is calculated to be

$$g_{\xi\approx 0}(\vec{r},\vec{r'},t) = -\frac{1}{2}\sum_{N}\delta(\xi-\xi_N) < N|\delta(\vec{R}-\vec{r})\mathcal{K}(t)\mathcal{K}^\dagger(0)\delta(\vec{R}(0)-\vec{r'})|N > . \tag{2.52}$$

Thus the linearized kernel has the same form as (2.35)

$$K(\vec{r},\vec{r'}) = V(\vec{r})T\sum_{\omega_\nu}\int d\xi d\xi' \frac{g_{\xi\approx 0}(\vec{r},\vec{r'},\Omega)}{(\xi-i\omega_\nu)(\xi'+i\omega_\nu)}, \tag{2.53}$$

or, identical with (2.39)

$$K(\vec{r},\vec{r'}) = 2\pi V(\vec{r})T\sum_{\omega_\nu}\int_0^\infty dte^{-2|\omega_\nu|t}g_{\xi\approx 0}(\vec{r},\vec{r'};t). \tag{2.54}$$

2.6 Proximity effect theory of Takahashi and Tachiki

The starting point of the Takahashi and Tachiki theory (1986) is the Gorkov linearized integral equation for the pair potential

$$\triangle(\vec{r}) = V(\vec{r})T\sum_{\omega}\int d^3r' Q_\omega(\vec{r},\vec{r'})\triangle(\vec{r'}), \tag{2.55}$$

which contains a position-dependent pairing interaction and a summation over Matsubara's frequencies $\omega = (2n+1)\pi T, (\hbar = k_B = 1)$. The kernel has been expressed in terms of the correlation function as

$$Q_\omega(\vec{r},\vec{r'}) = 2\pi\sum_{\sigma}\int_0^\infty dt e^{-2|\omega|t} g^\sigma_{\xi=0}(\vec{r},\vec{r'};t), \tag{2.56}$$

where $\xi = 0$ corresponds to the Fermi energy and the sum over spins. The one-electron correlation function is already evaluated in the previous section, namely,

$$g^\sigma_{\xi=0}(\vec{r},\vec{r'};t) = -\frac{1}{2}\sum_n \delta(\xi-\xi_\mu) < \mu|\delta(\vec{R}(t)-\vec{r})\mathcal{K}^\dagger(t)\mathcal{K}(0)\delta(\vec{R}(0)-\vec{r'})|\mu >, \tag{2.57}$$

where $\mu = (n,\sigma), n$ is the translation part and σ the spin part. $\vec{R}$ is the position operator and $\mathcal{K}$ is the time reversal operator.

For $t \to 0, g^\sigma_{\xi=0}(\vec{r},\vec{r'};t)$ reduces to the initial condition

$$\lim_{t\to 0} g^\sigma_{\xi=0}(\vec{r},\vec{r'};t) = \frac{1}{2}\delta(\vec{r}-\vec{r'})N(\vec{r}), \tag{2.58}$$

where $N(\vec{r})$ is the position-dependent density of states at the Fermi level.

The equation of motion for the time reversal operator $\mathcal{K}(t)$ satisfies

$$\frac{\partial}{\partial t}\mathcal{K}(t) = 2iI_m(\vec{r}(t))\sigma_z\mathcal{K}(t) - i\frac{d\theta(t)}{dt}\mathcal{K}(t), \tag{2.59}$$

$$\theta(t) = \frac{2e}{c}\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{A}(\vec{s}).d\vec{s}. \tag{2.60}$$

I_m is the mean-field exchange potential, σ_z is the Pauli matrix along the z-component and the phase $\theta(t)$ results from the semiclassical phase approximation. The diffusion equation of $\delta(\vec{R}(t)-\vec{r})$ is given by

$$\frac{\partial}{\partial t}\delta(\vec{R}(t)-\vec{r}) = D(\vec{r})\nabla^2\delta(\vec{R}(t)-\vec{r}), \tag{2.61}$$

$D(\vec{r})$ being the position-dependent electronic diffusion constant. Using (2.57), (2.59) and (2.61), the equation of motion for the correlation function (2.57) yields

$$[\frac{\partial}{\partial t} + 2iI_m(\vec{r})(\sigma_z)_{\sigma\sigma}]g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t) = -\mathcal{L}g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t);\ (t>0), \tag{2.62}$$

where the differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = -D(\vec{r})(\nabla - \frac{2ie}{c}\vec{A})^2. \tag{2.63}$$

In order to construct the differential equation for $Q_\omega(\vec{r},\vec{r'})$, we introduce the auxiliary function $R_\omega(\vec{r},\vec{r'})$ defined by

$$R_\omega(\vec{r},\vec{r'}) = 2\pi\sum_\sigma(\sigma_z)_{\sigma\sigma}\int_0^\infty dte^{-2|\omega|t}g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t). \tag{2.64}$$

The time derivative of (2.56) is

$$\frac{\partial}{\partial t}Q_\omega(\vec{r},\vec{r'}) = -2|\omega|Q_\omega(\vec{r},\vec{r'}) + 2\pi\sum_\sigma\int_0^\infty dte^{-2|\omega|t}\frac{\partial}{\partial t}g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t), \tag{2.65}$$

we integrate the last term on the right-side by parts and use (2.58) to yield

$$\frac{\partial}{\partial t}Q_\omega(\vec{r},\vec{r'}) = -2\pi N(\vec{r})\delta(\vec{r}-\vec{r'}), \tag{2.66}$$

while the direct substitution of (2.62) into (2.65) gives

$$\frac{\partial}{\partial t}Q_\omega(\vec{r},\vec{r'}) = -[2|\omega| + \mathcal{L}]Q_\omega(\vec{r},\vec{r'}) - 2iI_m(\vec{r})R_\omega(\vec{r},\vec{r'}). \tag{2.67}$$

Combining (2.66) and (2.67) we obtain

$$[2|\omega| + \mathcal{L}]Q_\omega(\vec{r}, \vec{r'}) + 2iI_m(\vec{r})R_\omega(\vec{r}, \vec{r'}) = 2\pi N(\vec{r})\delta(\vec{r} - \vec{r'}). \tag{2.68}$$

The differential equation of $Q_\omega(\vec{r}, \vec{r'})$ is coupled with the auxiliary kernel $R_\omega(\vec{r}, \vec{r'})$ this means we require the differential equation for $R_\omega(\vec{r}, \vec{r'})$.

The time derivative of (2.64) is

$$\frac{\partial}{\partial t}R_\omega(\vec{r}, \vec{r'}) = -2|\omega|R_\omega(\vec{r}, \vec{r'}) + 2\pi\sum_\sigma(\sigma_z)_{\sigma\sigma}\int_0^\infty dte^{-2|\omega|t}\frac{\partial}{\partial t}g^\sigma_{\xi=0}(\vec{r}, \vec{r'}). \tag{2.69}$$

Integrating (2.69) by parts implies

$$\frac{\partial}{\partial t}R_\omega(\vec{r}, \vec{r'}) = 0, \tag{2.70}$$

since the trace of Pauli matrix is zero, using (2.62) in (2.69) gives

$$\frac{\partial}{\partial t}R_\omega(\vec{r}, \vec{r'}) = -[2|\omega| + \mathcal{L}]R_\omega(\vec{r}, \vec{r'}) - 2iI_m(\vec{r})Q_\omega(\vec{r}, \vec{r'}). \tag{2.71}$$

From (2.70) and (2.71), we have

$$[2|\omega| + \mathcal{L}]R_\omega(\vec{r}, \vec{r'}) + 2iI_m(\vec{r})Q_\omega(\vec{r}, \vec{r'}) = 0 \tag{2.72}$$

The equations (2.68) and (2.72) form a set of coupled differential equations. We now define the eigenvalue equation for the operator $\mathcal{L}$:

$$\mathcal{L}\psi_\lambda(\vec{r}) = \varepsilon_\lambda\psi_\lambda(\vec{r}), \tag{2.73}$$

the eigenfunctions $\psi_\lambda(\vec{r})$ obey the de Gennes boundary conditions at the interfaces. The quantities $\psi_\lambda(\vec{r})/\sqrt{N(\vec{r})}$ and $\sqrt{N(\vec{r})}D(\vec{r})(\nabla - 2ie\vec{A}(\vec{r})/c)\psi_\lambda(\vec{r})$ are continuous at the interfaces. Further, the eigenfunctions have the orthogonality and closure properties as follows

$$\int d^3r\psi^*_\lambda(\vec{r})\psi_{\lambda'}(\vec{r}) = \delta_{\lambda\lambda'}, \tag{2.74}$$

and

$$\sum_{\lambda} \psi^*_{\lambda}(\vec{r})\psi_{\lambda}(\vec{r'}) = \delta(\vec{r} - \vec{r'}). \qquad (2.75)$$

By using the eigenfunctions $\psi_\lambda(\vec{r})$ as a basis set we expand Q_ω and R_ω in terms of the eigenfunctions $\psi_\lambda(\vec{r})$ of the operator $\mathcal{L}$ with corresponding eigenvalues ε_λ,

$$Q_\omega(\vec{r}, \vec{r'}) = \sqrt{N(\vec{r})N(\vec{r'})} \sum_{\lambda\lambda'} a_{\lambda\lambda'} \psi_{\lambda'}(\vec{r'})\psi_\lambda(\vec{r}), \qquad (2.76)$$

and

$$R_\omega(\vec{r}, \vec{r'}) = \sqrt{N(\vec{r})N(\vec{r'})} \sum_{\lambda\lambda'} b_{\lambda\lambda'} \psi_{\lambda'}(\vec{r'})\psi_\lambda(\vec{r}). \qquad (2.77)$$

We also define the pair function $F(\vec{r}) = \triangle(\vec{r})/V(\vec{r})$ to solve the linearized integral equation of the gap function (2.55). We try to expand the pair function $F(\vec{r})$ in terms of the eigenfunction $\psi_\lambda(\vec{r})$

$$F(\vec{r}) = \sqrt{N(\vec{r})} \sum_{\lambda} c_\lambda \psi_\lambda(\vec{r}). \qquad (2.78)$$

The expansion coefficients $a_{\lambda\lambda'}$ and $b_{\lambda\lambda'}$ are determined from the set of coupled differential equations, (2.68) and (2.72). Inserting (2.76) and (2.97) into (2.68) and (2.72), respectively, and multiplying by $\psi_{\lambda'}(\vec{r'})$ and $\psi_\lambda(\vec{r})$ and then integrating over $\vec{r}$ and $\vec{r'}$, we have

$$[2|\omega| + \varepsilon_\lambda]a_{\lambda\lambda'} + 2i \sum_{\zeta} < \lambda|I_m|\zeta > b_{\zeta\lambda'} = 2\pi\delta_{\lambda\lambda'}, \qquad (2.79)$$

$$[2|\omega| + \varepsilon_\lambda]b_{\lambda\lambda'} + 2i \sum_{\zeta} < \lambda|I_m|\zeta > a_{\zeta\lambda'} = 0, \qquad (2.80)$$

where the matrix element $< \lambda|I_m|\zeta >$ is denoted by

$$< \lambda|I_m|\zeta > = \int d^3r \psi^*_\lambda(\vec{r}) I_m(\vec{r}) \psi_\zeta(\vec{r}). \qquad (2.81)$$

We rewrite (2.80) as

$$b_{\zeta\lambda'} = -2i \frac{\sum_{\zeta'} < \zeta|I_m|\zeta' > a_{\zeta'\lambda'}}{(2|\omega| + \varepsilon_\zeta)}, \qquad (2.82)$$

and by inserting the above equation into (2.79), we obtain the equation

$$\sum_{\zeta} \Gamma_{\lambda\zeta}(\omega) a_{\zeta\lambda'} = 2\pi\delta_{\lambda\lambda'}, \tag{2.83}$$

where the matrix element $\Gamma_{\lambda\lambda'}(\omega)$ satisfies the relation

$$\Gamma_{\lambda\lambda'}(\omega) = (2|\omega| + \varepsilon_\lambda)\delta_{\lambda\lambda'} + 4\sum_{\zeta} \frac{< \lambda|I_m|\zeta >< \zeta|I_m|\lambda' >}{(2|\omega| + \varepsilon_\zeta)}. \tag{2.84}$$

The coefficient $a_{\lambda\lambda'}$ can be determined from (2.83)

$$a_{\lambda\lambda'} = 2\pi\Gamma^{-1}_{\lambda\lambda'}(\omega). \tag{2.85}$$

where $\Gamma^{-1}(\omega)$ is an inverse matrix of $\Gamma(\omega)$. Substituting (2.85) back into (2.76) we obtain the expression for $Q_\omega(\vec{r}, \vec{r'})$

$$Q(\vec{r}, \vec{r'}) = 2\pi\sqrt{N(\vec{r})N(\vec{r'})} \sum_{\lambda\lambda'} \psi_\lambda(\vec{r})\Gamma^{-1}_{\lambda\lambda'}(\omega)\psi^*_{\lambda'}(\vec{r'}). \tag{2.86}$$

The linearized integral equation, (2.55), is written in terms of the pair function $F(\vec{r})$ as

$$F(\vec{r}) = T\sum_{\omega} \int d^3r' Q_\omega(\vec{r}, \vec{r'}) V(\vec{r'}) F(\vec{r'}),$$

when inserting (2.86) into this equation, we have

$$F(\vec{r}) = 2\pi T\sqrt{N(\vec{r})} \sum_{\omega} \sum_{\lambda\lambda'} \psi_\lambda(\vec{r})\Gamma^{-1}_{\lambda\lambda'}(\omega) \int d^3r' \psi^*_{\lambda'}(\vec{r'})\sqrt{N(\vec{r'})} V(\vec{r'}) F(\vec{r'}). \tag{2.87}$$

Using the eigenfunction expansion of the pair function, (2.78), then (2.87) reads as

$$\sum_{\zeta} c_\zeta \psi_\zeta(\vec{r}) = 2\pi T \sum_{\omega} \sum_{\zeta\lambda\lambda'} \psi_\lambda(\vec{r})\Gamma^{-1}_{\lambda\lambda'}(\omega) < \lambda'|NV|\zeta > c_\zeta. \tag{2.88}$$

Multiplying both sides of (2.88) by $\psi^*_{\zeta'}(\vec{r})$ and integrating over $\vec{r}$ yields

$$\sum_{\lambda'} [\delta_{\lambda\lambda'} - 2\pi T \sum_{\omega} \sum_{\zeta} \Gamma^{-1}_{\lambda\zeta}(\omega) < \zeta|NV|\lambda' >] c_{\lambda'} = 0. \tag{2.89}$$

When the matrix equation of the coefficient c_λ has nontrivial solutions, the secular equation must satisfy

$$\det|\delta_{\lambda\lambda'} - 2\pi T \sum_{\omega}\sum_{\zeta}\Gamma^{-1}_{\lambda\zeta}(\omega) < \zeta|NV|\lambda' > | = 0. \tag{2.90}$$

The secular equation (2.90) is the central result in determining the upper critical field, parallel and perpendicular to the layer planes of superlattices, as a function of temperature. The highest magnetic field solution among the solutions corresponds to the upper critical field. Numerical calculations were performed by Takahashi-Tachiki and Takanaka in S/N systems to show that (i) only the lowest eigenstate is sufficient to take into account for the parallel upper critical field $H_{c2||}$ of superconductors when the diffusion constant is varied. (ii) one can calculate $H_{c2||}$ with the spatial variation of the BCS interaction constant using the lowest eigenvalue eigenfunction for the harmonic oscillator and the secular equation. (iii) If the density of states changes spatially, it is necessary to obtain all eigenvalues and eigenfunctions.

2.7 Generalized Takahashi- Tachiki proximity effect theory

In the previous section the Takahashi-Tachiki theory of proximity coupled superconductors has been formulated. They have extended the de Gennes theory using an eigenfunction expansion to determine the secular equation from the kernel $Q_\omega(\vec{r},\vec{r'})$ of the linearized integral equation for the pair potential $\triangle(\vec{r})$. The solutions of the secular equation give the upper critical field of the dirty type II superconductor as a function of temperature. In their work, the kernel $Q_\omega(\vec{r},\vec{r'})$ is derived from a set of coupled differential equations including diamagnetic and paramagnetic effects. However, in the real systems the interactions such as the magnetic impurity scattering and spin-orbit scattering will have also included, as studied by Auvil, Ketterson and Song (1989) and Jin and Ketterson (1989).

We consider the spin-dependent potential $U_{\alpha\beta}(\vec{r})$ as being composed of four terms

- $U(\vec{r})\delta_{\alpha\beta}$ is the direct potential,
- $I(\sigma_z)_{\alpha\beta}$ is the Pauli spin paramagnetism where I denotes the exchange energy which arises from the interaction of the electron magnetic moment with a magnetic field,
- $\mathcal{H}_{imp} = \sum_i \Gamma(\vec{r} - \vec{r}_i)\vec{\sigma}.\vec{S}_i$ is the pair-breaking Hamiltonian of the diluted magnetic impurity. Here $\Gamma(\vec{r} - \vec{r}_i)$ is the exchange interaction between the electron spin $\vec{\sigma}$ and the impurity spin $\vec{S}_i$. The impurity sites are located randomly so the impurity spins have no interaction between them,
- $\mathcal{H}_{so} = \frac{e}{4(mc)^2}\nabla U(\vec{r}).(\vec{\sigma} \times \vec{p})$ is the spin-orbit Hamiltonian.

The effect of these interaction will be treated on the time reversal operator $\mathcal{K}(t)$ in the Heisenberg representation

$$\mathcal{K}(t) = e^{-i\mathcal{H}t}\mathcal{K}(0)e^{-i\mathcal{H}t}. \tag{2.91}$$

The equation of motion of $\mathcal{K}(t)$ is governed by

$$\frac{\partial}{\partial t}\mathcal{K}(t) = (-i\frac{d\theta(t)}{dt} + 2iI\sigma_z - \frac{1}{\tau_m})\mathcal{K}(t), \tag{2.92}$$

where the first term refers to the orbital diamagnetism involving the vector potential,

$$\theta(t) = \frac{2e}{c}\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{A}(\vec{s}).d\vec{s},$$

the second term accounts the spin paramagnetism and the last term arises from the magnetic impurities atoms with the spin-flip scattering time τ_m. Formally, the spin-orbit coupling term commutes with the time reversal operator. We use the phenomenological approach to incorporate them at the initial time, say $t = 0$, there is no evolution of $\mathcal{K}(t)$ a combination of $\mathcal{K}$ and its transpose $\tilde{\mathcal{K}}$ satisfy this requirement, thus (2.92) becomes

$$\frac{\partial}{\partial t}\mathcal{K}(t) = (-i\frac{d\theta(t)}{dt} + 2iI\sigma_z - \frac{1}{\tau_m})\mathcal{K}(t) - \frac{1}{\tau_{so}}(\mathcal{K}(t) - \tilde{\mathcal{K}}(t)). \tag{2.93}$$

We have the one-electron correlation function evaluated at the Fermi surface, (2.57),

$$g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t) = -\frac{1}{2}\sum_{n}\delta(\xi - \xi_{n,\sigma}) < n,\sigma|\delta(\vec{R}(t) - \vec{r})\mathcal{K}^{\dagger}(t)\mathcal{K}(0)\delta(\vec{R}(0) - \vec{r'})|n,\sigma > .$$

Using (2.61) and (2.93), the equation of motion for $g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t)$ obeys

$$\begin{aligned}[\frac{\partial}{\partial t} + 2iI(\vec{r})(\sigma_z)_{\sigma\sigma} + \frac{1}{\tau_m(\vec{r})} + \frac{1}{\tau_{so}(\vec{r})}]g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t) - \frac{1}{\tau_{so}(\vec{r})}g^{-\sigma}_{\xi=0}(\vec{r},\vec{r'};t) \\ = -\mathcal{L}g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t);\ (t > 0),\end{aligned} \tag{2.94}$$

where the differential operator $\mathcal{L}$ is defined by (2.63).

The kernels $Q_\omega(\vec{r},\vec{r'})$ and $R_\omega(\vec{r},\vec{r'};t)$, (2.56) and (2.64), have a set of coupled differential equations as follows

$$[2|\omega| + \mathcal{L} + \frac{1}{\tau_m(\vec{r})} + \frac{2}{\tau_{so}(\vec{r})}]Q_\omega(\vec{r},\vec{r'}) + 2iI(\vec{r})R_\omega(\vec{r},\vec{r'}) = 2\pi N(\vec{r})\delta(\vec{r} - \vec{r'}), \tag{2.95}$$

$$[2|\omega| + \mathcal{L} + \frac{1}{\tau_m(\vec{r})}]R_\omega(\vec{r},\vec{r'}) + 2iI(\vec{r})Q_\omega(\vec{r},\vec{r'}) = 0. \tag{2.96}$$

To solve the coupled differential equations, (2.95) and (2.96), we adopt the eigenfunction expansion method of Takahashi and Tachiki and the boundary conditions of de Gennes. We choose the eigenvalue equation of the operator $\mathcal{L}$ as

$$(\mathcal{L} + \frac{1}{\tau_m(\vec{r})})\psi_\lambda(\vec{r}) = \varepsilon_\lambda\psi_\lambda(\vec{r}), \tag{2.97}$$

and let us expand the kernels $Q_\omega(\vec{r},\vec{r'})$ and $R_\omega(\vec{r},\vec{r'})$ and the pair function $F(\vec{r})$ in terms of eigenfunctions with the undetermined coefficients $a_{\lambda\lambda'}, b_{\lambda\lambda'}$ and c_λ respectively;

$$Q_\omega(\vec{r},\vec{r'}) = \sqrt{N(\vec{r})N(\vec{r'})}\sum_{\lambda\lambda'} a_{\lambda\lambda'}\psi_\lambda(\vec{r})\psi^*_{\lambda'}(\vec{r'})$$

$$R_\omega(\vec{r},\vec{r'}) = \sqrt{N(\vec{r})N(\vec{r'})}\sum_{\lambda\lambda'} b_{\lambda\lambda'}\psi_\lambda(\vec{r})\psi^*_{\lambda'}(\vec{r'}),$$

$$F(\vec{r}) = \sqrt{N(\vec{r})}\sum_{\lambda} c_\lambda\psi_\lambda(\vec{r}).$$

The linearized integral equation of the pair potential can be rewritten as

$$F(\vec{r}) = T\sum_\omega \int d^3r' Q_\omega(\vec{r},\vec{r'})V(\vec{r'})F(\vec{r'}).$$

The pair of coupled differential equations, (2.95) and (2.96), can be transformed into the set of the expansion coefficients as

$$(2|\omega| + \varepsilon_\lambda)a_{\lambda\lambda'} + \sum_\zeta [< \lambda|\frac{2}{\tau_{so}}|\zeta > a_{\zeta\lambda'} + 2i < \lambda|I|\zeta > b_{\zeta\lambda'}] = 2\pi\delta_{\lambda\lambda'}, \qquad (2.98)$$

$$(2|\omega| + \varepsilon_\lambda)b_{\lambda\lambda'} + 2i\sum_\zeta < \lambda|I|\zeta > a_{\zeta\lambda'} = 0. \qquad (2.99)$$

Only the coefficient $a_{\lambda\lambda'}$ is needed to be evaluated then by defining

$$\Gamma_{\lambda\lambda'}(\omega) = (2|\omega| + \varepsilon_\lambda)\delta_{\lambda\lambda'} + < \lambda|\frac{2}{\tau_{so}}|\lambda' > +4\sum_\zeta \frac{< \lambda|I|\zeta >< \zeta|I|\lambda' >}{(2|\omega| + \varepsilon_\zeta)}. \qquad (2.100)$$

We obtain

$$\sum_\zeta \Gamma_{\lambda\zeta}(\omega)a_{\zeta\lambda'} = 2\pi\delta_{\lambda\lambda'}, \qquad (2.101)$$

and the solution to (2.101) is

$$a_{\lambda\lambda'} = 2\pi\Gamma^{-1}_{\lambda\lambda'}(\omega). \qquad (2.102)$$

with the given $a_{\lambda\lambda'}$ the kernel $Q_\omega(\vec{r},\vec{r'})$ will be set up and we can solve the integral equation of the pair potential to determine the secular equation

$$\det|\delta_{\lambda\lambda'} - 2\pi T\sum_\omega\sum_\zeta \Gamma^{-1}_{\lambda\zeta}(\omega) < \zeta|NV|\lambda' > | = 0. \qquad (2.103)$$

This equation is a generalized Takahashi-Tachiki result, the lowest eigenvalue of which yields the highest transition temperature T_c.

A single layer

To test the validity of an eigenfunction expansion method of Takahashi and Tachiki for dirty coupled superconductors. We regard the single layer which is infinite in the x-y plane, occupies the range $0 \leq z \leq L$, the bulk limit $L \to \infty$ will be then taken later. The magnetic field is directed along the z-axis by using the polar coordinates so the vector potential $\vec{A} = H\rho\hat{\rho}/2$. The solutions to the eigenvalue equation,(2.97), are of the form

$$\psi(\rho, \varphi, z) \cong \cos(\frac{n\pi z}{L})e^{im\varphi}L_{mN}(\rho),$$

with the eigenvalue

$$\varepsilon_\lambda = \frac{1}{\tau_m} + \frac{1}{\tau_{so}} + D(\frac{n\pi}{L})^2 + [(N + \frac{1}{2}) + \frac{1}{2}(|m| - m)](\frac{4DeH}{c}),$$

where $\lambda = (n, N, m), n = 0, 1, 2, ..., N = 0, 1, 2, ...$ and $m = 0, \pm 1, \pm 2, ...$

The lowest eigenvalue, ε_λ is obtained by setting $\lambda = (0, 0, 0)$ i.e.,

$$\varepsilon_0 = \frac{1}{\tau_m} + \frac{1}{\tau_{so}} + \frac{2DeH}{c}. \tag{2.104}$$

Because of the orthogonality of $\varphi's$ and the constant value of τ_m, τ_{so}, I and NV. Eq. (2.100) is reduced to

$$\Gamma_{\lambda\lambda'} = [(2|\omega| + \varepsilon_\lambda) + \frac{2}{\tau_{so}} + \frac{4I^2}{(2|\omega| + \varepsilon_\lambda)}]\delta_{\lambda\lambda'},$$

and the secular equation,(2.103), becomes

$$\det|\delta_{\lambda\lambda'}(1 - 2\pi TNV\sum_\omega \Gamma^{-1})| = 0. \tag{2.105}$$

Before performing the frequency summation, we use the partial fraction method to write

$$\Gamma^{-1} = \frac{1}{(2|\omega| + \varepsilon_0) + \frac{2}{\tau_{so}} + \frac{4I^2}{(2|\omega|+\varepsilon_0)}},$$
$$= \frac{A_+}{2|\omega| + \omega_-} + \frac{A_-}{2|\omega| + \omega_+}, \tag{2.106}$$

where

$$A_\pm = \frac{1}{2}(1 \pm \frac{1/\tau_{so}}{\sqrt{(1/\tau_{so})^2 - (2I)^2}}), \tag{2.107}$$

and

$$\omega_\pm = \frac{1}{\tau_m} + \frac{1}{\tau_{so}} + \frac{2DeH}{c} \pm \sqrt{(\frac{1}{\tau_{so}})^2 - (2I)^2}. \tag{2.108}$$

Finally, we obtain the transition temperature equation which includes all pair-breaking parameters as follows

$$\ln\frac{T_c}{T_{c0}} = \psi(\frac{1}{2}) - A_+\psi(\frac{1}{2} + \frac{\omega_-}{4\pi T_c}) - A_-\psi(\frac{1}{2} + \frac{\omega_+}{4\pi T_c}), \tag{2.109}$$

with the BCS transition temperature T_{c0} denotes by

$$\frac{1}{NV} = \ln\frac{1.14\omega_D}{T_{c0}}.$$

The result (2.109) shows that the Pauli spin paramagnetism, I, coupled to the spin-orbit scattering. Although the magnetic impurity scattering is independent of the Pauli spin paramagnetic term, it may enhance the superconducting critical temperature. Note that the result does not depend on the thickness, L, this corresponds to the thin layer limit for the sandwich systems.

2.8 Usadel equations

Almost all theoretical work on type II superconductors has been laid on the Gorkov formulation of a set of coupled equation for the normal and anomalous Green's functions. Eilenberger (1968) derived transport-like equations by transforming the set of Gorkov's equations to the set for integrated Green's function. Usadel (1970) studied Eilenberger's equations for a dirty superconductor. Diffusion-like equation is derived in a short mean-free path limit. According to the Eilenberqer transport-like equation

$$[2\omega + \vec{v}.(\nabla - 2ie\vec{A}(\vec{r}))]f(\omega, \vec{r}, \vec{k}) = 2\triangle(\vec{r})g(\omega, \vec{r}, \vec{k})$$

$$+\frac{n_i}{(2\pi)^2}\int \frac{d^2k'}{v'}|u(\vec{k}, \vec{k'})|^2[g(\omega, \vec{r}, \vec{k})f(\omega, \vec{r}, \vec{k'}) - f(\omega, \vec{r}, \vec{k})g(\omega, \vec{r}, \vec{k'})], \qquad (2.110)$$

where $g(\omega, \vec{r}, \vec{k})$ and $f(\omega, \vec{r}, \vec{k})$ are the energy integrated Green's functions, $\omega = (2n+1)\pi T$ is the Matsubara frequency, $\vec{v}$ is the Fermi velocity, $\vec{A}(\vec{r})$ is a vector potential of the magnetic field, n_i the density of impurities and $u(\vec{k})$ the Fourier transformed impurity potential. The functions f and g are connected by the normalization condition

$$g(\omega, \vec{r}, \vec{k}) = \sqrt{1 - f(\omega, \vec{r}, \vec{k})f^*(\omega, \vec{r}, \vec{k})} \qquad (2.111)$$

The self-consistency condition is given by

$$\triangle(\vec{r})\ln\frac{T}{T_c} + 2\pi T\sum_{\omega>0}[\frac{\triangle(\vec{r})}{\omega} - \oint\frac{d\Omega_{\vec{k}}}{4\pi}f(\omega, \vec{r}, \vec{k})] = 0, \qquad (2.112)$$

where T_c denotes the transition temperature of the free field bulk superconductor, for the diluted impurity one can assume that the motion of electron nearly isotropic in the momentum space, therefore we expand the functions $f(\omega, \vec{r}, \vec{k})$ and $g(\omega, \vec{r}, \vec{k})$ in terms of the spherical harmonics by retaining only $l = 0$ and 1 terms,

$$f(\omega, \vec{r}, \vec{k}) = f_0(\omega, \vec{r}) + \hat{k}.\vec{f}(\omega, \vec{r}) \qquad (2.113)$$

$$g(\omega, \vec{r}, \vec{k}) = g_0(\omega, \vec{r}) + \hat{k}.\vec{g}(\omega, \vec{r}) \qquad (2.114)$$

here $\hat{k} = \vec{k}/k$ and f_0, g_0 are scalar functions while $\vec{f}$ and $\vec{g}$ are vector ones.

Using (2.111),(2.113),(2.114) and separating along the order of spherical harmonics we have

$$g_0(\omega,\vec{r}) = \sqrt{1-|f_0(\omega,\vec{r})|^2} \tag{2.115}$$

$$\vec{g}(\omega,\vec{r}) = -\frac{f_0^*(\omega,\vec{r})\vec{f}(\omega,\vec{r}) + f_0(\omega,\vec{r})\vec{f}^*(\omega,\vec{r})}{2g_0(\omega,\vec{r})}. \tag{2.116}$$

Integrating (2.110) over all solid angle we get

$$2\omega f_0(\omega,\vec{r}) + \frac{v}{3}(\nabla - 2ie\vec{A}(\vec{r})).\vec{f}(\omega,\vec{r}) = 2\triangle(\vec{r})g_0(\omega,\vec{r}), \tag{2.117}$$

and multiplying through (2.110) by $\hat{k}$ and integrating over all solid angle gives

$$\begin{aligned} 2\omega\vec{f}(\omega,\vec{r}) + v(\nabla - 2ie\vec{A}(\vec{r}))f_0(\omega,\vec{r}) &= 2\triangle(\vec{r})\vec{g}(\omega,\vec{r}) \\ &+\frac{1}{\tau_{tr}}[\vec{g}(\omega,\vec{r})f_0(\omega,\vec{r}) - \vec{f}(\omega,\vec{r})g_0(\omega,\vec{r})], \end{aligned} \tag{2.118}$$

where τ_{tr} is the transport scattering time.

The vector functions $\vec{f}$ and $\vec{g}$ of (2.117) and (2.118) can be eliminated by using (2.116), with straightforward calculations we obtain the Usadel equation

$$\begin{aligned} 2\omega f_0(\omega,\vec{r}) - D(\nabla - 2ie\vec{A}(\vec{r})).[g_0(\omega,\vec{r})(\nabla - 2ie\vec{A}(\vec{r}))f_0(\omega,\vec{r}) \\ +\frac{f_0(\omega,\vec{r})}{2g_0(\omega,\vec{r})}\nabla|f_0(\omega,\vec{r})|^2] = 2\triangle(\vec{r})g_0(\omega,\vec{r}), \end{aligned} \tag{2.119}$$

where $D = v^2\tau_{tr}/3$ is the diffusion coefficient.

Finally the self-consistency condition of the order parameter in the lowest order is given by

$$\triangle(\vec{r})\ln\frac{T}{T_c} + 2\pi T\sum_{\omega>0}[\frac{\triangle(\vec{r})}{\omega} - f_0(\omega,\vec{r})] = 0. \tag{2.120}$$

We shall show that in the vicinity of T_c, the Usadel equations lead to the de Gennes kernel of the linearized self-consistency equation. Near T_c, $f_0 \to 0$ while $g_0 \to 1$ then (2.119) can be linearized as

$$[2\omega - D(\nabla - 2ie\vec{A})^2]f_0(\omega,\vec{r}) = 2\triangle(\vec{r}). \tag{2.121}$$

Fourier transforming of this equation with neglecting the vector potential gives

$$f_0(\omega, \vec{q}) = \frac{2\triangle(\vec{q})}{2\omega + Dq^2}. \tag{2.122}$$

The self-consistency condition of the order parameter (2.120) can be expressed as

$$\triangle(\vec{q}) = K(\vec{q})\triangle(\vec{q}), \tag{2.123}$$

where the de-Gennes - Gorkov kernel $K(\vec{q})$ in the dirty limit reads as

$$K(\vec{q}) = 2\pi T \sum_{\omega} \frac{1}{2\omega + Dq^2}. \tag{2.124}$$

We see that the diffusion-like Usadel equations are equivalent to the Gorkov Green's function and the de-Gennes correlation function method.

2.9 Transition temperature of superconductor- ferromagnet superlattices

Superconductivity and ferromagnetism are two antagonistic orderings. With an advance of nano-fabricated multilayers, the interaction of superconducting (S) and ferromagnetic (F) layers exhibits phenomena such as coexistence of superconductivity and ferromagnetism, reentrant behavior, and the oscillatory critical current. The ferromagnetic exchange field tends to polarize the electron spins, then breaking the Cooper pairs inside F and thus strongly suppresses the superconducting order parameter in the vicinity of an S/F interface. Besides the pair breaking effect, the possibility of $\pi-$ phase difference between two neighboring S layers is an interesting feature, these features are oscillatory of T_c with a variation of F thicknesses. In this section we review the work of Radovic et al,(1991b), who calculated T_c by solving the Usadel equations in the exact multimode method.

We start with the Usadel's dirty limit version of the Eilenberger equations

$$-\frac{D}{2}\nabla[G(\vec{r},\omega)\nabla F(\vec{r},\omega) - F(\vec{r},\omega)\nabla G(\vec{r},\omega)] = G(\vec{r},\omega)\triangle(\vec{r}) - \omega F(\vec{r},\omega), \quad (2.125)$$

and the normalization condition

$$G^2(\vec{r},\omega) + F(\vec{r},\omega)F^\dagger(\vec{r},\omega) = 1. \quad (2.126)$$

Here, D is the diffusion coefficient, $\omega = (2n+1)\pi T$ with n being an integer, $\triangle(\vec{r})$ is the pair potential and the functions $F(\vec{r},\omega)$ and $G(\vec{r},\omega)$ are Gorkov's Green functions integrated over energy.

We consider the superlattice consisting of an alternating S and F layers with thicknesses d_s and d_f, the modulation taking along the x-axis and has the repeated structure $L = d_s + d_m$.

Near the second order phase transition, $G = sgn(\omega)$, the Usadel equations reduce to a linearized form,

$$(|\omega| - \frac{1}{2}\frac{d^2}{dx^2})F_s(x,\omega) = \triangle_s(x), \quad (2.127)$$

for S layers and

$$(\frac{d^2}{dx^2} - k_f^2)F_f(x,\omega) = 0, \quad (2.128)$$

with

$$k_f^2 = \frac{2}{D_f}(|\omega| + iIsgn(\omega)),$$

for F layers with $\triangle = 0$ means that there is no the formation of Cooper pairs in F but $F_f \neq 0$ due to the proximity of S. We consider the case where the ferromagnetic exchange field is so strong, $|I| >> T_{cS}$, then

$$k_f^2 = \frac{2i}{D_f}Isgn(\omega). \quad (2.129)$$

The supplementary equation in order to complete the Usadel equations, (2.127) and (2.128) is

$$\triangle_s(x) = \lambda\pi T \sum_{\omega} F_s(x,\omega), \tag{2.130}$$

where the BCS coupling constant,

$$\lambda = (\ln \frac{1.134\omega_D}{T_{cS}})^{-1}, \tag{2.131}$$

T_{cS} is the bulk critical temperature and ω_D the Debye cutoff frequency.

The functions $F_s(x,\omega)$ and $F_f(x,\omega)$ are connected by the boundary conditions at each S/F interface

$$F_s(x,\omega) = F_f(x,\omega), \tag{2.132}$$

$$\frac{d}{dx}F_s(x,\omega) = \eta\frac{d}{dx}F_f(x,\omega) \tag{2.133}$$

where the phenomenological parameter $\eta = \sigma_f/\sigma_s$ is the ratio of the normal state conductivities. The periodicity of the superlattice is subject to the Bloch condition

$$F(x+L) = e^{i\varphi}F(x), \tag{2.134}$$

here the phase difference φ takes values $0 \le \varphi \le \pi$. Using (2.132)-(2.134) we get the set of boundary conditions

$$F_s(0,\omega) = e^{-i\varphi}F_f(d_s+d_f,\omega), \tag{2.135}$$

$$F_s(d_s,\omega) = F_f(d_s,\omega), \tag{2.136}$$

$$\frac{d}{dx}F_s(d_s,\omega) = \eta\frac{d}{dx}F_f(d_s,\omega), \tag{2.137}$$

$$\frac{d}{dx}F_s(0,\omega) = e^{-i\varphi}\eta\frac{d}{dx}F_f(d_s+d_f,\omega). \tag{2.138}$$

The Usadel equation for F, (2.128) has the solution

$$F_f(x,\omega) = C_1e^{k_fx} + C_2e^{-k_fx}, \tag{2.139}$$

The coefficient C_1 and C_2 are determined by using (2.135) and (2.136) and we find

$$C_1 = \frac{e^{-k_f d_s}}{2\sinh(k_f d_f)}(e^{i\varphi}F_s(0,\omega) - e^{-k_f d_f}F_s(d_s,\omega)),$$

$$C_2 = -\frac{e^{k_f d_s}}{2\sinh(k_f d_f)}(e^{i\varphi}F_s(0,\omega) - e^{k_f d_f}F_s(d_s,\omega)),$$

The boundary conditions, (2.137) and (2.138), when using (2.139), give

$$\frac{d}{dx}F_s(0,\omega) = \eta k_f[\coth(k_f d_f)F_s(0,\omega) - \frac{e^{-i\varphi}}{\sinh(k_f d_f)}F_s(d_s,\omega)], \tag{2.140}$$

$$\frac{d}{dx}F_s(d_s,\omega) = -\eta k_f[\coth(k_f d_f)F_s(d_s,\omega) - \frac{e^{i\varphi}}{\sinh(k_f d_f)}F_s(0,\omega)], \tag{2.141}$$

In order to solve (2.127) and (2.130) in an exact multimode method, we introduce the Fourier transforms of the Usadel function of S, namely,

$$F_s(x,\omega) = \sum_{m=-\infty}^{\infty} F_s(q_m,\omega)\cos(q_m x), \tag{2.142}$$

$$F_s(q_m,\omega) = \frac{1}{d_s}\int_0^{d_s} dx F_s(x,\omega)\cos(q_m x), \tag{2.143}$$

as well as the superconducting order parameter

$$\triangle_s(x) = \sum_{m=-\infty}^{\infty} \triangle_s(q_m)\cos(q_m x), \tag{2.144}$$

$$\triangle_s(q_m) = \frac{1}{d_s}\int_0^{d_s} dx \triangle_s(x)\cos(q_m x), \tag{2.145}$$

with an eigenmode $q_m = m\pi/d_S$, m being integer, this implies the orthogonality relation

$$\int_0^{d_s} dx \cos(q_m x)\cos(q_{m'} x) = d_s \delta_{mm'}. \tag{2.146}$$

The Usadel equation (2.127) and the superconducting order parameter, (2.130) are transformed as follows

$$\triangle_s(q_m) = (|\omega| + \frac{1}{2}D_s q_m^2)F_s(q_m,\omega) - \frac{D_s}{2d_s}[(-1)^m \frac{d}{dx}F_s(d_s,\omega) - \frac{d}{dx}F_s(0,\omega)], \tag{2.147}$$

$$\triangle_s(q_m) = \lambda\pi T \sum_{\omega} F_s(q_m,\omega), \tag{2.148}$$

respectively. Inserting the boundary conditions (2.140) and (2.141) into (2.147) give the relation between $F_s(q_m,\omega)$ and $\triangle_s(q_m)$ as

$$\triangle_s(q_m) = \sum_{m'=-\infty}^{\infty} \Gamma_{mm'}(\omega) F_s(q_{m'},\omega), \tag{2.149}$$

where the matrix element of $\Gamma(\omega)$

$$\Gamma_{mm'}(\omega) = (|\omega| + \frac{1}{2}D_s q_m^2)\delta_{mm'} + \frac{D_s\eta k_f}{2d_s}([1+(-1)^{m+m'}]\coth(k_f d_f)$$

$$-[(-1)^m e^{i\varphi} + (-1)^{m'} e^{-i\varphi}]\frac{1}{\sinh(k_f d_f)}). \tag{2.150}$$

We must now express $F_s(q_m,\omega)$ in terms of $\triangle_s(q_m)$,

$$F_s(q_m,\omega) = \sum_{m'=-\infty}^{\infty} \Gamma_{mm'}(\omega)\triangle_s(q_{m'}). \tag{2.151}$$

Combining the above equation with (2.148) we obtain

$$\sum_{m'=-\infty}^{\infty} A_{mm'}\triangle_S(q_{m'}) = 0, \tag{2.152}$$

where

$$A_{mm'} = \delta_{mm'} - \lambda\pi T \sum_{\omega} \Gamma^{-1}_{mm'}(\omega), \tag{2.153}$$

here one has

$$\Gamma^{-1}_{mm'} = \frac{1}{|\omega| + \frac{1}{2}D_s q_m^2}\delta_{mm'} - \frac{c_\omega}{(|\omega| + \frac{1}{2}D_s q_m^2)(|\omega| + \frac{1}{2}D_s q_{m'}^2)}\frac{X}{Y}, \tag{2.154}$$

$$X = [1+(-1)^{m+m'}][a_\omega c_\omega + \coth(k_f d_f)] - [(-1)^m + (-1)^{m'}]b_\omega c_\omega$$

$$-[(-1)^m e^{i\varphi} + (-1)^{m'} e^{-i\varphi}]\frac{1}{\sinh(k_f d_f)} \tag{2.155}$$

$$Y = 1 + 2c_\omega[a_\omega \coth(k_f d_f) - b_\omega \frac{\cos\varphi}{\sinh(k_f d_f)}] + c_\omega^2(a_\omega^2 - b_\omega^2), \tag{2.156}$$

with

$$a_\omega = \sum_{m=-\infty}^{\infty} \frac{1}{|\omega| + \frac{1}{2} D_s q_m^2} = \sqrt{\frac{2d_s^2}{D_s|\omega|}} \coth \sqrt{\frac{2d_s^2|\omega|}{D_s}}, \tag{2.157}$$

$$b_\omega = \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{|\omega| + \frac{1}{2} D_s q_m^2} = \sqrt{\frac{2d_s^2}{D_s|\omega|}} \frac{1}{\sinh \sqrt{\frac{2d_s^2|\omega|}{D_s}}}, \tag{2.158}$$

$$c_\omega = \frac{D_s \eta k_f}{2d_s}, \tag{2.159}$$

we note that k_f is frequency dependent. The coherence length of Cooper pairs in S layers denotes by

$$\xi_s = \sqrt{\frac{D_s}{2\pi T_{cS}}}, \tag{2.160}$$

while the penetration length of Cooper pairs in F layers is

$$\xi_f = \sqrt{\frac{4D_f}{I}}. \tag{2.161}$$

The nontrivial solutions to (2.153) obey the secular equation

$$\det |A| = 0, \tag{2.162}$$

which provides the transition temperature.

In the single-mode approximation or the Cooper-de Gennes thin film limit, when the S layers are very thin, the single-mode approximation is valid. In this case only the $(0,0)$ component of $\Gamma^{-1}_{mm'}(\omega)$, (2.154), is nonvanishing,

$$\Gamma^{-1}_{00}(\omega) = \frac{1}{|\omega| + 2\pi T_{cS}\rho} \tag{2.163}$$

where the pair breaking parameters

$$\rho = \frac{\xi_s^2 \eta k_f}{d_s} \tanh \frac{k_f d_f}{2}, \tag{2.164}$$

for $\varphi = 0$ (0- phase) and

$$\rho = \frac{\xi_s^2 \eta k_f}{d_s} \coth \frac{k_f d_f}{2}, \tag{2.165}$$

for $\varphi = \pi$(π-phase). So the transition temperature reduces to the well-known expression

$$\ln \frac{T_c}{T_{cS}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{T_{cS}}{T_c}\rho), \tag{2.166}$$

where ψ is the digamma function. This expression looks like the bulk superconductor in the presence of the diluted paramagnetic impurity which was obtained by Abrikosov and Gorkov in 1961. The reduction of the transition temperature due to the Cooper pair is destroyed by the spin of impurity atoms. Then (2.166) shows that the short-period S/F superlattices behave like a bulk superconductor in a uniform exchange field.

2.10 Perpendicular upper critical field $H_{c2\perp}$ of superconductor- ferromagnet superlattices

In the previous section the Usadel equations are solved exactly to determine the transition temperature of S/F superlattices. Here we calculate the perpendicular upper critical field $H_{c2\perp}$ of S/F superlattices by solving the Usadel equations in the multimode method following Radovic, Ledvij and Dobrosavljevic-Grujic (1991c).

Consider the S/F superlattice parallel to the x-y plane in a perpendicular magnetic field $\vec{H} = H\hat{z}$. Near the second order phase transition the Usadel equations for the anomalous functions $F_s(\vec{r},\omega)$ and $F_f(\vec{r},\omega)$ are

$$\frac{D_s}{2}\vec{\Pi}^2 F_s(\vec{r},\omega) - |\omega| F_s(\vec{r},\omega) = -\triangle(\vec{r}), \tag{2.167}$$

for S and

$$\frac{D_f}{2}\vec{\Pi}^2 F_f(\vec{r},\omega) - (|\omega| + iIsgn(\omega)) F_f(\vec{r},\omega) = 0, \tag{2.168}$$

for F.

The self-consistency equation supplemented with (2.167) and (2.168) is

$$\triangle_s(\vec{r}) = \lambda_s \pi T \sum_{\omega} F_s(\vec{r}, \omega) \tag{2.169}$$

here, there is no pairing in F, $\lambda_f = 0$, But $F_f \neq 0$ due to the proximity of S. $D_{s,f}$ is the diffusion coefficient, I is the ferromagnetic exchange field inside F, $\omega = (2n+1)\pi T$ with n being integer, the gauge-invariant operator has the components as

$$\vec{\Pi} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + \frac{2\pi i H}{\phi_0} x, \frac{\partial}{\partial z}), \tag{2.170}$$

in which the gauge $\vec{A} = (0, Hx, 0)$ is chosen, and $\phi_0 = hc/2e$ is the flux quantum. The boundary conditions at each S/F interfaces are

$$F_s(\vec{r}, \omega) = F_f(\vec{r}, \omega), \tag{2.171}$$

$$\frac{d}{dz} F_s(\vec{r}, \omega) = \eta \frac{d}{dz} F_f(\vec{r}, \omega), \tag{2.172}$$

with $\eta = \sigma_f/\sigma_s$ the ratio of normal state conductivities. The periodicity of superlattices having the repeated structure $L = d_s + d_f$. Both F_s and F_f satisfy the Bloch condition

$$F(\vec{r} + L\hat{z}) = e^{i\varphi} F(\vec{r}), \tag{2.173}$$

Assume the Usadel functions $F_{s,f}$ can be separated according to the fact that the modulation of Abrikosov vortex lattices occur only on the z-direction, this assumption is true because of the perpendicular magnetic field normal to the x-y plane,

$$F_{s,f}(\vec{r}, \omega) = f(x, y) g_{s,f}(z, \omega). \tag{2.174}$$

For F with the relation

$$k_f^2 = \frac{2}{D_f}(|\omega| + iIsgn(\omega)),$$

of the propagating momentum in the ferromagnet, (2.168) becomes

$$\frac{d^2}{dz^2}g_f(z,\omega) = q_f^2 g_f(z,\omega), \qquad (2.175)$$

and

$$[\frac{\partial^2}{\partial x^2} + (\frac{\partial}{\partial y} + \frac{2i\pi H}{\phi_0}x)^2]f(x,y) = (q_f^2 - k_f^2)f(x,y). \qquad (2.176)$$

The lowest eigenvalue of $f(x,y)$ gives the perpendicular upper critical field,

$$\frac{2\pi}{\phi_0}H_{c2\perp} = q_f^2 - k_f^2. \qquad (2.177)$$

Since we have assumed $f(x,y)$ to be identical for both metals then for S layers we get

$$\frac{D_s}{2}(\frac{d^2}{dz^2} - \frac{2\pi}{\phi_0}H_{c2\perp})F_s(z,\omega) - |\omega|F_s(z,\omega) = -\triangle(z). \qquad (2.178)$$

Relating $F_s(z,\omega)$ to $F_f(z,\omega)$ via the boundary conditions, (2.171)- (2.173), the results are

$$\frac{d}{dz}F_s(0,\omega) = \eta q_f[\coth(q_f d_f)F_s(0,\omega) - \frac{e^{-i\varphi}}{\sinh(q_f d_f)}F_s(d_s,\omega)], \qquad (2.179)$$

$$\frac{d}{dz}F_s(d_s,\omega) = -\eta q_f[\coth(q_f d_f)F_s(d_s,\omega) - \frac{e^{i\varphi}}{\sinh(q_f d_f)}F_s(0,\omega)]. \qquad (2.180)$$

Introduce the Fourier transforms of $F_s(z,\omega)$

$$F_s(z,\omega) = \sum_{m=-\infty}^{\infty} F_s(Q_m,\omega)\cos(Q_m z), \qquad (2.181)$$

$$F_s(Q_m,\omega) = \frac{1}{d_s}\int_0^{d_s} dz F_s(z,\omega)\cos(Q_m z), \qquad (2.182)$$

as well as for $\triangle_s(z)$ and $\triangle_s(Q_m)$, the eigenmode has the relation

$$Q_m = \frac{m\pi}{d_s}. \qquad (2.183)$$

We repeat the whole argument in solving the superconducting order parameter $\triangle_S(z)$, (2.169) together with the Usadel equation,(2.178), as in the previous section, we arrive at the multimode solution

$$\sum_{m'=-\infty}^{\infty} A_{mm'} = 0, \tag{2.184}$$

with

$$A_{mm'} = \delta_{mm'} - \lambda\pi T \sum_{\omega} \Gamma^{-1}_{mm'}, \tag{2.185}$$

here one has

$$\Gamma^{-1}_{mm'} = \frac{1}{|\tilde{\omega}| + \frac{1}{2}D_s q_m^2}\delta_{mm'} - \frac{c_\omega}{(|\tilde{\omega}| + \frac{1}{2}D_s q_m^2)(|\tilde{\omega}| + \frac{1}{2}D_s q_{m'}^2)}\frac{X}{Y}, \tag{2.186}$$

$$X = [1 + (-1)^{m+m'}][a_\omega c_\omega + \coth(q_f d_f)] - [(-1)^m + (-1)^{m'}]b_\omega c_\omega$$

$$-[(-1)^m e^{i\varphi} + (-1)^{m'} e^{-i\varphi}]\frac{1}{\sinh(q_f d_f)}, \tag{2.187}$$

$$Y = 1 + 2c_\omega[a_\omega \coth(q_f d_f) - b_\omega \frac{\cos\varphi}{\sinh(q_f d_f)}] + c_\omega^2(a_\omega^2 - b_\omega^2), \tag{2.188}$$

where

$$|\tilde{\omega}| = |\omega| + \frac{\pi D_s}{\phi_0} H_{c2\perp}, \tag{2.189}$$

$$a_\omega = \sum_{m=-\infty}^{\infty} \frac{1}{|\tilde{\omega}| + \frac{1}{2}D_s Q_m^2} = \sqrt{\frac{2d_s^2}{D_s|\tilde{\omega}|}} \coth\sqrt{\frac{2d_s^2|\tilde{\omega}|}{D_s}}, \tag{2.190}$$

$$b_\omega = \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{|\tilde{\omega}| + \frac{1}{2}D_s Q_m^2} = \sqrt{\frac{2d_s^2}{D_s|\tilde{\omega}|}} \frac{1}{\sinh\sqrt{\frac{2d_s^2|\tilde{\omega}|}{D_s}}}, \tag{2.191}$$

$$c_\omega = \frac{D_s \eta q_f}{2d_s}, \tag{2.192}$$

The nontrivial solutions of (2.184)

$$\det|A| = 0, \tag{2.193}$$

gives the perpendicular upper critical field $H_{c2\perp}$ as functions of temperature. In doing the numerical calculations, the following approximations are made

- Strong ferromagnets $I \gg T_{cS}$, $k_f^2 \approx 2iIsgn(\omega)/D_f$,
- Neglecting the orbital magnetic field in F layers, i.e., $q_f^2 = k_f^2$ or the critical field $H_{c2\perp}$ exists only in S layers.

2.11 Effect of spin orientation dependence on transition temperatures in ferromagnet/superconductor/ ferromagnet trilayers

The studies of proximity effects between superconductor/ ferromagnet (S/F) artificial superlattices suggest the possibility of the FFLO phase. The oscillatory behavior of the critical temperature in the ferromagnetic layer is a direct evidence of such state in which there exists the π-phase superconducting state, the adjacent superconducting layers have opposite phase. The interplay between superconductivity and magnetism in F/S/F sandwiches will be examined in details in this section. The dependence of the critical temperature on the mutual orientation of ferromagnetic moments of the outer layers is calculated for both the parallel and antiparallel orientations of ferromagnetic moments by solving the Usadel equations in the single mode approximation, i.e., the Cooper limit when the thickness of a superconductor is smaller than or of the order of the superconducting coherence length, $d_s < \xi_s$. The theoretical studies of the mutual orientation of magnetizations alignment has been considered by Tagirov(1999) for the case of the arbitrary F-layer thickness d_f, by taking the effect of the finite boundary transparency interface into account and by Buzdin, Vedyayev and Ryzhanova (1999) and Baladie, Buzdin, Ryzhonova and Vedyayev (2001) for the case of the thick F layers with the highly transparent F/S interfaces.

The calculated results show that the superconducting transition temperature for the parallel magnetization alignment is always lower than the antiparallel one and demonstrate the occurrence of the superconducting π-phase even in tri-layer structures.

We shall examine the F/S/F structure assuming that the dirty limit conditions are held in the superconducting layer. Let the superconductor and ferromagnet located in x-y plane, the modulation of the pair amplitude is taking along the x-direction. The origin is defined at the middle of the superconducting layer with thickness be $2d_s$ whereas d_f is the thickness of ferromagnet layers.

We start with the Usadel equations that describe the diffusion motion of electrons in dirty superconductors. For the proximity system one has

$$(|\omega| - \frac{1}{2}D_s\frac{d^2}{dx^2})F_s(x,\omega) = \triangle(x), \tag{2.194}$$

in S and

$$(\frac{d^2}{dx^2} - k_f^2)F_f(x,\omega) = 0, \tag{2.195}$$

in F with the the propagation wave vector in ferromagnets obeys

$$k_f^2 = \frac{2}{D_f}(|\omega| + iI(x)sgn(\omega)). \tag{2.196}$$

here $D_{s(f)} = v_{s(f)}l_{s(f)}/3$ is the diffusion coefficient in S(F) regions, $v_{s,(f)}$ and $l_{s(f)}$ are the Fermi velocity and the electron mean free path in a given layer, $\omega = (2n+1)\pi T$ is the Matsubara frequency, $F(x,\omega)$ is the spatial Gorkov's anomalous function, the pair potential, $\triangle_s$ is assumed to exist only in the S layer whereas the pairing in F layers be zero but the leakage of the pair amplitude from S to F is allowed due to the proximity. The ferromagnetic exchange field, $I(x)$, may either parallel or antiparallel alignment in F layers; for the parallel orientation,

$$I(x > d_s) = I = I(x < -d_s), \tag{2.197}$$

and for the antiparallel one

$$I(x > d_s) = I = -I(x < -d_s). \tag{2.198}$$

The Usadel equations, (2.194) and (2.195), must be supplemented to the self-consistency condition for the superconducting order parameter

$$\triangle_s(x) \ln \frac{T_{cS}}{T_c} = \pi T_c \sum_{\omega} (\frac{\triangle_s(x)}{|\omega|} - F_s(x, \omega)), \tag{2.199}$$

with T_{cS} is the bulk transition temperature and the summation is taken for both positive and negative integers.

For simplicity we considering the case of highly transparent F/S interfaces (it means a small potential barrier at the S/F interfaces). The boundary conditions at the interfaces $x = \pm d_s$ are

$$F_s(x, \omega) = F_f(x, \omega), \tag{2.200}$$

$$\frac{d}{dx} F_s(x, \omega) = \eta \frac{d}{dx} F_f(x, \omega), \tag{2.201}$$

where $\eta = \sigma_f / \sigma_s$ is the ratio of the normal-state conductivities, and $\eta \ll 1$ will be assumed to follow from the hypothesis of the weak proximity effect.

We seek the solution of (2.194) in the single-mode approximation which is familiar in the Abrikosov and Gorkov theory for the magnetic impurity

$$F_s(x, \omega) = \frac{\triangle_s \cos(kx)}{|\omega| + \rho} + f(x, \omega), \tag{2.202}$$

where $\triangle_s(x) = \triangle_s \cos(kx)$ and $\rho = D_s k^2/2$ is the pair-breaking parameter that is to be determined from the self-consistency condition, (2.199). Upon substituting (2.202) into (2.194), we get the equation for $f(x, \omega)$,

$$(|\omega| - \frac{1}{2} D_s \frac{d^2}{dx^2}) f(x, \omega) = 0.$$

The above equation has the solution as

$$f(x,\omega) = A_\omega e^{k_s x} + B_\omega e^{-k_s x}, \tag{2.203}$$

where $k_s = \sqrt{2|\omega|/D_s}$.

Applying (2.202) in (2.199) with splitting the sum over ω into two contributions, $\omega < 0$ and $\omega > 0$, and then rearranging the summation we obtain

$$\triangle_s(x) \ln \frac{T_{cS}}{T_c} = 2\pi T_c \sum_{\omega>0} [(\frac{1}{|\omega|} - \frac{1}{|\omega| + \rho})\triangle_s(x) - \frac{1}{2}(f(x,\omega) + f(x,-\omega))]. \tag{2.204}$$

Observing that $\triangle_s(x)$ behaves in an oscillatory manner while $F_s(x,\omega)$ behaves as an exponential decay thus to obtain the reduced transition temperature equation, we therefore must impose the condition

$$f(x,\omega) + f(x,-\omega) = 0, \tag{2.205}$$

then (2.204) becomes

$$\ln \frac{T_{cS}}{T_c} = 2\pi T_c \sum_{\omega>0} (\frac{1}{|\omega|} - \frac{1}{|\omega| + \rho}). \tag{2.206}$$

Since we have assumed that the Cooper limit for the superconducting layer, $d_s \ll \xi_s$, where $\xi_s = \sqrt{D_s/2\pi T_{cS}}$ being the coherence length, then (2.203) and (2.205) imply

$$(A_\omega + B_\omega) + (A_{-\omega} + B_{-\omega}) = 0. \tag{2.207}$$

AP-Phase

For the case of an antiparallel alignment of magnetizations in F-layers, i.e, $I(x > d_s) = I = -I(x < -d_s)$. The solutions of (2.195) are

$$F_f(x,\omega) = C_\omega \cosh(k_f[x - d_s - d_f]), \tag{2.208}$$

for $d_s \le x \le d_s + d_f$, and

$$F_f(x,\omega) = D_\omega \cosh(k_f^*[x + d_s + d_f]), \qquad (2.209)$$

for $-(d_s + d_f) \le x \le -d_s$, here $k_f = (1+i)\sqrt{I/D_f}$ is the approximation of (2.196) in the strong exchange field limit, $I \gg T_{cS}$. Note that (2.208) and (2.209) satisfy the condition of no supercurrent passing through the free boundaries at $x = \pm(d_s + d_f)$, i.e.,

$$\frac{d}{dx}F_f(x,\omega) = 0$$

Using the boundary conditions (2.200) and (2.201) with the Usadel equations for S, (2.202) and(2.203), and for F, (2.208) and (2.209). The equations for the coefficients A_ω and B_ω are

$$[k_s + \eta k_f \tanh(k_f d_f)]A_\omega e^{k_s d_s} - [k_s - \eta k_f \tanh(k_f d_f)]B_\omega e^{-k_s d_s}$$
$$= [k \tan(k d_s) - \eta k_f \tanh(k_f d_f)]\frac{\triangle_s(d_s)}{|\omega| + \rho}, \qquad (2.210)$$

at the interface $x = d_s$ and

$$[k_s + \eta k_f^* \tanh(k_f^* d_f)]A_\omega e^{k_s d_s} - [k_s - \eta k_f \tanh(k_f d_f)]B_\omega e^{-k_s d_s}$$
$$= -[k \tan(k d_s) - \eta k_f^* \tanh(k_f^* d_f)]\frac{\triangle_s(d_s)}{|\omega| + \rho}, \qquad (2.211)$$

at $x = -d_s$. The set of coupled equations, (2.210) and (2.211), give $A_\omega = A_{-\omega}$ and $B_\omega = B_{-\omega}$ then (2.207) becomes

$$A_\omega + B_\omega = 0. \qquad (2.212)$$

Solving (2.210) and (2.211) for the coefficients A_ω, B_ω in the single- mode approximation and using (2.212) we obtain the equation for the pair-breaking parameter $\rho = D_s k^2/2$,

$$k \tan(k d_s) = \frac{1}{2}\eta[k_f \tanh(k_f d_f) + k_f^* \tanh(k_f^* d_f)]. \qquad (2.213)$$

The reduced transition temperature in AP-phase can now be written in terms of the digamma function

$$\ln\frac{T_c}{T_{cS}} = \psi(\frac{1}{2}) - Re\psi(\frac{1}{2} + \frac{\rho}{2\pi T_c}). \tag{2.214}$$

P-Phase

For the parallel alignment of magnetizations, the pair-breaking parameter equation (2.213) is replaced by

$$k\tan(kd_s) = \eta k_f \tanh(k_f d_f), \tag{2.215}$$

here we have changed from k_f^* to k_f.

In the limit of thick ferromagnetic layers, $d_f \gg \xi_f$, where $\xi_f = \sqrt{4D_f/I}$ is the penetration length in the ferromagnets, we approximate $\tanh(k_f d_f) \approx 1$, the superconducting layer could be justified in the Cooper limit $d_s/\xi_s \ll 1$. As a result, we get the pair-breaking parameters (2.213) and (2.215) in an approximate form

$$\rho_{AP} = \frac{\eta D_s}{2d_s}\sqrt{\frac{I}{D_f}},$$
$$\rho_P = \frac{\eta D_s}{2d_s}(1+i)\sqrt{\frac{I}{D_f}}.$$

According to (2.214) we arrive at the final results

$$\ln\frac{T_c}{T_{cS}}|_{AP} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{\eta D_s}{4\pi d_s T_c}\sqrt{\frac{I}{D_f}}), \tag{2.216}$$

and

$$\ln\frac{T_c}{T_{cS}}|_P = \psi(\frac{1}{2}) - Re\psi(\frac{1}{2} + (1+i)\frac{\eta D_s}{4\pi d_s T_c}\sqrt{\frac{I}{D_f}}). \tag{2.217}$$

Both equations show that T_c is substantially higher for AP configuration. This means that the AP-phase is energetically preferable to the P phase.

Chapter 3

Upper Critical Field and Transition Temperature in Ferromagnet/Superconductor /Ferromagnet Sandwiches

3.1 Introduction

Superconductivity and ferromagnetism are two antagonistic orderings, it is well known that the former occurs in an antiparallel spin configuration of electron spins, while the latter tends to break the spins of the Cooper pair which leads to the parallel spin orientation of electrons. The question whether superconductivity can coexist with ferromagnetism has been considered long time ago. Fulde and Ferrel (1964) and Larkin and Ovchinnikov (1965) who demonstrated that the inhomogeneous superconducting state may occur in a narrow range of the spin-exchange field. Unfortunately, the Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) phase has not been detected yet. The interplay between superconductivity and magnetism is possible when the two orders are spatially separated. Much attention has been paid to the proximity effect of layered structures consisting of superconductors (S) and ferromagnets (F) in which the nonmonotonic critical temperature is the most striking effect.

Experimentally, this nonmonotonic behavior was observed by Wong et al.(1986) in V/Fe multilayers as a function of the Fe thickness when the V thicknesses were fixed. Instead, the sharp decrease of T_c with increasing Fe thickness, the upturn of T_c at large d_{Fe} for some fixed d_V was found. Subsequent study of the V/Fe system by Koorevaar et al. (1994) and on Nb/Gd triple layers by Strunk et al. (1994) showed no evidence for an oscillatory behavior. Jiang et al. (1995) also studied the Nb/Gd multilayers and observed an oscillatory behavior of T_c as a function of the ferromegnetic Gd layer thickness d_{Gd} for fixed Nb thickness d_{Nb}.

For an explanation of the nonmonotonic T_c behavior, Strunk et al. explained the observed step-like behavior of T_c could be attributed to the change in the underlying pair-breaking mechanism due to the transition of the Gd layer from a paramagnetic to a ferromagnetic state with increasing d_{Gd}. Contrary to Jiang et al. who concluded that their results provide the first evidence for the predicted theoretically $\pi-$phase in S/F multilayers. The situations seem contradicting. Later on, Muhge et al. (1996, 1997) observed for Fe/Nb/Fe trilayers a nonmonotonic dependence of the superconducting transition temperature T_c with increasing d_{Fe} for fixed d_{Nb} which look similar to the observed T_c in Nb/Gd multilayers as reported by Jiang et al. Because of the Fe/Nb/Fe trilayer system, there is only one S layer then the model based on the $\pi-$phase can be ruled out. It was concluded that $T_c(d_{Fe})$ occurs due to the existence of magnetically dead Fe layers near the interface and their properties change drastically upon the onset of ferromagnetic order. The study of this effect has been performed on Nb/Fe bilayers by Muhge et al.(1998) and on Nb/Fe multilayers by Verbanck et al. (1998). Both groups observed the step-like behavior of T_c versus Fe layers thickness.

Until now the superconductor/ferromagnet proximity effect is far from being qualitatively understood (see, Garifullin, 2002). The superconducting T_c increasing with the thickness of the ferromagnetic layer contradicts the physical intuition, since it is expected that the strong exchange field in the ferromagnet should strongly suppresses the superconductivity. Obviously, there is no consensus in the literature concerning the origin of the nonmonotonic T_c behavior. In particular, the results between experiments and theories were not conclusive.

Theoretical study of proximity effect can be divided into two formulation; the Usadel dirty-limit version of Eilenberger theory (Usadel,1970; Eilenberger, 1968) which is the differential equation for transport-like motion and the de Gennes-Takahashi-Tachiki theory based on the correlation function method (de Gennes, 1964; Takahashi and Tachiki, 1986), the latter theory apparently is an integral equation of the former one. Radovic et al.(1991 a,b,c) developed the Usadel equation to calculate the superconducting transition temperature T_c and the perpendicular upper critical field $H_{c2\perp}$. The oscillatory dependence of T_c and $H_{c2\perp}$ on the ferromagnetic layers thickness was first predicted theoretically. Fominov, Chtchelkatchev and Golubov (2002) developed a general method for investigating the nonmonotonic T_c as a function of the S/F bilayer parameters. Various types of nonmonotonic behavior of T_c as a function of d_f, such as a minimum of T_c and even reentrant superconductivity were found.

To explain the T_c oscillation leads Tagirov (1999) proposed a superconducting spin switch device based on the proximity effect of F/S/F trilayers, the spin- dependent pairing function is influenced by the alignment of magnetizations thus the T_c depends on the orientation of the ferromagnet exchange field. Calculations show that the antiparallel configuration of the magnetization is energetically preferable to the parallel configuration. The same result was concluded by Buzdin, Vedyayev and Ryzhanova (1999) and Baladie et al.(2001). It is worth mentioning that the T_c behavior is still controversial in some range of the ferromagnetic thickness. Demler, Arnold and Beasley (1997) suggested that spin-orbit scattering plays a major role in the superconducting proximity effect. The reason is that the exchange interaction coupled to the spin-orbit scattering and yields large modification of the oscillatory T_c. Oh et al.(2000) extended the work of Demler, Arnold and Beasley to calculate the T_c of S/F multilayers, the results indicate the oscillatory behavior of T_c is reduced by the spin-orbit scattering.

Besides the Usadel equations and its application, the Takahashi-Tachiki proximity effect theory first succeeded in explaining the upper critical field of superconductor (S)/normal-metal (N) multilayers. In their formalism the eigenfunction expansion method and the de Gennes boundary conditions were used to obtain the phase diagram of the upper critical field versus temperature by means of numerical calculations. Auvil, Ketterson and Song (1989) generalized the Takahashi-Tachiki theory to include the pair-breaking effects such as orbital diamagnetism, Pauli spin paramagnetism, spin-orbit scattering and magnetic impurity scattering. Auvil and Ketterson (1988) showed that the critical temperature of S/N systems in the limit of small layer thickness is equivalent to the diagonal approximation method, or the Cooper-de Gennes limit. Kuboya and Takanaka (1998) numerically calculated the transition temperature and the upper cirtical field of superconductor/ferromagnetic superlattices. The difficult task in applying the Takahashi- Tachiki theory to the proximity effect of S/F systems is the requirement of the numerical calculation, then the Usadel equations seem to be popularly used more than the Takahashi-Tachiki theory. Apparently, the equivalence between two theories was shown by Lodder and Koperdraad (1993) but limited only to the S/N system.

Motivated by the theoretical and experimental works, the attention is paid to the F/S/F proximity structure. We theoretically study the transition temperature T_c and perpendicular upper critical field $H_{c2\perp}$ of F/S/F trilayers. Because of the trilayer structure, there is one superconducting layer, the occurrence of the $\pi-$phase, due to the interference of the pairing function between adjacent superconducting layers, can be ruled out. We will show that the trilayers structure can demonstrate the superconducting $\pi-$phase by regarding the mutual orientation of ferromagnetic moments of the outer layers. Previous calculations of T_c were involved with thick or thin layers, the interface boundaries having high transparency, the exchange energy was often assumed to be much larger than the superconducting condensation energy. The method for solving the problem were usually treated in the single-mode approximation (Tagirov, 1998, 1999; Buzdin, Vedyayev, Ryzhanova, 1999; Baladie et al. 2001). We develop the method for solving the problem by using the multimode method exactly and show that the Usadel equations can be derived from the Takahashi-Tachiki theory even in the presence of pair-breaking effects. The parameters entering in our calculations such as the role of the finite transparency at boundary interfaces as reported by Lazar et al. (2000) and the arbitrary exchange energy which account for both weak and strong ferrromagnets will be treated in a generalized way.

This chapter is organized as follows. In §3.2, we derive the Usadel equations of an S/F system from the Takahashi-Tachiki theory. In §3.3, the Usadel equations including the pair-breaking effects is also derived for the first time. In §3.4, we calculate exactly the dependence of the transition temperature T_c as a function of the mutual orientation of ferromagnetic exchange fields in magnetic layers by solving the Usadel equations in a multimode method. The calculation of the perpendicular upper critical field $H_{c2\perp}$ is performed in §3.5. We note here that until now there has no theoretical study $H_{c2\perp}$ of F/S/F trilayers using the multimode solutions.

3.2 Equivalence between Takahashi-Tachiki and Usadel proximity effect theories

Based on the early work of Lodder and Koperdraad (1993) who have shown that the proximity effect theories, the theory of Takahashi and Tachiki (1986) used the de Gennes correlation function method, and the Usadel's dirty-limit version of the Eilenberger theory (Usadel, 1970) are completely equivalent to the S/N system. Here we will show that both formulations are also equivalent even for the S/F structure.

We start with an integral equation for the superconducting order parameter, which according to Takahashi-Tachiki, is

$$\triangle(\vec{r}) = V(\vec{r})T\sum_{\omega}\int d^3r' Q_\omega(\vec{r},\vec{r'})\triangle(\vec{r'}), \tag{3.1}$$

where $\omega = (2n+1)\pi T$, with n is integer, $V(\vec{r})$ is the position-dependent pairing interaction, and the kernel $Q_\omega(\vec{r},\vec{r'})$ is expressed as an integral of the correlation function $g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t)$,

$$Q_\omega(\vec{r},\vec{r'}) = 2\pi\sum_{\sigma}\int_0^\infty dt e^{-2|\omega|t} g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t). \tag{3.2}$$

Here the subscript $\xi = 0$ means the Fermi energy, so the equation of motion for the spin-dependent correlation function satisfies

$$[\frac{\partial}{\partial t} + 2iI(\vec{r})(\sigma_z)_{\sigma\sigma}]g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t) = -\mathcal{L}g^{\sigma}_{\xi=0}(\vec{r},\vec{r'};t); \quad (t>0),$$

with the gauge-invariant differential operator $\mathcal{L}$ denotes by

$$\mathcal{L} = -D(\vec{r})(\nabla - \frac{2ie}{c}\vec{A}(\vec{r}))^2. \tag{3.3}$$

Therefore $I(\vec{r})$ is the position-dependent exchange potential of conduction electrons, σ_z is the Pauli matrix, $D(\vec{r})$ is the diffusion coefficient and $\vec{A}(\vec{r})$ the vector potential. We also denote $N(\vec{r})$ the density of states at the Fermi surface.

Because the electron spin coupled with the exchange interaction which yields the spin split Fermi surface it is necessary to define the auxiliary kernel $R_\omega(\vec{r},\vec{r'})$,

$$R_\omega(\vec{r},\vec{r'}) = 2\pi \sum_\sigma (\sigma_z)_{\sigma\sigma} \int_0^\infty dt e^{-2|\omega|t} g^\sigma_{\xi=0}(\vec{r},\vec{r'};t), \tag{3.4}$$

then one has the set of coupled differential equations

$$[2|\omega| + \mathcal{L}]Q_\omega(\vec{r},\vec{r'}) + 2iI(\vec{r})R_\omega(\vec{r},\vec{r'}) = 2\pi N(\vec{r})\delta(\vec{r}-\vec{r'}), \tag{3.5}$$

$$[2|\omega| + \mathcal{L}]R_\omega(\vec{r},\vec{r'}) + 2iI(\vec{r})Q_\omega(\vec{r},\vec{r'}) = 0. \tag{3.6}$$

The case where $I(\vec{r}) = 0$ was considered by Lodder and Koperdraad, we further investigate for the $I(\vec{r}) \neq 0$ case. The kernels $Q_\omega(\vec{r},\vec{r'})$ and $R_\omega(\vec{r},\vec{r'})$ can be decomposed into the up-and down-spin contributions

$$Q_\omega(\vec{r},\vec{r'}) = Q^+_\omega(\vec{r},\vec{r'}) + Q^-_\omega(\vec{r},\vec{r'}), \tag{3.7}$$

$$R_\omega(\vec{r},\vec{r'}) = Q^+_\omega(\vec{r},\vec{r'}) - Q^-_\omega(\vec{r},\vec{r'}). \tag{3.8}$$

Applying (3.7), (3.8) to (3.5), (3.6) and by adding and subtracting these equations, we obtain the uncoupled differential equations for $Q^+_\omega(\vec{r},\vec{r'})$ and $Q^-_\omega(\vec{r},\vec{r'})$ separately

$$[2|\omega| + \mathcal{L} \pm 2iI(\vec{r})]Q^\pm_\omega(\vec{r},\vec{r'}) = \pi N(\vec{r})\delta(\vec{r}-\vec{r'}). \tag{3.9}$$

We observe that (3.9) gives the relation for the kernel $Q^\pm_\omega(\vec{r},\vec{r'})$,

$$Q^+_\omega(\vec{r},\vec{r'};-I) = Q^-_\omega(\vec{r},\vec{r'};I). \tag{3.10}$$

Consequently, the superconducting order parameter (3.1) has the symmetry with respect to I,

$$\triangle(\vec{r}) = \triangle(\vec{r}; |I|), \tag{3.11}$$

this allows us to rewrite (3.1) as

$$\triangle(\vec{r}) = 2V(\vec{r})T \sum_{\omega} \int d^3r' Q^+_{\omega}(\vec{r}, \vec{r'}) \triangle(\vec{r'}). \tag{3.12}$$

Before going further we note that the kernel $Q^+_{\omega}(\vec{r}, \vec{r'})$ behaves like Green's function. To compare with the Usadel equations as used by Radovic et al. in S/F multilayers, we denote the pair function $F(\vec{r}, \omega)$ by the relation

$$F(\vec{r}, \omega) = \frac{2}{\pi N(\vec{r})} \int d^3r' Q^+_{\omega}(\vec{r}, \vec{r'}) \triangle(\vec{r'}). \tag{3.13}$$

By virtue of (3.13) we get the self-consistency condition for the pair potential

$$\triangle(\vec{r}) = \pi T N(\vec{r}) V(\vec{r}) \sum_{\omega} F(\vec{r}, \omega), \tag{3.14}$$

as well as the diffusion-like differential equation

$$[\frac{1}{2}\mathcal{L} + (|\omega| + iI(\vec{r}))] F(\vec{r}, \omega) = \triangle(\vec{r}). \tag{3.15}$$

Equations (3.14) and (3.15) are identical with the Usadel equations. In applying these equations to the S/F proximity effect problems, parameters for each layer are treated separately. The exchange potential exists only in ferromagnets while the pairing interaction and the gap function are taken to be zero. The set of equations used in determining the critical temperature T_c and the perpendicular upper critical field $H_{c2\perp}$ are

$$\triangle_s(\vec{r}) = \pi \lambda_s T \sum_{\omega} F_s(\vec{r}, \omega), \tag{3.16}$$

and

$$[-\frac{D_s}{2}(\nabla - \frac{2ie}{c}\vec{A}(\vec{r}))^2 + |\omega|]F_s(\vec{r},\omega) = \triangle_s(\vec{r}), \tag{3.17}$$

for the superconducting layer and

$$[-\frac{D_f}{2}(\nabla - \frac{2ie}{c}\vec{A}(\vec{r}))^2 + (|\omega| + iIsgn(\omega))]F_f(\vec{r},\omega) = 0, \tag{3.18}$$

for the ferromagnetic layer. The coupling constant $\lambda_s = N_s V_s$ in the weak-coupling approximation obeys

$$\frac{1}{\lambda_s} = \ln(\frac{1.134\omega_D}{T_{cS}}),$$

where T_{cS} is the isolated superconducting critical temperature and ω_D the Debye cutoff frequency.

In solving the Usadel equations, the boundary conditions are incorporated to relate F_s with F_f at each S/F interface, following Radovic et al.

$$F_s(\vec{r},\omega) = F_f(\vec{r},\omega), \tag{3.19}$$

$$\nabla F_s(\vec{r},\omega) = \eta \nabla F_f(\vec{r},\omega), \tag{3.20}$$

in which the parameter η denotes the ratio of normal-state conductivity of materials. In the theory of Radovic et al., the perfect interface transparency was assumed. However, Aart et al. (1997), were the first who argued the important role of the interface transparency. They discussed their experimental results using the boundary conditions which have been derived by Kupriyanov and Lukichev (1988). The boundary condition (3.19) is modified to be

$$-D_f(\hat{n}_f.\nabla F_f) = \frac{v_f T_f}{2}(F_s - F_f), \tag{3.21}$$

where $\hat{n}_f$ is the unit vector outward normal to the interface, v_f is the Fermi velocity inside ferromagnets and T_f is the dimensionless interface transparency parameter ($T_f \in [0,\infty]$). The difference between (3.19) and (3.21) is that the latter allows the jump of the anomalous Green function at the interface while the former assumes the pairing function to be continuous across the interface.

3.3 Usadel equations including pair-breaking effects

We begin with the linearized integral equation for the superconducting order parameter,

$$\triangle(\vec{r}) = V(\vec{r})T\sum_{\omega}\int d^3r' Q_\omega(\vec{r},\vec{r'})\triangle(\vec{r'}), \tag{3.22}$$

the kernels $Q_\omega(\vec{r},\vec{r'})$ and $R_\omega(\vec{r},\vec{r'})$ in the presence of pair-breaking scatterers are governed by the set of coupled differential equations, following Auvil, Ketterson and Song (1989),

$$[2|\omega| + \mathcal{L} + \frac{1}{\tau_m} + \frac{1}{\tau_{so}}]Q_\omega(\vec{r},\vec{r'}) + 2iIR_\omega(\vec{r},\vec{r'}) = 2\pi N\delta(\vec{r}-\vec{r'}), \tag{3.23}$$

$$[2|\omega| + \mathcal{L} + \frac{1}{\tau_m}]R_\omega(\vec{r},\vec{r'}) + 2iIQ_\omega(\vec{r},\vec{r'}) = 0, \tag{3.24}$$

where the gauge-invariant differential operator is denoted by

$$\mathcal{L} = -D(\nabla - \frac{2ie}{c}\vec{A})^2. \tag{3.25}$$

Therefore the position-dependent parameters $D, \vec{A}, \tau_m, \tau_{so}, I$ and N represent the diffusion coefficient, the vector potential, the magnetic impurity scattering time, the spin-orbit scattering time, the Pauli spin paramagnetism and the density of states at the Fermi surface.

To proceed further, we will derive the Usadel equations by including pair-breaking scatterers through the generalized de Gennes- Takahashi-Tachiki proximity effect theory. To accomplish this we introduce the Usadel anomalous functions $F^{\pm}(\vec{r},\omega)$;

$$F^{+}(\vec{r},\omega)=\frac{1}{\pi N(\vec{r})}\int d^3r' Q_{\omega}(\vec{r},\vec{r'})\triangle(\vec{r'}), \tag{3.26}$$

$$F^{-}(\vec{r},\omega)=\frac{1}{\pi N(\vec{r})}\int d^3r' R_{\omega}(\vec{r},\vec{r'})\triangle(\vec{r'}). \tag{3.27}$$

It is obvious that the order parameter (3.22) becomes

$$\triangle(\vec{r})=\pi T N(\vec{r})V(\vec{r})\sum_{\omega}F^{+}(\vec{r},\omega). \tag{3.28}$$

Applying (3.26) and (3.27) to the set of coupled differential equations (3.23)and (3.24), one arrives at the following equations

$$[2|\omega|+\mathcal{L}+\frac{1}{\tau_m}+\frac{1}{\tau_{so}}]F^{+}(\vec{r},\omega)+2iIF^{-}(\vec{r},\omega)=2\triangle(\vec{r}), \tag{3.29}$$

$$[2|\omega|+\mathcal{L}+\frac{1}{\tau_m}]F^{-}(\vec{r},\omega)+2iIF^{+}(\vec{r},\omega)=0. \tag{3.30}$$

The set of coupled differential equations (3.29) and (3.30) are the Usadel equations in the presence of the magnetic impurity scattering τ_m, the spin-orbit scattering τ_{so}, and the spin exchange potential I.

Note that the extended Usadel equations as derived by Demler , Arnold and Beasley have considered only the spin-orbit scattering effect. This result shows the generalization of the Usadel equations whose the magnetic impurity scattering is included for the first time.

3.4 Transition temperature of ferromagnet/ superconductor/ ferromagnet trilayers

We assume that the dirty-limit conditions are held, and calculate exactly the transition temperature T_c of the FSF trilayer within the context of the linearized Usadel equations by using a multimode method. We investigate the influence of the mutual orientation of the ferromagnetic exchange field in the parallel and antiparallel configurations. The ferromagnetic layers occupy the regions $d_s/2 \leq |x| \leq d_s/2 + d_f$ whereas $-d_s/2 \leq x \leq d_s/2$ is the region for the superconducting layer. Near T_c the Usadel equations for the anomalous function take the form

$$\frac{D_s}{2}\frac{d^2}{dx^2}F_s(x,\omega) - |\omega|F_s(x,\omega) = -\triangle(x), \tag{3.31}$$

for the superconducting region and

$$\frac{D_f}{2}\frac{d^2}{dx^2}F_f(x,\omega) - (|\omega| + iI(x)sgn(\omega))F_f(x,\omega) = 0, \tag{3.32}$$

for the ferromagnetic regions. The self-consistency condition for the order parameter is given by

$$\triangle(x) = \pi\lambda T\sum_{\omega} F_s(x,\omega), \tag{3.33}$$

with the BCS coupling constant λ satisfies

$$\frac{1}{\lambda} = \ln(\frac{1.134\omega_D}{T_{cS}}).$$

In the above $D_s = v_s l_s/3$ and $D_f = v_f l_f/3$ are the diffusion coefficients in S and F which expressed in terms of the Fermi velocity and the mean free path. T_{cS} is the isolated critical temperature and ω_D the Debye cutoff frequency, $\omega = (2n+1)\pi T$, with $n = 0, \pm1, \pm2, ..$ are the Matsubara frequency, $I(x)$ is the ferrromagnetic exchange field.

The functions F_s and F_f are connected by the boundary conditions of no supercurrent at the outer surfaces, $|x| = d_s/2 + d_f$,

$$\frac{d}{dx}F_f(\frac{d_s}{2} + d_f, \omega) = 0 = \frac{d}{dx}F_f(-\frac{d_s}{2} - d_f, \omega), \tag{3.34}$$

as well as at the SF interfaces $x = \pm d_s/2$;

$$\frac{d}{dx}F_s(x,\omega) = \frac{N_f D_f}{N_s D_s}\frac{d}{dx}F_f(x,\omega), \tag{3.35}$$

$$F_s(x,\omega) = F_f(x,\omega) \mp \frac{2D_f}{v_f T_f}\frac{d}{dx}F_f(x,\omega). \tag{3.36}$$

Here $N_{s,f}$ is the density of states in a given layer, v_f is the Fermi velocity in ferromagnets, T_f is the dimensionless interface transparency parameter $T_f \in [0,\infty]$ and the $\mp$ refers to the right and to the left ferromagnets, respectively.

Let us denote the propagating momentum in the ferromagnet by

$$k_f = \sqrt{\frac{2}{D_f}[|\omega| + iIsgn(\omega)]}, \tag{3.37}$$

and seek the solution to (3.32) in two cases.

For the case of the parallel alignment of magnetizations (P-phase): $I(x > d_s/2) = I = I(x < -d_s/2)$, the solution of the Usadel equation (3.32) that satisfies the condition of no supercurrent at the outer surfaces is

$$F_f(x,\omega) = C_\omega \cosh(k_f[|x| - \frac{d_s}{2} - d_f]), \tag{3.38}$$

for both F layers, and the boundary conditions at $x = \pm d_s/2$ can be written in a closed form with respect to F_s :

$$\frac{d}{dx}F_s(\frac{d_s}{2},\omega) = -W(\omega)F_s(\frac{d_s}{2},\omega), \tag{3.39}$$

$$\frac{d}{dx}F_s(-\frac{d_s}{2},\omega) = W(\omega)F_s(-\frac{d_s}{2},\omega), \tag{3.40}$$

with

$$W(\omega) = \frac{N_f D_f}{N_s D_s}\frac{k_f \tanh(k_f d_f)}{1 + \frac{2D_f k_f}{v_f T_f}\tanh(k_f d_f)}. \tag{3.41}$$

For the case of the antiparallel alignment of magnetizations (AP-phase): $I(x > d_s/2) = I = -I(x < -d_s/2)$, the difference from the parallel case is that $k_f \to k_f^*$ for the left ferromagnetic layer ($x < -d_s/2$) then we obtain the boundary conditions at interfaces $x = \pm d_s/2$:

$$\frac{d}{dx}F_s(\frac{d_s}{2},\omega) = -W(\omega)F_s(\frac{d_s}{2},\omega), \tag{3.42}$$

$$\frac{d}{dx}F_s(-\frac{d_s}{2},\omega) = W^*(\omega)F_s(-\frac{d_s}{2},\omega), \tag{3.43}$$

here W^* means the complex conjugation of W.

In order to calculate T_c in the multimode method, we must solve the Usadel equation in the superconducting layer (3.31) together with the self-consistency equation for the order parameter (3.33). We propose the method for solving the problem exactly.

We employ the Takahashi-Tachiki differential equation for the diffusive kernel (3.9) in the superconducting region, namely

$$\frac{D_s}{2}\frac{d^2}{dx^2}\bar{Q}_\omega(x,x') - |\omega|\bar{Q}_\omega(x,x') = -\delta(x-x'), \tag{3.44}$$

here $\bar{Q}_\omega(x,x') = 2Q^{\pm}_\omega(x,x')/\pi N_s$ plays the role of mathematical Green's functions and has the similar boundary conditions as $F_s(x,\omega)$;

$$\frac{d}{dx}\bar{Q}_\omega(\frac{d_s}{2},x') = -W(\omega)\bar{Q}_\omega(\frac{d_s}{2},x'), \tag{3.45}$$

$$\frac{d}{dx}\bar{Q}_\omega(-\frac{d_s}{2},x') = W(\omega)\bar{Q}_\omega(-\frac{d_s}{2},x'), \tag{3.46}$$

for P-phase, while we replace $W \rightarrow W^*$ in (3.46) for AP-phase. The anomalous function $F_s(x,\omega)$ can be expressed in an integral equation form

$$F_s(x,\omega) = \int_{-d_s/2}^{d_s/2} dx'\bar{Q}_\omega(x,x')\triangle(x'). \tag{3.47}$$

Introducing the eigenfunction expansion of $\bar{Q}_\omega(x,x')$,

$$\bar{Q}_\omega(x,x') = \sum_{m=-\infty}^{\infty}\bar{Q}_\omega(q_m,x')\cos(q_m x), \tag{3.48}$$

$$\bar{Q}_\omega(q_m,x') = \frac{1}{d_s}\int_{-d_s/2}^{d_s/2} dx\bar{Q}_\omega(x,x')\cos(q_m x), \tag{3.49}$$

where $q_m = 2m\pi/d_s$ with $m = 0, \pm1, \pm2, ...$ are the eigenmodes in the superconducting region and provides the orthogonality relation

$$\frac{1}{d_s}\int_{-d_s/2}^{d_s/2} dx \cos(q_m x)\cos(q_{m'} x) = \delta_{mm'}. \tag{3.50}$$

Performing the Fourier transform of (3.44) yields

$$d_s(|\omega| + \frac{D_s}{2}q_m^2)\bar{Q}_\omega(q_m, x') = \cos(q_m x') + \frac{D_s}{2}(-1)^m[\frac{d}{dx}\bar{Q}_\omega(\frac{d_s}{2}, x') - \frac{d}{dx}\bar{Q}_\omega(-\frac{d_s}{2}, x')]. \tag{3.51}$$

In the last term of (3.51) we use (3.48) the symbol

$$\alpha_\omega = \frac{D_s}{d_s}W(\omega); \quad P - phase, \tag{3.52}$$

$$\alpha_\omega = \frac{D_s}{d_s}ReW(\omega); \quad AP - phase, \tag{3.53}$$

to obtain the eigenfunction expansion of the diffusive kernel $\bar{Q}_\omega$ as follows

$$\bar{Q}_\omega(q_m, x') = \frac{\cos(q_m x')}{d_s(|\omega| + \frac{D_s}{2}q_m^2)} - \frac{\alpha_\omega}{|\omega| + \frac{D_s}{2}q_m^2}\sum_{l=-\infty}^{\infty}(-1)^{m+l}\bar{Q}_\omega(q_l, x'). \tag{3.54}$$

Solving (3.54) for $\bar{Q}_\omega(q_m, x')$ algebraically we have

$$\bar{Q}_\omega(q_m, x') = \frac{\cos(q_m x')}{d_s(|\omega| + \frac{D_s}{2}q_m^2)} - \frac{\alpha_\omega}{1 + \alpha_\omega\beta_\omega}\sum_{l=-\infty}^{\infty}\frac{(-1)^{m+l}\cos(q_l x')}{d_s(|\omega| + \frac{D_s}{2}q_m^2)(|\omega| + \frac{D_s}{2}q_l^2)}, \tag{3.55}$$

where the frequency-dependent coefficient β_ω is denoted by

$$\beta_\omega = \sum_{m=-\infty}^{\infty}\frac{1}{|\omega| + \frac{D_s}{2}q_m^2} = \sqrt{\frac{d_s^2}{2D_s|\omega|}}\coth(\sqrt{\frac{d_s^2|\omega|}{2D_s}}). \tag{3.56}$$

We proceed further by transforming the order parameter equation (3.33) and the integral equation of the anomalous function F_s (3.47), the results are

$$\triangle(q_m) = \pi\lambda T\sum_\omega F_s(q_m, \omega), \tag{3.57}$$

and

$$F_s(q_m,\omega) = \sum_{m'=-\infty}^{\infty} L_{mm'}(\omega)\triangle(q_{m'}), \tag{3.58}$$

where we define

$$L_{mm'}(\omega) = \int_{-d_s/2}^{d_s/2} dx' \bar{Q}_\omega(q_m, x')\cos(q_{m'}x'), \tag{3.59}$$

Substituting (3.55) into (3.59) and then utilizing (3.57) and (3.58) we obtain

$$\sum_{m'=-\infty}^{\infty} A_{mm'}\triangle(q_{m'}) = 0. \tag{3.60}$$

Here

$$A_{mm'} = \delta_{mm'} - \lambda\pi T \sum_\omega L_{mm'}(\omega), \tag{3.61}$$

$$L_{mm'}(\omega) = \frac{\delta_{mm'}}{(|\omega| + \frac{D_s}{2}q_m^2)} - \frac{\alpha_\omega}{1+\alpha_\omega\beta_\omega}\frac{(-1)^{m+m'}}{(|\omega| + \frac{D_s}{2}q_m^2)(|\omega| + \frac{D_s}{2}q_{m'}^2)}. \tag{3.62}$$

Equation (3.60) has a nontrivial solution when

$$\det|A| = 0. \tag{3.63}$$

The secular equation (3.63) provides the transition temperature T_c as the largest solution.

In the single-mode approximation when the superconducting layer thickness is very small comparable to their coherence length, i.e., $d_s/\xi_s \ll 1$, the all other elements of $A_{mm'}$ are vanished except A_{00}, then (3.60) reduces to

$$A_{00} = 0,$$

or

$$\ln\frac{T_c}{T_{cS}} = \sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{2}+\frac{\alpha_\omega}{2\pi T_c}} - \frac{1}{n+\frac{1}{2}}\right). \tag{3.64}$$

In the case of strong ferromagnets $I \gg T_{cS}$, we can write (3.64) in terms of the digamma function

$$\ln \frac{T_c}{T_{cS}} = \psi(\frac{1}{2}) - Re\ \psi(\frac{1}{2} + \frac{T_{cS}}{T_c}\rho), \tag{3.65}$$

where the pair-breaking parameter

$$\rho = \frac{2\phi^2}{(d_s/\xi_s)^2}, \tag{3.66}$$

with

$$\phi_P^2 = \frac{d_s}{2}W, \qquad \phi_{AP}^2 = \frac{d_s}{2}ReW.$$

We examine the effect of the interface transparency T_f in two limiting cases.

In the limit of the low interface transparency $T_f \ll 1$ with an arbitrary ferromagnetic thicknesses d_f, we find the same result as Tagirov, i.e.,

$$\phi_P^2 = \phi_{AP}^2 = T_f \frac{N_f}{N_s} \frac{d_s v_f}{4D_s}. \tag{3.67}$$

This result shows that $T_c^P = T_c^{AP}$ for any ferromagnetic thickness thus the mutual orientation of magnetizations does not have influence on the transition temperature.

In the limit of the high interface transparency $T_f \gg 1$ with a very thick ferromagnet, we obtain the similar result as Baladie et al.,

$$\phi_P^2 = (1+i)\frac{d_s}{2}\frac{N_f D_f}{N_s D_s}\sqrt{\frac{I}{D_f}}, \tag{3.68}$$

$$\phi_{AP}^2 = \frac{d_s}{2}\frac{N_f D_f}{N_s D_s}\sqrt{\frac{I}{D_f}}. \tag{3.69}$$

We conclude that the parallel alignment of magnetizations strongly suppresses T_c of trilayer structures meanwhile the antiparallel one enhances the superconductivity. Thus $T_c^{AP} > T_c^P$ or the antiparallel phase is more favorable than the parallel phase.

3.5 Perpendicular upper critical field of ferromagnet/ superconductor/ferromagnet trilayers

In this section we study the magnetic field effect on the F/S/F trilayer by calculating the perpendicular upper critical field $H_{c2\perp}$ as a function of temperature for any thickness d_s or d_f. The trilayer structure lies in the x-y plane with the modulation of the pairing amplitude is taking along the z-axis parallel to the external magnetic field $\vec{H} = H\hat{z}$. We choose the gauge $\vec{A} = (0, Hx, 0)$, the gauge-invariant operator $\mathcal{L}$ reads as

$$\mathcal{L} = -D_{s,f}[\frac{\partial^2}{\partial x^2} + (\frac{\partial}{\partial y} - \frac{2i\pi}{\phi_0}Hx)^2 + \frac{\partial^2}{\partial z^2}], \tag{3.70}$$

where $\phi_0 = hc/2|e|$ is the flux quantum.

Assuming that the anomalous functions $F_{s,f}$ can be separated according to

$$F_{s,f}(\vec{r}, \omega) = f(x, y)g_{s,f}(z, \omega), \tag{3.71}$$

here we have supposed that $f(x, y)$ is a frequency-independent function because of the Abrikosov vortex lattice occurs only in the z-direction.

For ferromagnetic layers, $d_s/2 \leq |z| \leq d_s/2 + d_f$, we have the Usadel equation

$$\frac{1}{2}\mathcal{L}F_f(\vec{r}, \omega) + [|\omega| + iI(z)sgn(\omega)]F_f(\vec{r}, \omega) = 0. \tag{3.72}$$

we denote the propagating momentum in ferromagnets by

$$k_f = \sqrt{\frac{2}{D_f}[|\omega| + iIsgn(\omega)]}, \tag{3.73}$$

and examine the solution to (3.72) when the spin-exchange fields align either parallel or antiparallel with respect to the magnetization axis. For the case of the parallel alignment of magnetizations: $I(z > d_s/2) = I = I(z < -d_s/2)$, (3.72) can be written as

$$\frac{d^2}{dz^2} g_f(z,\omega) = p_f^2 g_f(z,\omega), \tag{3.74}$$

and

$$\left[\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial y} - \frac{2i\pi}{\phi_0} Hx\right)^2\right] f(x,y) + (p_f^2 - k_f^2) f(x,y) = 0. \tag{3.75}$$

We seek the solution to $f(x,y)$ by taking $f(x,y) = e^{ik_y y} f(x)$ and $x_0 = \phi_0 k_y / 2\pi H$ and introduce the new variable $\zeta = \sqrt{2\pi H/\phi_0} x$ and $\eta = (p_f^2 - k_f^2)\phi_0/2\pi H$, then we have

$$f(\zeta) = e^{-\frac{1}{2}\zeta^2} H_n(\zeta),$$

where $H_n(\zeta)$ is the Hermite polynomial of order n, we can show that $\eta = 2n+1$ thus

$$p_f^2 - k_f^2 = (2n+1)\frac{2\pi H}{\phi_0},$$

the lowest eigenvalue gives the highest magnetic field i.e.,

$$\frac{2\pi H_{c2\perp}}{\phi_0} = p_f^2 - k_f^2. \tag{3.76}$$

Since we have assumed that $f(x,y)$ is the same for both layers then for the superconductor layer we consider only the z-dependent part of $F_s(\vec{r},\omega)$

$$\frac{D_s}{2}\left(\frac{d^2}{dz^2} - \frac{2\pi}{\phi_0} H_{c2\perp}\right) F_s(z,\omega) - |\omega| F_s(z,\omega) = -\triangle(z). \tag{3.77}$$

Relating $F_s(z,\omega)$ to $g_f(z,\omega)$ through the interface boundaries $z = \pm d_s/2$, we arrive at the relations

$$\frac{d}{dz} F_s\left(\frac{d_s}{2},\omega\right) = -W(\omega) F_s\left(\frac{d_s}{2},\omega\right), \tag{3.78}$$

$$\frac{d}{dz} F_s\left(-\frac{d_s}{2},\omega\right) = W(\omega) F_s\left(-\frac{d_s}{2},\omega\right), \tag{3.79}$$

with

$$W(\omega) = \frac{N_f D_f}{N_s D_s} \frac{p_f \tanh(p_f d_f)}{1 + \frac{2D_f p_f}{v_f T_f} \tanh(p_f d_f)}. \tag{3.80}$$

For the case of the antiparallel alignment of magnetizatons: $I(z > d_s/2) = I = -I(z < -d_s/2)$, we replace $k_f \rightarrow k_f^*$ for the lower ferromagnet $(z < -d_s/2)$ thus

$$\frac{d}{dz} F_s(\frac{d_s}{2}, \omega) = -W(\omega) F_s(\frac{d_s}{2}, \omega), \tag{3.81}$$

$$\frac{d}{dz} F_s(-\frac{d_s}{2}, \omega) = W^*(\omega) F_s(-\frac{d_s}{2}, \omega), \tag{3.82}$$

are the boundary conditions at interface $z = \pm d_s/2$.

In analogous to (3.77) we introduce the differential equation for the diffusive kernel $\bar{Q}_\omega(z, z')$

$$\frac{D_s}{2}(\frac{d^2}{dz^2} - \frac{2\pi}{\phi_0} H_{c2\perp}) \bar{Q}_\omega(z, z') - |\omega| \bar{Q}_\omega(z, z') = -\delta(z - z'), \tag{3.83}$$

this enables us to express the anomalous function $F_s(z, \omega)$ as an integral equation of $\bar{Q}_\omega(z, z')$

$$F_s(z, \omega) = \int_{-d_s/2}^{d_s/2} dx' \bar{Q}_\omega(z, z') \triangle(z'). \tag{3.84}$$

The kernel $\bar{Q}_\omega(z, z')$ has the similar boundary conditions as $F_s(z, \omega)$.

Transforming the kernel $\bar{Q}_\omega(z, z')$ by virtue of the eigenfunction expansion method, we have

$$\bar{Q}_\omega(z, z') = \sum_{m=-\infty}^{\infty} \bar{Q}_\omega(q_m, z') \cos(q_m z), \tag{3.85}$$

$$\bar{Q}_\omega(q_m, z') = \frac{1}{d_s} \int_{-d_s/2}^{d_s/2} dz \bar{Q}_\omega(z, z') \cos(q_m z), \tag{3.86}$$

with $q_m = 2m\pi/d_s$, m is integer. The Fourier transforms of the anomalous function $F_s(z, \omega)$, (3.84) and the order parameter equation (3.33) imply

$$F_s(q_m, \omega) = \sum_{m'=-\infty}^{\infty} L_{mm'}(\omega) \triangle(q_{m'}), \tag{3.87}$$

and

$$\triangle(q_m) = \pi\lambda T \sum_{\omega} F_s(q_m, \omega), \tag{3.88}$$

where the matrix element $L_{mm'}(\omega)$ is given by

$$L_{mm'}(\omega) = \int_{-d_s/2}^{d_s/2} dz' \bar{Q}_\omega(q_m, z') \cos(q_{m'} z'), \tag{3.89}$$

Combining (3.87) and (3.88) to obtain the equation for $\triangle(q_m)$,

$$\sum_{m'=-\infty}^{\infty} A_{mm'} \triangle(q_{m'}) = 0. \tag{3.90}$$

Here

$$A_{mm'} = \delta_{mm'} - \lambda\pi T \sum_{\omega} L_{mm'}(\omega), \tag{3.91}$$

$$L_{mm'}(\omega) = \frac{\delta_{mm'}}{(|\tilde{\omega}| + \frac{D_s}{2} q_m^2)} - \frac{\alpha_\omega}{1 + \alpha_\omega \beta_\omega} \frac{(-1)^{m+m'}}{(|\tilde{\omega}| + \frac{D_s}{2} q_m^2)(|\tilde{\omega}| + \frac{D_s}{2} q_{m'}^2)}, \tag{3.92}$$

$$|\tilde{\omega}| = |\omega| + \frac{\pi D_s}{\phi_0} H_{c2\perp}, \tag{3.93}$$

$$\alpha_\omega = \frac{D_s}{d_s} W(\omega); \quad P - phase, \tag{3.94}$$

$$\alpha_\omega = \frac{D_s}{d_s} Re W(\omega); \quad AP - phase, \tag{3.95}$$

and

$$\beta_\omega = \sum_{m=-\infty}^{\infty} \frac{1}{|\tilde{\omega}| + \frac{D_s}{2} q_m^2} = \sqrt{\frac{d_s^2}{2D_s|\tilde{\omega}|}} \coth(\sqrt{\frac{d_s^2|\tilde{\omega}|}{2D_s}}). \tag{3.96}$$

The nontrivial solutions to (3.90) provide the secular equation

$$\det|A| = 0, \tag{3.97}$$

in which the largest solution gives the perpendicular upper critical field $H_{c2\perp}$ as a function of temperature.

Our analytical expressions should be justified in the single-mode approximation where the thin superconducting layer limit ($d_s \ll \xi_s$), ξ_s being the superconducting coherence length, is taken into account as well as the strong exchange field limit ($I \gg T_{cs}$), where T_{cs} is the bulk superconducting transition temperature in the zero field. We also neglect the critical field $H_{c2\perp}$ inside the ferromagnetic layer i.e., $p_f^2 = k_f^2$. As a result, the secular equation (3.97) implies

$$A_{00} = 0,$$

or

$$\ln \frac{T}{T_{cs}} = \psi(\frac{1}{2}) - Re\ \psi(\frac{1}{2} + \frac{1}{2\pi T}[\frac{\pi D_s}{\phi_0} H^P_{c2\perp}(T) + \frac{D_s W}{d_s}]), \tag{3.98}$$

for the P-phase and

$$\ln \frac{T}{T_{cs}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{1}{2\pi T}[\frac{\pi D_s}{\phi_0} H^{AP}_{c2\perp}(T) + \frac{D_s}{d_s} Re\ W]), \tag{3.99}$$

for the AP-phase, where $\psi(z)$ is the digamma function and therefore

$$W = \frac{N_f D_f}{N_s D_s} \frac{k_f \tanh(k_f d_f)}{1 + \frac{2 D_f k_f}{v_f T_f} \tanh(k_f d_f)}, \tag{3.100}$$

with $k_f = (1+i)\sqrt{I/D_f}$.

In the following, we will examine the effect of the interface transparency T_f on the perpendicular upper critical field $H_{c2\perp}(T)$ in two limiting cases.

(i) In the case of the low transparency limit $T_f \ll 1$, the unity factor in the denominator of W, (3.100), may be dropped regardless the ferromagnetic layer thickness d_f, then W becomes

$$W = \frac{N_f D_f}{N_s D_s} \frac{v_f T_f}{2 D_f}, \tag{3.101}$$

this result shows that $H^P_{c2\perp}(T) = H^{AP}_{c2\perp}(T)$ for all temperature range. Hence the critical field $H_{c2\perp}(T)$ is independent of the oriented magnetization alignment.

(ii) In the case of the nearly perfect transparency $T_f \gg 1$ with an arbitrary ferromagnetic layer thickness, the Taylor's series expansion of denominator of

(3.100) leads to

$$W = \frac{N_f D_f}{N_s D_s} k_f \tanh(k_f d_f)[1 - \frac{2D_f}{v_f T_f} k_f \tanh(k_f d_f)], \qquad (3.102)$$

we proceed further by considering the thin ferromagnetic layer thickness limit, in this case $\tanh(k_f d_f) \approx k_f d_f$ and (3.102) is simplified to be

$$W = \frac{N_f D_f}{N_s D_s}[\ \frac{2iId_f}{D_f} + \frac{2D_f}{v_f T_f}(\frac{2Id_f}{D_f})^2\ ]. \qquad (3.103)$$

Substituting (3.103) into (3.98) and (3.99), we obtain

$$\ln \frac{T}{T_{cs}} = \psi(\frac{1}{2}) - Re\ \psi(\frac{1}{2} + \frac{1}{2\pi T}[\frac{\pi D_s}{\phi_0} H^P_{c2\perp}(T) + 2iI\frac{N_f d_f}{N_s d_s}]\ ), \qquad (3.104)$$

for the P-phase and

$$\ln \frac{T}{T_{cs}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{1}{2\pi T}[\frac{\pi D_s}{\phi_0} H^{AP}_{c2\perp}(T) + 8\frac{I^2 d_f}{v_f T_f}\frac{N_f d_f}{N_s d_s}]\ ), \qquad (3.105)$$

for the AP-phase, respectively. The obtained results show that the critical field $H^P_{c2\perp}(T)$ does not depends on the parameter T_f while $H^{AP}_{c2\perp}(T)$ increases as T_f increases from the high limit $T_f \gg 1$ to the perfect limit $T_f \to \infty$. Because of the weak suppression character of the exchange field on the pairing function in the antiparallel aligned ferromagnetic layers, the paired electrons are preferable to situate in the superconducting layer than the ferromagnetic layers even though the perfect transparency limit is reached.

Chapter 4
Discussion and Conclusions

4.1 Numerical Results

In the previous chapter the superconducting transition temperature T_c and the perpendicular upper critical field $H_{c2\perp}$ of the proximity effect of F/S/F trilayers are calculated, regarding the mutual orientation of ferromagnetic exchange fields of the outer layers, in a multimode method by taking the role of the interface boundary transparency T_f and the arbitrary exchange energy I into account. The central results shown in §3.4 and §3.5 and we will write them again for practical purpose. We begin with the secular equation (3.97)

$$\det |A| = 0, \tag{4.1}$$

where the matrix element $A_{mm'}$ and their related expressions are

$$A_{mm'} = \delta_{mm'} - \lambda\pi T \sum_{\omega} L_{mm'}(\omega), \tag{4.2}$$

$$L_{mm'}(\omega) = \frac{\delta_{mm'}}{(|\tilde{\omega}| + \frac{D_s}{2} q_m^2)} - \frac{\alpha_\omega}{1 + \alpha_\omega \beta_\omega} \frac{(-1)^{m+m'}}{(|\tilde{\omega}| + \frac{D_s}{2} q_m^2)(|\tilde{\omega}| + \frac{D_s}{2} q_{m'}^2)}. \tag{4.3}$$

$$|\tilde{\omega}| = |\omega| + \frac{\pi D_s}{\phi_0} H_{c2\perp}, \tag{4.4}$$

$$\alpha_\omega = \frac{D_s}{d_s} W(\omega); \quad P - phase, \tag{4.5}$$

$$\alpha_\omega = \frac{D_s}{d_s} ReW(\omega); \quad AP - phase, \tag{4.6}$$

and

$$\beta_\omega = \sum_{m=-\infty}^{\infty} \frac{1}{|\tilde{\omega}| + \frac{D_s}{2} q_m^2} = \sqrt{\frac{d_s^2}{2D_s|\tilde{\omega}|}} \coth(\sqrt{\frac{d_s^2|\tilde{\omega}|}{2D_s}}). \tag{4.7}$$

The function $W(\omega)$ follows from the boundary condition at F/S interfaces and is expressed as

$$W(\omega) = \frac{N_f D_f}{N_s D_s} \frac{p_f \tanh(p_f d_f)}{1 + \frac{2D_f p_f}{v_f T_f} \tanh(p_f d_f)}, \tag{4.8}$$

here the propagating momentum inside ferromagnets, p_f, including the perpendicular upper critical field, $H_{c2\perp}$, has the relation according to (3.76)

$$p_f^2 = k_f^2 + \frac{2\pi}{\phi_0} H_{c2\perp}, \tag{4.9}$$

with

$$k_f = \sqrt{\frac{2}{D_f}(|\omega| + iIsgn(\omega))}. \tag{4.10}$$

In the above equations λ is the dimensionless coupling constant of the superconducting layer in which we have assumed that the pairing interaction does not exist in ferromagnets and in the case of the isolated superconducting metal with the hypothetical weak-coupling approximation, $\omega_D/T_{cs} \gg 1$, ω_D is the Debye cutoff frequency and T_{cs} the bulk critical temperature, one has

$$\lambda = (\ln \frac{1.14\omega_D}{T_{cs}})^{-1}, \tag{4.11}$$

$\omega = (2n+1)\pi T$ is the discrete frequency, $n = 0, \pm 1, \pm 2, ..$ $q_m = 2m\pi/d_s$, with m being integer, is the eigenmode, $d_{s(f)}$ represents the layer thicknesses of S(F) metals, $\phi_0 = hc/2|e|$ is the flux quantum, $D = vl/3$ is diffusion coefficient where v and l denote the Fermi velocity and the mean free path in a given layer, N is the constant density of states,T_f serve as an (dimensionless) interface boundary transparency and I the exchange energy of ferromagnetic layers.

It should be noted that the set of equations (4.1)-(4.11) are for determining the phase diagram (H, T) where the critical line $H_{c2\perp}$ at a given temperature T separates the regions between the superconducting and the normal phases where the pairing amplitude from the S sides can penetrate through the F side due to the proximity effect. If we take $H_{c2\perp}$ in (4.9) to be equal to zero, the problem reduces to investigating the nonmonotonic behavior of the transition temperature,T_c, over the ferromagnetic thickness.

In the following, we introducing the coherence length of S and F layers, respectively

$$\xi_{s,f} = \sqrt{\frac{D_{s,f}}{2\pi T_{cs}}}, \tag{4.12}$$

and denote the notations

$$\gamma = \frac{N_f D_f}{N_s D_s} = \frac{\sigma_f}{\sigma_s}, \tag{4.13}$$

$$\gamma_b = \frac{2D_f}{v_f \xi_f T_f} = \frac{2}{3}\frac{(l_f/\xi_f)}{T_f}, \tag{4.14}$$

here $\sigma_{s,f}$ means the normal-state conductivity for each material (S and F), usually $\sigma_f \ll \sigma_s$ then γ characterizes the weak proximity and γ_b may be interpreted as the boundary resistivity at FS interfaces. Within the framework of the Ginzburg-Landau theory, the perpendicular upper critical field at zero temperature $H_{c2\perp}(0)$ of an isolated superconducting film is given by

$$H_{c2\perp}(0) = \frac{\phi_0}{2\pi\xi_{GL}^2(0)}, \tag{4.15}$$

the temperature dependent Ginzburg-Landau coherence length $\xi_{GL}^2(T)$, or the GL magnetic coherence length, is related to the superconducting coherence length ξ_s,(4.12) through

$$\xi_{GL}(T) = \frac{\pi\xi_s}{2}(1 - T/T_{cs})^{-1/2}. \tag{4.16}$$

Combining (4.15)and (4.16) gives the relation

$$\frac{\pi D_s}{\phi_0} H_{c2\perp}(T) = 2\pi T_{cs} \frac{2H_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}. \tag{4.17}$$

By virtue of (4.12)-(4.14), we can rewritten $W(\omega)$,(4.8), as follows

$$W(\omega) = \frac{\gamma/\xi_f}{B_f(\omega) + \gamma_b}, \tag{4.18}$$

where

$$B_f(\omega) = (p_f \xi_f \tanh(p_f d_f))^{-1}, \tag{4.19}$$

then the matrix element $A_{mm'}$, (4.2), can be expressed in terms of dimensionless reduced parameters such as $T/T_{cs}, d_s/\xi_s, d_f\xi_f, H_{c2\perp}(T)/H_{c2\perp}(0)$,

$$A_{mm'} = \delta_{mm'} - \lambda \frac{T}{T_{cs}} \sum_{\omega \geq 0}^{\infty} (\frac{\delta_{mm'}}{E_m} - (-1)^{m+m'} \frac{\Omega(\omega)}{E_m E_{m'}}), \tag{4.20}$$

here

$$E_m = \frac{\omega}{2\pi T_{cs}} + \frac{2H_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)} + \frac{2(m\pi)^2}{(d_s/\xi_s)^2}, \tag{4.21}$$

$$\Omega^P(\omega) = \frac{E_0^2 Rea_\omega + E_0|a_\omega|^2 b_\omega}{E_0^2 + 2E_0 b_\omega Rea_\omega + |a_\omega|^2 b_\omega^2}, \ P - phase; \tag{4.22}$$

$$\Omega^{AP}(\omega) = \frac{E_0 Rea_\omega}{E_0 + b_\omega Rea_\omega}, \ AP - phase; \tag{4.23}$$

with

$$a_\omega = \frac{(\gamma\xi_s/\xi_f)}{(d_s/\xi_s)} \frac{1}{B_f(\omega) + \gamma_b}, \tag{4.24}$$

$$b_\omega = \sqrt{\frac{1}{2}(\frac{d_s}{\xi_s})^2[\frac{\omega}{2\pi T_{cs}} + \frac{2H_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}]} \coth(\sqrt{\frac{1}{2}(\frac{d_s}{\xi_s})^2[\frac{\omega}{2\pi T_{cs}} + \frac{2H_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}]}). \tag{4.25}$$

The set of equations (4.20)-(4.26) are suitable for numerical calculations, if we take $H_{c2\perp}$ equal to zero, the problem then reduces to determinations of the transition temperature T_c as a function of a ferromagnetic layer thickness, d_f.

In numerical calculation, it is sufficient to taking only the (0,0) component of $A_{mm'}$, (4.20), therefore the reduced length d_s/ξ_s and d_f/ξ_f are arbitrary values. Because of all other modes are included via the summation over m which appeared in β_ω, (4.7), or b_ω, (4.25). Then the secular equation (4.1) becomes $A_{00} = 0$. Using (4.11) and (4.20), we get the equation for the phase diagram $(H_{c2\perp}, T)$,

$$\ln\frac{T}{T_{cs}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{2H_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}\frac{T_{cs}}{T}) - \frac{T}{T_{cs}}\sum_{\omega\geq 0}\frac{\Omega(\omega)}{E_0^2}, \tag{4.26}$$

where $\psi(z)$ is the digamma function

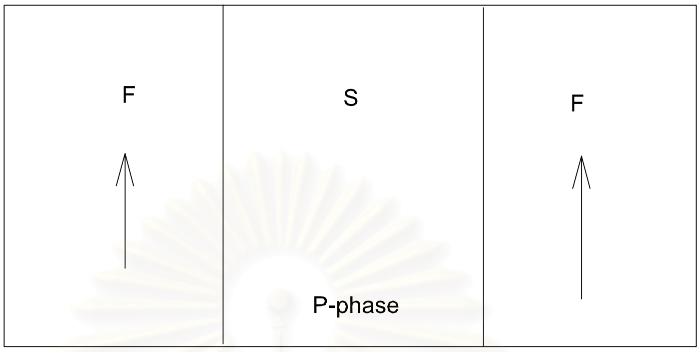
$$\psi(y) - \psi(x) = \sum_{n=0}^{\infty}(\frac{1}{n+x} - \frac{1}{n+y}). \tag{4.27}$$

For the P and AP magnetization alignments, one has from (4.26)

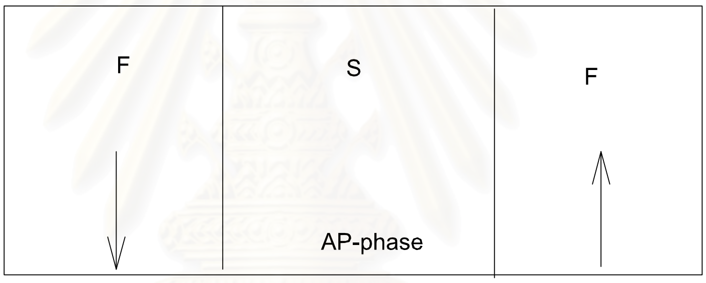
$$\ln\frac{T}{T_{cs}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{2H^P_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}\frac{T_{cs}}{T}) - \frac{T}{T_{cs}}\sum_{\omega\geq 0}\frac{\Omega^P(\omega)}{E_0^2}, \tag{4.28}$$

$$\ln\frac{T}{T_{cs}} = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{2H^{AP}_{c2\perp}(T)}{\pi^2 H_{c2\perp}(0)}\frac{T_{cs}}{T}) - \frac{T}{T_{cs}}\sum_{\omega\geq 0}\frac{\Omega^{AP}(\omega)}{E_0^2}, \tag{4.29}$$

where $\Omega^{P(AP)}(\omega)$ is given by (4.22) and (4.33). The parameters used here correspond to Fominov et al.,(2002) and are following $d_s = 22\ nm, \xi_s = 8.9\ nm, \xi_f = 7.6\ nm, \gamma = 0.125, I = 6.8\pi T_{cs}$, and varying only γ_b, it can be seen from (4.14) that the parameter γ_b depends linearly on the ratio l_f/ξ_f while is proportional inversely to the interface boundary transparency parameter T_f. As point out by Garifullin et al.,(2002) and Tagirov et al.,(2002) who observed the re-entrant behavior of the superconducting transition temperature in Fe/V/Fe trilayered system, the electron mean free path l_f is essential for observing the re-entrant superconductivity where the transparency parameter T_f is fit to the experimental data. We illustrate this phenomenon by plot $T_c(d_f)$ for several values of γ_b which is achieved by taking $H_{c2\perp} = 0$ in (4.28) and (4.29). The results for T_c/T_{cs} versus d_f/ξ_f are showed in Figs.(4.2)-(4.8). We obtain various types of $T_c(d_f)$ behavior, Fig.(4.2) is of the parallel alignment while Fig.(4.3) the antiparallel one, (i) the monotonic decay of T_c at a very low γ_b, superconductivity completely vanishes at a particular d_f, (ii) the re-entrant superconductivity at a moderate γ_b, T_c drops to zero at finite d_f and it restores again. (iii) the nonmonotonic decay of T_c at a large γ_b, superconductivity existing throughout the ferromagnet but it is a particular d_f that T_c has a minimum value. Note that the P-case of F/S/F trilayers is equivalent to the F/S bilayers when the relation $d_s^{tri} = 2d_s^{bi}$ is held and Fig.(4.2) reproduces the result of the F/S bilayers of Fominov et al., (2002). The $T_c(d_f)$ in AP-case is shown in Fig.(4.3), one can see that $T_c^{AP} \geq T_c^P$, i.e., the AP-phase is more favorable than the P-phase, the physical reason is simply; the ferromagnetic exchange fields strongly suppress the Cooper pair in the parallel magnetization alignment and weakly suppress in the antiparallel one. Figs.(4.4)-(4.8) are drawn to compare T_c between the P-and AP-phases for several γ_b, at small γ_b. T_c^{AP} is much larger than T_c^P and decreases when γ_b increases until γ_b is larger than unity, T_c^{AP} equals T_c^P. For the perpendicular upper critical field $H_{c2\perp}(T)$, Figs.(4.9)-(4.10) show the linear temperature dependence of $H_{c2\perp}$ near T_{cs}. Therefore the parameters are the same as in the calculations of $T_c(d_f)$ and we have chosen $d_f/\xi_f = 0.25$ which corresponds to the real situation ($d_s \gg d_f$), i.e., thick S and thin F layers. For the P-case, Fig.(4.8) shows that the small value of γ_b does not have influence on the critical field $H_{c2\perp}$, the effect of γ_b on $H_{c2\perp}$ become significant at a moderate value, the reduced critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ increases due to the pairing amplitude is confined in the S layer. For the AP-case, the result is different from the P-case as shown in Fig.(4.9), the reduced critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ decreases as γ_b increase from zero to the moderate value but it is higher when γ_b is large.



(a)



(b)

Fig. 4.1 Schematic pictures of ferromagnet(F)/ superconductor(S)/ ferromagnet layered structures, F and S occupy the regions $d_s/2 \leq |x| \leq d_s/2 + d_f$ and $-d_s/2 \leq\ x\ \leq\ d_s/2$, respectively. (a) the parallel magnetization alignment (P-phase) of F and (b) the antiparallel one (AP-phase).

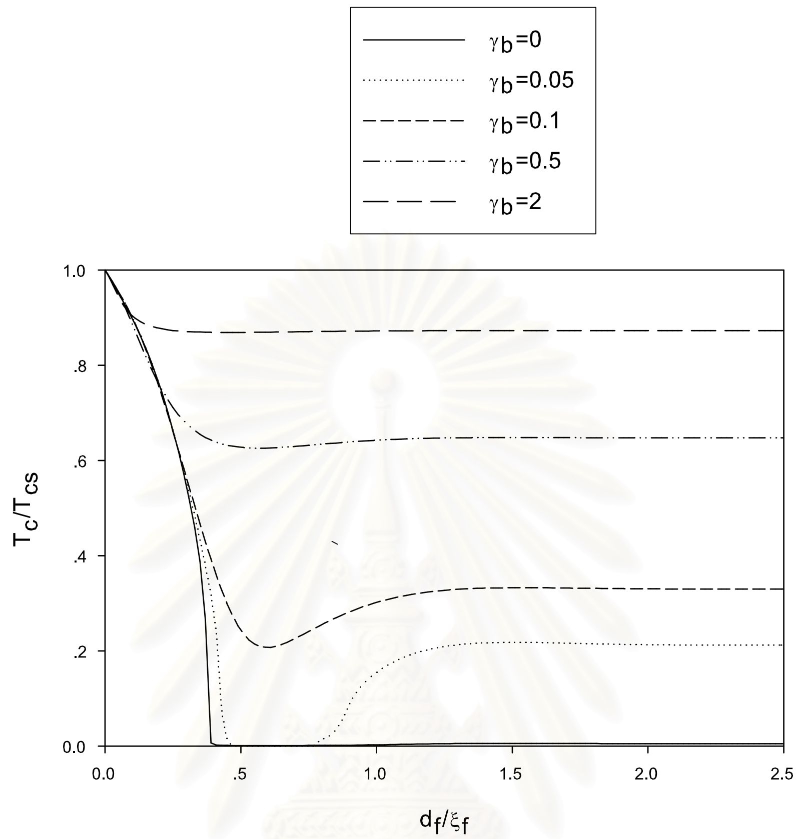


Fig. 4.2 The reduced transition temperature T_c/T_{cs} as a function of the reduced ferromagnetic layer thickness d_f/ξ_f in the P-phase for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ and varying the boundary resistivity γ_b. The $T_c(d_f)$ curves show the monotonic decay for $\gamma_b = 0$, the nonmonotonic decay for $\gamma_b = 0.1, 0.5, 2$ and the re-entrant superconductivity for $\gamma_b = 0.05$.

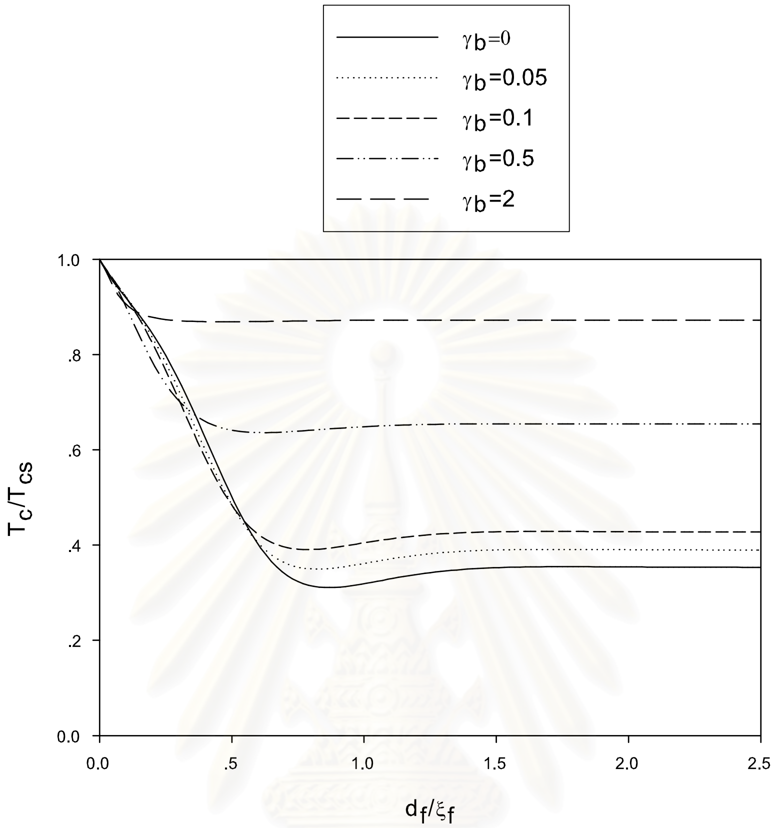


Fig. 4.3 The reduced transition temperature T_c/T_{cs} as a function of the reduced ferromagnetic layer thickness d_f/ξ_f in the AP-phase with varying the boundary resistivity γ_b. The parameters are the same as in Fig. 4.2. The $T_c(d_f)$ curves show only the nonmonotonic behavior.

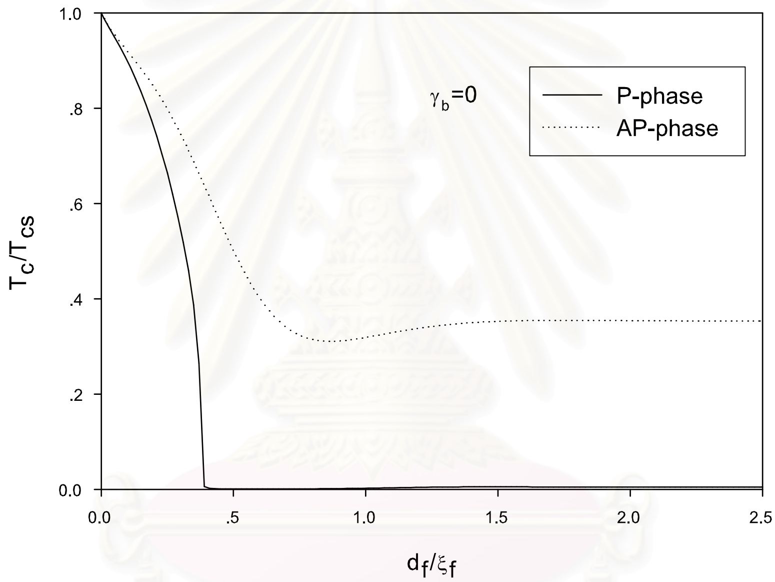


Fig. 4.4 The dependence of the reduced transition temperature T_c/T_{cs} on the normalized ferromagnetic thickness d_f/ξ_f between the parallel and antiparallel magnetization alignments (P-and AP-phases), for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ with the boundary resistivity $\gamma_b = 0$.

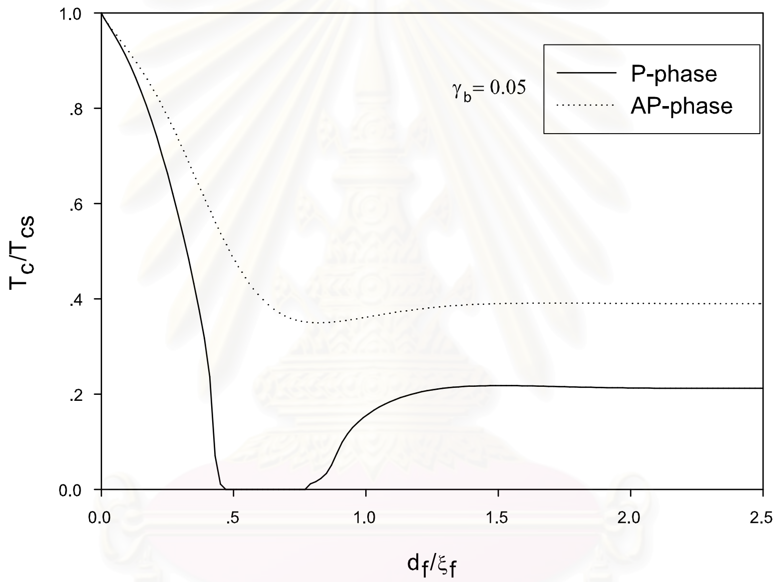


Fig. 4.5 The dependence of the reduced transition temperature T_c/T_{cs} on the normalized ferromagnetic thickness d_f/ξ_f between the parallel and antiparallel magnetization alignments (P-and AP-phases), for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ with the boundary resistivity $\gamma_b = 0.05$

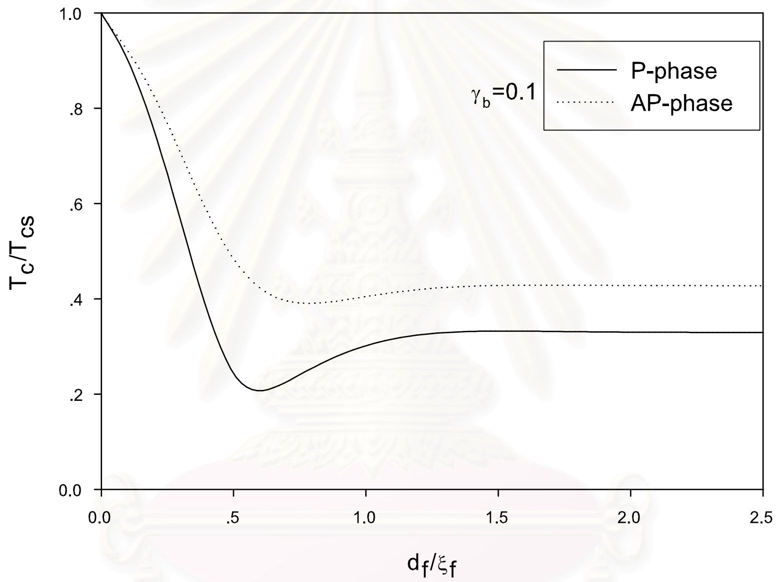


Fig. 4.6 The dependence of the reduced transition temperature T_c/T_{cs} on the normalized ferromagnetic thickness d_f/ξ_f between the parallel and antiparallel magnetization alignments (P-and AP-phases), for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ with the boundary resistivity $\gamma_b = 0.1$

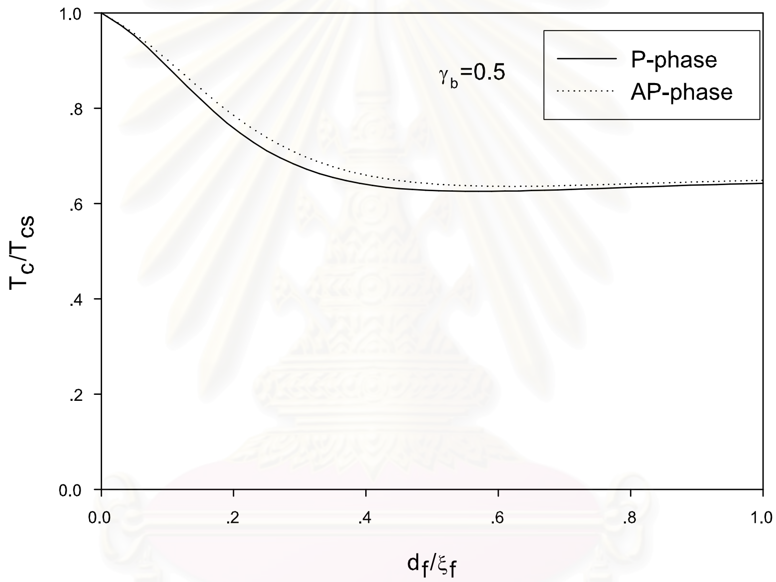


Fig. 4.7 The dependence of the reduced transition temperature T_c/T_{cs} on the normalized ferromagnetic thickness d_f/ξ_f between the parallel and antiparallel magnetization alignments (P-and AP-phases), for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ with the boundary resistivity $\gamma_b = 0.5$

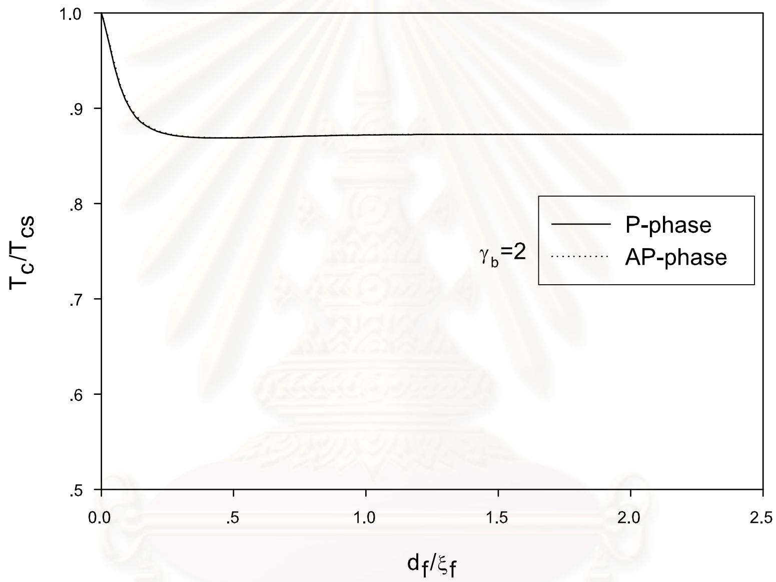


Fig. 4.8 The dependence of the reduced transition temperature T_c/T_{cs} on the normalized ferromagnetic thickness d_f/ξ_f between the parallel and antiparallel magnetization alignments (P-and AP-phases), for $\gamma = 0.125,\ d_s/\xi_s = 2.5,\ I = 6.8\pi T_{cs},\ \xi_f = 7.6\ nm$ with the boundary resistivity $\gamma_b = 2$.

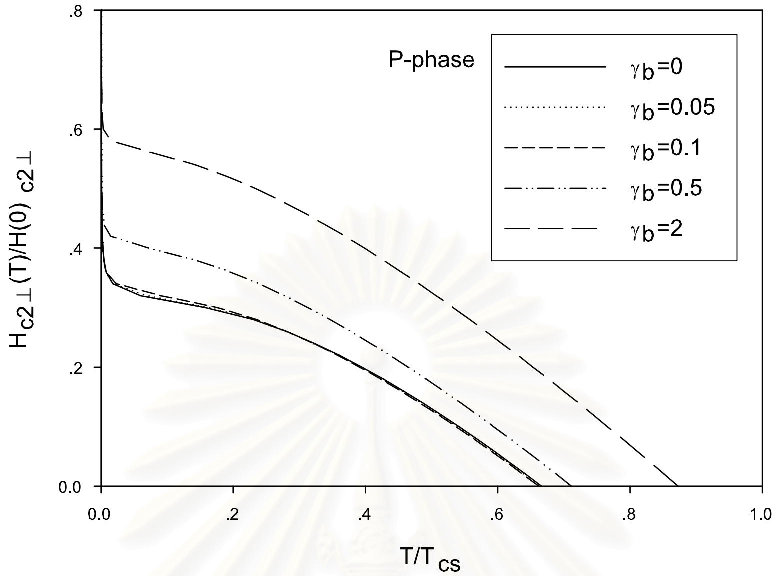


Fig. 4.9 The normalized perpendicular upper critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ versus the reduced temperature T/T_{cs} in the P-phase for $\gamma = 0.125$, $d_s/\xi_s = 2.5, d_f/\xi_f = 0.25$, $I = 6.8\pi T_{cs}$, and varying the boundary resistivity γ_b. The effect of γ_b on the phase diagram $(H_{c2\perp}, T)$ becomes significant at a moderate value, e.g., $\gamma_b = 0.5, 2$, the normalized critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ increases due to the pairing amplitude is confined in the S layer.

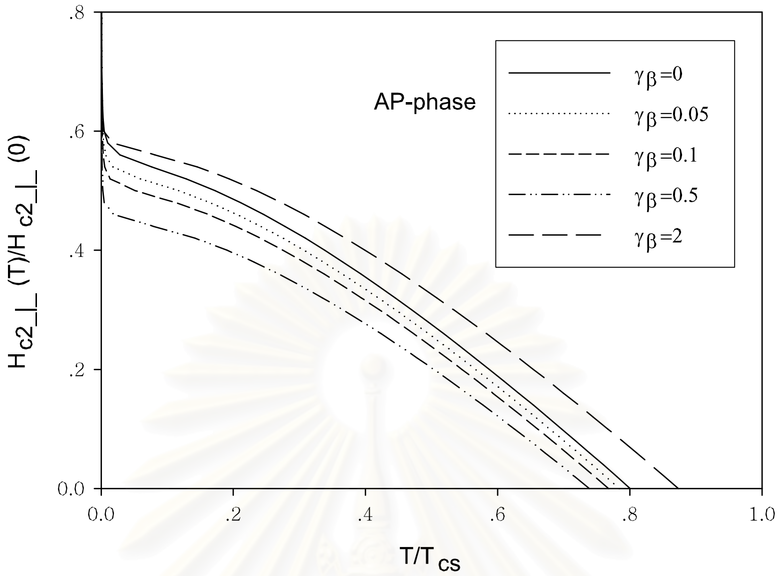


Fig. 4.10 The normalized perpendicular upper critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ versus the reduced temperature T/T_{cs} in the AP-phase for $\gamma = 0.125$, $d_s/\xi_s = 2.5, d_f/\xi_f = 0.25$, $I = 6.8\pi T_{cs}$, and varying the boundary resistivity γ_b. The normalized critical field $H_{c2\perp}(T)/H_{c2\perp}(0)$ decreases as γ_b increase from zero to the moderate value, e.g., $\gamma_b = 0, 0.05, 0.1, 0.5$, but it is higher when γ_b is large.

4.2 Discussion

We study the influence of the proximity effect on FSF trilayers by considering the effect of spin orientation dependence on the transition temperature T_c and $H_{c2\perp}$ and calculate them as a function of the mutual orientation of ferromagnetic exchange fields. Therefore we treat the cases of parallel and antiparallel orientation of ferromagnetic moments. The Usadel equations are solved in the exact multimode method.

In this work, the Usadel equations, the transport-like differential equation, are derived from the Takahashi-Tachiki theory, the integral equation of the de Gennes correlation function. Moreover the generalized Usadel equations including the pair-breaking effects such as orbital diamagnetism, Pauli spin paramagnetism, spin-orbit scattering and magnetic impurity scattering are obtained for the first time.

We propose the exact method for calculating T_c and $H_{c2\perp}$ as a function of the SF parameters by solving the Usadel equations with making use the Takahashi-Tachiki differential equation. The obtained results are identical with Radovic et al.,'s method, the multimode method. Therefore we treat the role of finite transparency at boundary interfaces as well as the arbitrary exchange energy which account for both weak and strong ferromagnets.

We obtain the dependence of T_c and $H_{c2\perp}$ on the mutual orientation of ferromagnetic exchange fields in parallel and antiparallel configurations. The multimode solution to the Usadel equations is expressed through the secular equation which contains the SF parameters such as the superconducting layer thickness d_s, the ferromagnetic layer thickness d_f, the density of states $N_{s,f}$, the diffusion coefficients $D_{s,f}$, the exchange energy I and the dimensionless interface transparency parameter T_f.

When the superconducting layer thickness is thin enough comparable to their coherence length $d_s/\xi_s \ll 1$, all other elements of the secular equation vanish except for the zeroth mode, the single-mode approximation. In the strong ferromagnet limit, we can write down the result in terms of the digamma function. We proceed further by examining the effect of the interface transparency in some limiting cases and find that the high interface transparency limit together with a thick ferromagnet layer gives the transition temperature T_c of the antiparallel phase enhances than the parallel one. The important of the multimode method arises when we deal with the perpendicular upper critical field $H_{c2\perp}$ since the exact solution is required. We have assumed that the anomalous function $F_{s,f}(\vec{r},\omega)$ behaves the same in both layers. This allows us to treat the anomalous function having proximity only in the normal direction. The solution to the anomalous function in the layer plane gives the relation between the magnetic field and the propagating momentum, the perpendicular upper critical field is achieved as the lowest eigenvalue of the harmonic oscillator.

4.3 Conclusions

The proximity effect of superconductors (S) and ferromagnets (F) has unusual features such as the superconducting $\pi-$ phase, the oscillation of the superconducting transition temperature T_c in ferromagnetic layers, of the SF multilayers.

Particular interest is paid to the FSF trilayers proximity structure where the mutual orientations of magnetization can be aligned either parallel (P) or antiparallel (AP). We have calculated the transition temperature T_c and the perpendicular upper critical field $H_{c2\perp}$ as a function of the mutual orientation of the ferromagnetic exchange fields in magnetic layers by solving the Usadel equations in a multimode method. Numerical results of the transition temperature T_c and the perpendicular upper critical field $H_{c2\perp}$ show that the AP configuration enhances the superconductivity than the P one. We obtain the nonmonotonic and the re-entrant behaviors of T_c depending on the material parameters. Investigations of the critical field $H_{c2\perp}$ reveal that the low boundary resistivity γ_b does have influence only on the AP-phase which is due to the weak suppression character of the ferromagnetic exchange fields.

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Curriculum Vitae

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