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TENSOR PRODUCTS OF MODULES OVER SEMIFIELDS

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เราเรียกระบบ $(K,+,\cdot)$ ว่า **กึ่งฟีลด์** ก็ต่อเมื่อ (i) $(K,+)$ เป็นกึ่งกลุ่มสลับที่ที่มีเอกลักษณ์ 0 (ii) $(K\backslash\{0\},\cdot)$ เป็นกลุ่มสลับที่ที่มีเอกลักษณ์ 1 และ $k\cdot 0 = 0\cdot k = 0$ สำหรับทุกๆ $k\in K$ และ (iii) $x(y+z) = xy+xz$ สำหรับทุกๆ $x,y,z\in K$ **มอดูลบนกึ่งฟีลด์** K คือกลุ่มสลับที่ M ที่มี 0 เป็นเอกลักษณ์ และมีฟังก์ชัน $(k,m)\mapsto km$ จาก $K\times M$ ไปยัง M ซึ่งสำหรับทุกๆ $k,k_1,k_2\in K$ และ $m,m_1,m_2\in M$ ได้ว่า (i) $k(m_1+m_2) = km_1+km_2$, (ii) $(k_1+k_2)m = k_1m+k_2m$ และ (iii) $(k_1k_2)m = k_1(k_2m)$ นอกจากนี้ถ้า $1_Km = m$ ทุก $m\in M$ เมื่อ 1_K คือเอกลักษณ์ของ $(K\setminus\{0\},\cdot)$ แล้วเราเรียก M ว่า **ปริภูมิเวกเตอร์บน** K กำหนดให้ X เป็นเซตย่อยของปริภูมิเวกเตอร์ M บนกึ่งฟีลด์ K และ $\langle X\rangle$ เป็นกลุ่มย่อยของ M ที่ก่อกำเนิดโดย $KX = \{kx \mid k\in K$ และ $x\in X\}$ เรากล่าวว่า X **แผ่ทั่ว** M เมื่อ $\langle X\rangle = M$ เซตย่อย X เป็นเซต **อิสระเชิงเส้น** ถ้า X สอดคล้องข้อใดข้อหนึ่งของเงื่อนไขต่อไปนี้คือ (i) $X=\emptyset$ หรือ (ii) $|X|=1$ และ $X\neq\{0\}$ หรือ (iii) $|X|>1$ และ $x\notin\langle X\setminus\{x\}\rangle$ สำหรับทุกๆ $x\in X$ นอกจากนี้เรากล่าวว่าเซตย่อย X เป็น **ฐานหลัก** ของ M บน K เมื่อ X เป็นเซตอิสระเชิงเส้นที่แผ่ทั่ว M

ในงานวิจัยนี้เราศึกษามอดูลบนกึ่งฟีลด์ ซึ่งเป็นส่วนขยายของปริภูมิเวกเตอร์บนฟีลด์และเป็นอีกแขนงหนึ่งในพีชคณิตนามธรรม เราสามารถนิยามผลคูณเทนเซอร์ของมอดูลหรือฟีลด์และพิสูจน์ทฤษฎีบท สมบัติการส่งแบบเอกภพของผลคูณเทนเซอร์ของมอดูลบนกึ่งฟีลด์ได้ คุณศิริจันทร์ พหุพงศ์ทรัพย์ได้ศึกษาและขยายทฤษฎีบทบางบทในปริภูมิเวกเตอร์บนฟีลด์ไปยังทฤษฎีบทในปริภูมิเวกเตอร์บนกึ่งฟีลด์ K โดยที่ K มีสมบัติว่า สำหรับทุกๆ $\alpha,\beta\in K$ จะมี $\gamma\in K$ ซึ่ง $\alpha+\gamma=\beta$ หรือ $\beta+\gamma=\alpha$ เราจะศึกษาและขยายทฤษฎีบทอื่นๆบางบทในปริภูมิเวกเตอร์บนกึ่งฟีลด์นั้น ยิ่งไปกว่านั้นเราสามารถพิสูจน์ว่า ทุกฐานหลักอนันต์ของปริภูมิเวกเตอร์บนกึ่งฟีลด์มีจำนวนเชิงการนับของฐานเท่ากัน

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A system $(K, +, \cdot)$ is said to be a *semifield* if (i) $(K, +)$ is a commutative semigroup with identity 0, (ii) $(K \setminus \{0\}, \cdot)$ is an abelian group with identity 1 and $k \cdot 0 = 0 \cdot k = 0$ for all $k \in K$, and (iii) $x(y+z) = xy + xz$ for all $x, y, z \in K$. A *module over a semifield* K is an abelian additive group M with identity 0, for which there is a function $(k, m) \mapsto km$ from $K \times M$ into M such that for all $k, k_1, k_2 \in K$ and $m, m_1, m_2 \in M$, (i) $k(m_1 + m_2) = km_1 + km_2$, (ii) $(k_1 + k_2)m = k_1 m + k_2 m$ and (iii) $(k_1 k_2)m = k_1(k_2 m)$. Moreover, if $1_K m = m$ for all $m \in M$ where 1_K is the identity of $(K \setminus \{0\}, \cdot)$, then M is said to be a *vector space over* K. Let X be a subset of a vector space M over a semifield K and $\langle X \rangle$ be the subgroup of M generated by $KX = \{kx \mid k \in K \text{ and } x \in X\}$. We call that X *spans* M if $\langle X \rangle = M$. The set X is said to be a *linearly independent* set if it satisfies one of the following conditions: (i) $X = \emptyset$ or (ii) $|X| = 1$ and $X \neq \{0\}$, or (iii) $|X| > 1$ and $x \notin \langle X \setminus \{x\} \rangle$ for all $x \in X$. Furthermore, the set X is said to be a *basis* of M over K if X is a linearly independent set which spans M.

In this research, we study another area in abstract algebra, a module over a semifield which is a generalization of a vector space over a field. We can define tensor products of modules over semifields and prove the Universal Mapping Property of Tensor Products. Miss Sirichan Pahupongsab studied and generalized theorems in vector spaces over fields to those in vector spaces over a semifield K such that for all $\alpha, \beta \in K$ there exists a $\gamma \in K$ which causes $\alpha + \gamma = \beta$ or $\beta + \gamma = \alpha$. We carry on investigating and generalizing some other theorems in vector spaces over such a semifield. Besides, we can prove that every infinite basis of a vector space over a semifield has the same cardinality.

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Field of study **Mathematics** | Advisor's signature...............................
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CHAPTER I

INTRODUCTION

In [3], Sirichan Pahupongsab studied and generalized theorems in vector spaces over fields to those in vector spaces over semifields satisfying a certain property. Moreover, she considered linear transformations of vector spaces over semifields.

In this research, we carry on investigating and generalizing some other theorems in vector spaces over semifields with the same property. In addition, we study modules over any semifields and obtain similar theorems in ring modules. Furthermore, we explore tensor products of modules over semifields and tensor products of vector spaces over semifields.

This thesis contains 4 chapters. Chapter I is an introduction.

In Chapter II, we introduce some notation, definitions, theorems, corollaries and examples which are required in the following chapters.

In Chapter III, we study modules over semifields, homomorphisms of modules over semifields, tensor products of modules over semifields and multilinear maps. We also give examples in each topic.

In Chapter IV, we extend our work from [3] in order to obtain more theorems in vector spaces over semifields. Moreover, we discuss deeply in tensor products of vector spaces over semifields.

In this thesis references are denoted by square brackets [ ] and equations by round brackets ( ), for example, (1.2.3) denotes the equation 3 in Section 2 of Chapter I.

CHAPTER II

PRELIMINARIES

In this chapter, we present some notation, known definitions and theorems which will be referred later in this thesis.

2.1. Notation

We summarize standard notation being used throughtout this thesis.

$\mathbb{Z}$ is the set of all integers.

$\mathbb{Z}^+$ is the set of all positive integers.

$\mathbb{Q}$ is the set of all rational numbers.

$\mathbb{Q}^+$ is the set of all positive rational numbers.

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$.

$\mathbb{R}$ is the set of all real numbers.

$\mathbb{R}^+$ is the set of all positive real numbers.

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

$\aleph_0$ is the cardinal number of $\mathbb{Z}$.

2.2. Known Definitions and Theorems

In this section, we require the following definitions, theorems and examples that will be used in Chapter III and Chapter IV.

First, we follow notion of semifields given in [3] and [5].

Definition 2.2.1. [3] A system $(K, +, \cdot)$ is said to be a *semifield* if

(i) $(K, +)$ is a commutative semigroup with identity 0,

(ii) $(K\backslash\{0\}, \cdot)$ is an abelian group and $k \cdot 0 = 0 \cdot k = 0$ for all $k \in K$ and

(iii) $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in K$.

We always denote the identity of the group $(K \backslash \{0\}, \cdot)$ by 1_K and $x \cdot y$ by xy for all $x, y \in K$. Moreover, we call an element of K is a *scalar*.

Definition 2.2.2. [3] Let K be a semifield. A nonempty subset L of K is said to be a *subsemifield* of K if

(i) $0 \in L$ and $L \neq \{0\}$,

(ii) for all $x, y \in L$, with $y \neq 0$, implies $xy^{-1} \in L$, and

(iii) for all $x, y \in L, x + y \in L$.

Example 2.2.3. [3]

(i) Every field is a semifield.

(ii) $(\mathbb{Q}_0^+, +, \cdot)$ and $(\mathbb{R}_0^+, +, \cdot)$ are semifields which are not fields.

(iii) If we define the binary operation $*$ on $\mathbb{Q}_0^+$ by $x * y = \max\{x, y\}$ for all $x, y \in \mathbb{Q}_0^+$, then $(\mathbb{Q}_0^+, *, \cdot)$ is a semifield but not a field.

(iv) If we define two binary operations $\oplus$ and $\odot$ on $\mathbb{Z} \cup \{\varepsilon\}$, where ε is a new symbol which is not an integer, by $x \oplus y = \max\{x, y\}, x \oplus \varepsilon = \varepsilon \oplus x = x$ and $\varepsilon \oplus \varepsilon = \varepsilon$ and $x \odot y = x + y, x \odot \varepsilon = \varepsilon \odot x = \varepsilon, \varepsilon \odot \varepsilon = \varepsilon$ for all $x, y \in \mathbb{Z}$. Then $(\mathbb{Z} \cup \{\varepsilon\}, \oplus, \odot)$ is a semifield but not a field. Moreover, this example is still true if $\mathbb{Z}$ is replaced by $\mathbb{Q}$.

(v) $(\mathbb{Q}^+ \times \mathbb{Q}^+ \cup \{(0,0)\}, +, \cdot)$ is a semifield.

Later, we will deal with infinite sets so that we need the followings which are standard.

Definition 2.2.4. [2] Let α and β be cardinal numbers, A and B be disjoint sets such that $|A| = \alpha$ and $|B| = \beta$. The *sum* $\alpha + \beta$ is defined to be the cardinal number $|A \cup B|$. The *product* $\alpha\beta$ is defined to be the cardinal number $|A \times B|$.

Theorem 2.2.5. [2] ***Schroeder-Bernstein***

If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Theorem 2.2.6. [2] *If α and β are cardinal numbers such that $0 \neq \beta \leq \alpha$ and α is infinite, then $\alpha\beta = \alpha$; in particular, $\alpha\aleph_0 = \alpha$ and if β is finite $\aleph_0\beta = \aleph_0$.*

Corollary 2.2.7. [2] *If A is an infinite set and $\mathfrak{P}(A)$ the set of all finite subsets of A, then $|\mathfrak{P}(A)| = |A|$.*

The following familiar definitions and theorems regarding notion of free abelian groups are needed to define tensor products of modules over semifields in Section 3.3.

Theorem 2.2.8. *Let $\mathfrak{X}$ be a nonempty set and let*

$$\mathcal{FA}(\mathfrak{X}) = \big\{ f : \mathfrak{X} \to \mathbb{Z} \mid \exists F \subseteq \mathfrak{X} \text{ such that } |F| < \infty \text{ and } f(x) = 0 \text{ for all } x \in \mathfrak{X} \setminus F \big\}.$$

Define $+$ on $\mathcal{FA}(\mathfrak{X})$ by for any $f, g \in \mathcal{FA}(\mathfrak{X})$,

$$(f+g)(x) = f(x) + g(x) \text{ for all } x \in \mathfrak{X}.$$

Then $(\mathcal{FA}(\mathfrak{X}), +)$ is an abelian group.

Definition 2.2.9. Let $\mathfrak{X}$ be a nonempty set. For any $x \in \mathfrak{X}$, define $f_x : \mathfrak{X} \to \mathbb{Z}$ by

$$f_x(y) = \delta_{xy} = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

We can see that for each nonempty set $\mathfrak{X}$, $f_x \in \mathcal{FA}(\mathfrak{X})$ for all $x \in \mathfrak{X}$.

Definition 2.2.10. Let $\mathfrak{X}$ be a nonempty set. We define a function τ from $\mathfrak{X}$ into $\mathcal{FA}(\mathfrak{X})$ by $\tau(x) = f_x$ for all $x \in \mathfrak{X}$. We usually denote $\tau(x)$ by $\overline{x}$ for any $x \in \mathfrak{X}$ since τ is injective.

Proposition 2.2.11. *Let $\mathfrak{X}$ be a nonempty set and $f \in \mathcal{FA}(\mathfrak{X})$. Then either $f \equiv 0$ (the zero function) or $f = \sum_{x \in F} f(x) f_x$ where $\emptyset \neq F \subseteq \mathfrak{X}$, $|F| < \infty$ and $f(x) = 0$ for all $x \in \mathfrak{X} \setminus F$.*

Lemma 2.2.12. *For any $g \in \mathcal{FA}(\mathfrak{X}) \setminus \{0\}$, there exist unique distinct $x_1, \ldots, x_n \in \mathfrak{X}$ and unique non-zero integers $\alpha_1, \ldots, \alpha_n$ such that*

$$g = \alpha_1 \overline{x}_1 + \cdots + \alpha_n \overline{x}_n = \sum_{x \in \{x_1, \ldots, x_n\}} g(x) f_x .$$

Definition 2.2.13. A group A is a *free abelian group* on a nonempty set $\mathfrak{X}$ if

(i) A is an abelian group and

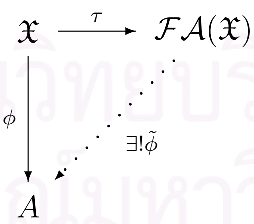
(ii) $\forall g \in A \setminus \{0\}$ $\exists!$ distinct $x_1, \ldots, x_n \in \mathfrak{X}$ $\exists! \alpha_1, \ldots, \alpha_n \in \mathbb{Z} \setminus \{0\}$,

$g = \alpha_1 x_1 + \cdots + \alpha_n x_n$.

We sometimes call $\mathfrak{X}$ a *basis for the free abelian group* A.

Note 2.2.14. Let $\mathfrak{X}$ be a nonempty set. Then $(\mathcal{FA}(\mathfrak{X}), +)$ is a free abelian group on $\tau(\mathfrak{X})$. Sometimes, we say, instead that $\mathcal{FA}(\mathfrak{X})$ is a free abelian group on $\mathfrak{X}$.

Proposition 2.2.15. *Let A be an (additive) abelian group, $\mathfrak{X}$ a nonempty set and $\phi : \mathfrak{X} \to A$ a function. Then there exists a unique $\tilde{\phi} : \mathcal{FA}(\mathfrak{X}) \to A$ such that $\tilde{\phi}$ is a group homomorphism and $\tilde{\phi} \circ \tau = \phi$, i.e., the following diagram commutes:*



CHAPTER III

MODULES OVER SEMIFIELDS

In this chapter, we investigate modules over semifields in various aspects. Definitions and theorems of modules over semifields are given in Section 3.1. Then, in Section 3.2, we study homomorphisms of modules over semifields. We discuss tensor products of modules over semifields in Section 3.3. Finally, we introduce and study multilinear maps of modules over semifields in Section 3.4.

3.1. Modules over Semifields

Roughly speaking, we can see from Definition 2.2.1 that the definition of a semifield is similar to the one of a commutative ring by interchanging roles between addition and multiplication. For this reason, we define modules over semifields in the same way as modules over rings.

Definition 3.1.1. Let K be a semifield. A *left K-module* or *left module over K* is an additive abelian group M together with a function $K \times M \to M$ (the image of (k, m) being denoted by km) such that for all $m, m_1, m_2 \in M$ and $k, k_1, k_2 \in K$,

(i) $k(m_1 + m_2) = km_1 + km_2$,

(ii) $(k_1 + k_2)m = k_1m + k_2m$ and

(iii) $(k_1k_2)m = k_1(k_2m)$.

Moreover, if $1_K m = m$ for all $m \in M$ where 1_K is the identity of $\big(K\setminus\{0\}, \cdot\big)$, then M is said to be a *left vector space over K* or *unitary left K-module*.

A *right K-module* is defined similarly via a function $M \times K \to M$ (the image of (m, k) being denoted by mk) and satisfies the obvious analogues of (i)–(iii). Besides,

M is a *right vector space over* K or *unitary right* K*-module* if it is a right K-module and $m1_K = m$ for all $m \in M$.

Note 3.1.2. Let K be a semifield. Then

M is a left K-module if and only if M is a right K-module

with $mk = km$ for all $k \in K$ and $m \in M$.

Proof. Assume that M is a left K-module. Let $k, k_1, k_2 \in K$ and $m, m_1, m_2 \in M$. Thus

$$(m_1 + m_2)k = k(m_1 + m_2) = km_1 + km_2 = m_1k + m_2k,$$

$$m(k_1 + k_2) = (k_1 + k_2)m = k_1m + k_2m = mk_1 + mk_2 \text{ and}$$

$$m(k_1k_2) = (k_1k_2)m = (k_2k_1)m = k_2(k_1m) = (k_1m)k_2 = (mk_1)k_2.$$

Therefore M is a right K-module.

Conversely, if M is a right K-module, then M is also a left K-module by the similar way. □

Example 3.1.3.

(i) $\mathbb{Q}^n$ is both a left and a right module over $\mathbb{Q}_0^+$ and also both a left and a right vector space over $\mathbb{Q}_0^+$ for all $n \in \mathbb{N}$.

(ii) $\mathbb{R}^n$ is both a left and a right module over $\mathbb{R}_0^+$ and also both a left and a right vector space over $\mathbb{R}_0^+$ for all $n \in \mathbb{N}$.

(iii) $\mathbb{Q} \times \mathbb{R}$ is both a left and a right module over $\mathbb{Q}_0^+$ and also both a left and a right vector space over $\mathbb{Q}_0^+$.

(iv) If $n \in \mathbb{N}$ and $M_1, \ldots, M_n$ are modules over a semifield K, then $M_1 \times \cdots \times M_n$ is a module over K under usual addition and scalar multiplication.

Definition 3.1.4. Let $K$and S be semifields. An abelian group $(M,+)$ is a $K-S$ *bimodule* provided that M is both a left K-module and a right S-module and $k(ms)=(km)s$ for all $k\in K, s\in S$ and $m\in M$.

We sometimes write $_KM_S$ to indicate the fact that M is a K-S bimodule. Simiilary, $_KM$ indicates a left K-module M and M_S a right S-module M. Sometimes, we simply write $_KM$ as "M is a left K-module". From now on, unless specified otherwise, "K-module"means "left K-module". Moreover, if M is a vectore space over K, then "vector space "means "left vector space ".

Example 3.1.5.

(i) If K is a semifield and M is a K-module, then M is a K-K bimoddule.

(ii) $\mathbb{R}^n$ is an $\mathbb{Q}_0^+$-$\mathbb{R}_0^+$ bimodule.

Proposition 3.1.6. *If M is a module over a semifield K, then the following statements hold:*

(i) $0m=0$ *for all* $m\in M$,

(ii) $k0=0$ *for all* $k\in K$,

(iii) $-(km)=k(-m)$ *for all* $k\in K$ *and* $m\in M$ *and*

(iv) $-(k(-m))=km$ *for all* $k\in K$ *and* $m\in M$.

Proof. This is straightforward. □

From now on, if M is a module over a semifield K, then we write $-km$ instead of $-(km)$ for all $k\in K$ and $m\in M$.

Definition 3.1.7. Let M be a module over a semifield K. A *submodule* of M is a subset of M which is, itself, a module over K with the addition and scalar multiplication of M. A submodule of a vector space over a semifield K is called a *subspace.*

Theorem 3.1.8. *Let N be a nonempty subset of a module M over a semifield K. Then N is a submodule of M if and only if $n_1 - n_2, kn \in N$ for all $n, n_1, n_2 \in N$ and $k \in K$.*

Proof. This is straightforward. □

Theorem 3.1.9. [3] *Let W be a nonempty subset of a vector space V over a semifield K. Then the following statements are equivalent.*

(i) *W is a subspace of V.*

(ii) *If $w, w_1, w_2 \in W$ and $k \in K$, then $w_1 - w_2, kw \in W$.*

(iii) *If $w_1, w_2 \in W$ and $k_1, k_2 \in K$, then $k_1w_1 - k_2w_2 \in W$.*

Example 3.1.10.

(i) $\mathbb{Q}^n$ is a submodule of $\mathbb{R}^n$ over $\mathbb{Q}_0^+$ and also a subspace of $\mathbb{R}^n$ over $\mathbb{Q}_0^+$.

(ii) $\mathbb{Q}\times\mathbb{R}$ is a submodule of $\mathbb{R}\times\mathbb{R}$ over $\mathbb{Q}_0^+$ and also a subspace $\mathbb{R}\times\mathbb{R}$ over $\mathbb{Q}_0^+$.

Theorem 3.1.11. *The intersection of any collections of submodules of a module M over a semifield is also a submodule of M.*

Proof. Let M be a module over a semifield K and $\{N_i \mid i \in I\}$ be any collections of submodules M. Let $n_1, n_2 \in \bigcap_{i\in I} N_i$ and $\alpha, \beta \in K$. Then $n_1, n_2, n \in N_i$ for all i. For each $i \in I$, since N_i is a submodule of M, we have $n_1 - n_2, \alpha n \in N_i$. Thus $n_1 - n_2, \alpha n \in \bigcap_{i\in I} N_i$. Therefore $\bigcap_{i\in I} N_i$ is a submodule of M. □

Corollary 3.1.12. *The intersection of any collections of subspaces of a vector space V over a semifield is also a subspace of V.*

Proposition 3.1.13. *Let M be a module over a semifield and X a subset of M. Moreover, let $\{N_i \mid i \in I\}$ be the family of all submodules of M containing X, then $\bigcap_{i\in I} N_i$ is the smallest submodule of M containing X.*

Moreover, if M *is a vector space over a semifield, then* $\bigcap_{i\in I} N_i$ *is the smallest subspace of* M *containing* X.

Proof. By Theorem 3.1.11, $\bigcap_{i\in I} N_i$ is a submodule of M. Since $X \subseteq N_i$ for all i, we have $\bigcap_{i\in I} N_i$ is a submodule of M containing X. Let W be a submodule of M containing X. Then $W \in \{N_i \mid i \in I\}$. Thus $\bigcap_{i\in I} N_i \subseteq W$. Therefore $\bigcap_{i\in I} N_i$ is the smallest submodule of M containing X. $\square$

Definition 3.1.14. If X is a subset of a module M over a semifield, then the intersection of all submodules of M containing X is called the *submodule of* M *generated by* X.

In particular, if M is a vector space over a semifield, then the intersection of all subspaces of M containing X is called the *subspace of* M *generated by* X.

Definition 3.1.15. Let M be a module over a semifield K and X a subset of M. Define $\langle X\rangle$ to be the smallest subgroup of M containing $KX = \{kx \mid k \in K \text{ and } x \in X\}$. Moreover, if $X = \emptyset$, then $\langle X\rangle = \{0\}$.

Note 3.1.16. Let M be a module over a semifield K and X a nonempty subset of M. Since KX is a subset of M, the subgroup of M generated by KX is, in fact,

$$\langle X\rangle = \{\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1(-x'_1) + \cdots + \beta_m(-x'_m) \mid m, n \in \mathbb{N}, x_i, x'_i \in X \text{ and } \alpha_i, \beta_i \in K\}.$$

Theorem 3.1.17. *Let* V *be a vector space over a semifield and* X *a subset of* V. *Then* $\langle X\rangle$ *is a subspace of* V *generated by* X.

Proof. Let V be a vector space over a semifield K. If $X = \emptyset$, then $\langle X\rangle = \{0\}$ so that $\langle X\rangle$ is a subspace of V generated by $\emptyset$. Thus we assume that $X \neq \emptyset$. First, we show that $\langle X\rangle$ is a subspace of V containing X. Let $\alpha \in K$ and $v, v' \in \langle X\rangle$. Then $v - v' \in \langle X\rangle$ since $\langle X\rangle$ is a subgroup of V. Since $v \in V$, there exist $\alpha_1, \ldots, \alpha_n, \acute{\alpha_1}, \ldots, \acute{\alpha_m} \in K$ and $x_1, \ldots, x_n, \acute{x_1}, \ldots, \acute{x_m} \in X$ such that $v = \alpha_1 x_1 + \cdots + \alpha_n x_n + \acute{\alpha_1}(-\acute{x_1}) + \cdots + \acute{\alpha_m}(-\acute{x_m})$. Then

$$\begin{aligned}\alpha v &= \alpha\left(\alpha_1 x_1 + \cdots + \alpha_n x_n + \acute{\alpha_1}(-\acute{x_1}) + \cdots + \acute{\alpha_m}(-\acute{x_m})\right) \\ &= (\alpha\alpha_1)x_1 + \cdots + (\alpha\alpha_n)x_n + (\alpha\acute{\alpha_1})(-\acute{x_1}) + \cdots + (\alpha\acute{\alpha_m})(-\acute{x_m}).\end{aligned}$$

Hence $\alpha v \in \langle X\rangle$. Thus $\langle X\rangle$ is a subspace of V. Since $1_K \in K$, we obtain that $X \subseteq KX \subseteq \langle X\rangle$. Therefore $\langle X\rangle$ is a subspace of V containing X. Let W be a subspace of V containing X. Thus $KX \subseteq KW \subseteq W$. Since W is a subspace of V and $KX \subseteq W$, we obtain that W is a subgroup of V containing KX. Then $\langle X\rangle \subseteq W$. Therefore $\langle X\rangle$ is the smallest subspace of V containing X, i.e., $\langle X\rangle$ is a subspace of V generated by X. $\square$

If V is a vector space over a semifield and X is a subset of V, then we can characterize the smallest subspace W of V generated by X according to Theorem 3.1.17, i.e., $W = \langle X\rangle = \{\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1(-\acute{x_1}) + \cdots + \beta_m(-\acute{x_m}) \mid m, n \in \mathbb{N}, x_i, \acute{x_i} \in X$ and $\alpha_i, \beta_i \in K\}$.

On the other hand, if V is a module over a semifield but not a vector space and X is a subset of V, we know only that the smallest submodule of V generated by X is just simply the intersection of all submodules of M containing X.

Notation 3.1.18. [3] Let V be a vector space over a semifield and X a subset of V. If $X = \{x_1, \ldots, x_n\}$, let $\langle x_1, \ldots, x_n\rangle$ denote $\left\langle\{x_1, \ldots, x_n\}\right\rangle$ and simply call $\langle x_1, \ldots, x_n\rangle$ the subspace of V generated by $x_1, \ldots, x_n$. We denote the cardinality of X by $|X|$.

Definition 3.1.19. Let V be a vector space over a semifield and X a subset of V. If $\langle X\rangle = V$, then we say that X *spans* V or V is a vector space generated by X.

Example 3.1.20. Let $e_1, \ldots, e_n \in \mathbb{Q}^n$ be defined by

$$e_1 = (1, 0, \ldots, 0, 0),\ e_2 = (0, 1, 0 \ldots, 0, 0),\ \ldots,\ e_n = (0, 0, \ldots, 0, 1).$$

(i) In Example 3.1.10 (i), $\mathbb{Q}^n$ is a subspace of $\mathbb{R}^n$ over $\mathbb{Q}_0^+$ and $\mathbb{Q}^n = \langle e_1, \ldots, e_n \rangle$.

(ii) $\mathbb{R}^n$ is a vector space over $\mathbb{R}_0^+$ and $\mathbb{R}^n = \langle e_1, \ldots, e_n \rangle$.

Definition 3.1.21. [3] A subset X of a vector space V over a semifield K is *linearly independent* if it satisfies one of the following conditions:

(i) $X = \emptyset$,

(ii) $|X| = 1$ and $X \neq \{0\}$, or

(iii) $|X| > 1$ and $x \notin \langle X \setminus \{x\} \rangle$ for all $x \in X$.

Moreover, X is said to be a *linearly dependent* set if X is not linearly independent.

Remark 3.1.22. [3] If X is a subset of a vector space V containing 0, then X is always linearly dependent.

Definition 3.1.23. [3] Let X be a subset of a vector space V over a semifield. Then X is a *basis* of V if X is a linearly independent set which spans V.

Note 3.1.24. If $V = \{0\}$, then $\emptyset$ is the only basis of V.

Example 3.1.25.

(i) In Example 3.1.20 (i), $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}_0^+$ and $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{Q}^n$.

(ii) In Example 3.1.20 (ii), $\mathbb{R}^n$ is a vector space over $\mathbb{R}_0^+$ and $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

3.2. Homomorphisms

In this section, we are interested in studying homomorphisms of modules over semifields which, again, are analogous to those of modules over rings.

Definition 3.2.1. Let M and N be modules over a semifield K and T a mapping from M into N. Then T is said to be a *(left)* K*-module homomorphism* if for all $\alpha \in K$ and $m, m_1, m_2 \in M$,

$$T(m_1 + m_2) = T(m_1) + T(m_2) \quad \text{and} \quad T(\alpha m) = \alpha T(m).$$

If a (left) K-module homomorphism T is injective, then we say that T is a (left) K*-module monomorphism*.

If a (left) K-module homomorphism T is surjective, then we say that T is a (left) K*-module epimorphism*.

If a (left) K-module homomorphism T is bijective, then we say that T is a (left) K*-module isomorphism*.

Moreover, we say that M is *isomorphic* to N, denoted by $M \cong N$, if there exists a (left) K-module isomorphism from M into N.

Furthermore, T is said to be a *(right)* K*-module homomorphism* if M_K and N_K are modules over K and for all $\alpha \in K$ and $m, m_1, m_2 \in M$,

$$T(m_1 + m_2) = T(m_1) + T(m_2) \quad \text{and} \quad T(m\alpha) = T(m)\alpha.$$

Definition 3.2.2. Let V and W be vector spaces over a semifield K. Then a K*-linear transformation* is a left K-module homomorphism from V into W.

If a K-linear transformation T is injective, then we say that T is a *monomorphism* (*over* K).

If a K-linear transformation T is surjective, then we say that T is an *epimorphism* (*over* K).

If a K-linear transformation is bijective, then we say that T is an *isomorphism* (*over* K).

Example 3.2.3.

(i) Let n and m be positive integers with $m > n$ and let $\mathbb{R}^m, \mathbb{R}^n$ be vector spaces over $\mathbb{R}_0^+$. Then the function $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$ is an $\mathbb{R}_0^+$-linear transformation.

(ii) Let n be a positive integer and $\mathbb{R}^n$ be a vector space over $\mathbb{Q}_0^+$. Then the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x_1, \ldots, x_n) = (0, x_1, \ldots, x_{n-1})$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a $\mathbb{Q}_0^+$-linear transformation.

(iii) Let n be a positive integer and $\mathbb{R}^n$ be a vector space over $\mathbb{Q}_0^+$. Then the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a $\mathbb{Q}_0^+$-linear transformation.

Lemma 3.2.4. *Let M and N be modules over a semifield K and T be a K-module homomorphism of M into N. Then*

$$T(0) = 0 \text{ and } T(-m) = -T(m) \text{ for all } m \in M.$$

Proof. This is obvious. □

Proposition 3.2.5. *Let V and W be vector spaces over a semifield K and $T : V \to W$. Then the following statements are equivalent:*

(i) *T is a K-linear transformation.*

(ii) *For all $v_1, v_2 \in V$ and $\alpha, \beta \in K$, $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$.*

(iii) *For all $v_1, v_2 \in V$ and $\alpha \in K$, $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$.*

Proof. This is straightforward. □

Definition 3.2.6. Let M and N be modules over a semifield K and T a K-module homomorphism of M into N. We define the *kernel* of T, denoted by $\ker T$, and *image* of T, denoted by $\mathrm{im}T$, as follows:

$$\ker T = \{m \in M \mid T(m) = 0\} = T^{-1}(\{0\}) \quad \text{and}$$

$$\mathrm{im}\ T = \{T(m) \mid m \in M\} = T(M).$$

Example 3.2.7.

(i) From Example 3.2.3 (i), $\ker T = \{(0, \dots, 0, x_{n+1}, \dots, x_m) \mid x_i \in \mathbb{R} \quad \text{for} \quad i = n+1, \dots, m\}$ and $\mathrm{im}\ T = \mathbb{R}^n$.

(ii) From Example 3.2.3 (ii), $\ker T = \{(0, \dots, 0, x_n) \mid x_n \in \mathbb{R}\}$ and $\mathrm{im}\ T = \{(0, x_1, \dots, x_{n-1}) \mid x_i \in \mathbb{R} \quad \text{for} \quad i = 1, \dots, n-1\} = \{0\} \times \mathbb{R}^{n-1}$.

(iii) From Example 3.2.3 (iii), $\ker T = \{(0, \dots, 0)\}$ and $\mathrm{im}\ T = \mathbb{R}^n$.

Proposition 3.2.8. *Let M and N be modules over a semifield K and $T : M \to N$ a K-module homomorphism. Then the following statements hold.*

(i) *T is injective if and only if $T(m) = 0$ implies that $m = 0$ for all $m \in M$.*

(ii) *If M' is a submodule of M, then $T(M')$ is a submodule of N. In particular, $\mathrm{im}\ T$ is a submodule of N.*

(iii) *If N' is a submodule of N, then $T^{-1}(N')$ is a submodule of M. In particular, $\ker T$ is a submodule of M.*

Proof. This is straightforward. □

Corollary 3.2.9. *If $T : M \to N$ is a K-module homomorphism, then*

T is injective if and only if $\ker T = \{0\}$.

Proposition 3.2.10. *Let M, N and W be modules over a semifield K. Then the following statements hold.*

(i) *If $T_1 : M \to N$ and $T_2 : N \to W$ are K-module homomorphisms, then $T_2 \circ T_1 : M \to W$ is also a K-module homomorphism.*

(ii) *If T is a K-module isomorphism from M onto N, then T^{-1} is a K-module isomorphism from N onto M.*

(iii) *If $T_1 : M \to N$ and $T_2 : N \to W$ are K-module isomorphisms, then $T_2 \circ T_1 : M \to W$ is also a K-module isomorphism.*

Proof. This is obvious. □

3.3. Tensor Products of Modules over Semifields

This section devotes to tensor products of modules over semifields. For given modules M and N over a semifield K, we know from Example 4.1.3 (iv) that $M \times N$ is a module over K. We would like to find another module over K arising from M and N which is different from $M \times N$.

We come across that the tensor product of M and N is the case. Moreover, the notion of free abelian groups (summarized in Section 2.2) plays a major role for construction the tensor product of modules over a semifield.

Definition 3.3.1. Let M_K and ${}_KN$ be modules over a semifield K and let $\mathcal{F}$ be the free abelian group $\mathcal{FA}(M \times N)$ on $M \times N$. Let L be the subgroup of $\mathcal{F}$ generated by all elements of the following forms (where $\alpha \in K, m, m' \in M$ and $n, n' \in N$):

(i) $f_{m+m',n} - f_{m,n} - f_{m',n}$, which is the same as $\tau(m+m', n) - \tau(m, n) - \tau(m', n)$,

(ii) $f_{m,n+n'} - f_{m,n} - f_{m,n'}$, which is the same as $\tau(m, n+n') - \tau(m, n) - \tau(m, n')$,

(iii) $f_{m\alpha,n} - f_{m,\alpha n}$, which is the same as $\tau(m\alpha, n) - \tau(m, \alpha n)$.

Since L is the subgroup of the abelian group $\mathcal{F}$, we obtain that the quotient group $\mathcal{F}/L$ exists. We call $\mathcal{F}/L$ the *tensor product* of M and N, and denoted by $M \otimes_K N$.

Note 3.3.2. If M_K and ${}_KN$ are modules over a semifield K, then $(M \otimes_K N, +)$ is an abelian group.

Definition 3.3.3. Let M_K and ${}_KN$ be modules over a semifield K. For each $m \in M$ and $n \in N$ define a function $f_{(m,n)} : M \times N \to \mathbb{Z}$ by

$$f_{(m,n)}(x,y) = \begin{cases} 1, & \text{if } (x,y) = (m,n), \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in M$ and $y \in N$.

Note 3.3.4. If M_K and ${}_KN$ are modules over a semifield K, then it is easy to see that $f_{(m,n)} \in \mathcal{F}$ for all $m \in M$ and $n \in N$.

Recall that $\mathcal{F}/L = \{f + L \mid f \in \mathcal{F}\}$ and $\mathcal{F} = \mathcal{FA}(M \times N)$. For any $m \in M$ and $n \in N$, we write $m \otimes n$ for $f_{(m,n)} + L = \tau(m,n) + L$. Moreover, we can see that $\mathcal{F}/L$ is generated by $\{m \otimes n \mid m \in M \text{ and } n \in N\}$ as a ring-module over $\mathbb{Z}$, i.e.,
$\forall f + L \in \mathcal{F}/L, f + L = \sum\limits_{(m,n) \in F \subseteq M \times N} \alpha_{(m,n)}(m \otimes n)$, where $\alpha_{(m,n)} \in \mathbb{Z}$ and $|F| < \infty$, and we simply write $f+L$ as $\sum\limits_{i=1}^{p} \alpha_{(m_i,n_i)}(m_i \otimes n_i)$ or $\sum\limits_{i=1}^{p} \alpha_i(m_i \otimes n_i)$ where $\alpha_{(m_i,n_i)} = \alpha_i \in \mathbb{Z}, m_i \in M$ and $n_i \in N$.

Proposition 3.3.5. *Let M_K and ${}_KN$ be modules over a semifield K. Then, for each $\alpha \in K, m, m' \in M$ and $n, n' \in N$,*

(i) $(m + m') \otimes n = m \otimes n + m' \otimes n$,

(ii) $m \otimes (n + n') = m \otimes n + m \otimes n'$ *and*

(iii) $m\alpha \otimes n = m \otimes \alpha n$.

Proof. Let $\alpha \in K, m, m' \in M$ and $n, n' \in N$. Recall that $\tau(m+m', n) - \tau(m,n) - \tau(m', n) \in L$. Thus, $\tau(m+m', n) - (\tau(m,n) + \tau(m',n)) \in L$. Then $\tau(m+m',n) + L = (\tau(m,n) + \tau(m',n)) + L = (\tau(m,n) + L) + (\tau(m',n) + L)$. That is $(m + m') \otimes n = m \otimes n + m' \otimes n$.

The other results are obtained by similar agument. ☐

Remark 3.3.6. Let M_K and ${}_KN$ be modules over a semifield K. Then

(i) $m \otimes 0 = 0 \otimes n = 0 \otimes 0 = 0$ for all $m \in M$ and $n \in N$,

(ii) $(-m) \otimes n = -(m \otimes n) = m \otimes (-n)$ for all $m \in M$ and $n \in N$, and

(iii) $am \otimes n = a(m \otimes n) = m \otimes an$ for all $m \in M, n \in N$ and $a \in \mathbb{Z}$.

Proof. The proofs of (i) and (ii) are obvious. We abtain (iii) by applying (ii) and induction. □

From Remark 3.3.6, we simply write $\sum_{i=1}^{p} a_i(m_i \otimes n_i)$ as $\sum_{i=1}^{p} m_i' \otimes n_i$ or $\sum_{i=1}^{p} m_i \otimes n_i'$ where $a_i \in \mathbb{Z}$, $m_i, \in M$, $n_i \in N$ and $m_i' = a_i m_i$, $n_i' = a_i n_i$. Consequently, if $x \in M \otimes_K N$, then there exist $p \in \mathbb{N}, m_i \in M$ and $n_i \in N$ such that $x = \sum_{i=1}^{p} m_i \otimes n_i$.

Now, we are ready to define a scalar multiplication on the tensor product of M and N where M_K and ${}_KN$ are modules over a semifield K. Recall that $M \otimes_K N = \mathcal{FA}(M \times N)/L$ where L is defined in Definition 3.3.1 and $\mathcal{FA}(M \times N)$ is a free abelian group on a basis $\tau(M \times N)$. Thus, we will define a scalar multiplication on $\tau(M \times N)$ first, and then extend linearly in order to obtain a scalar muliplication on $M \otimes_K N$.

Definition 3.3.7. Let M_K and ${}_KN$ be modules over a semifield K. For each $\alpha \in K$, $m \in M$ and $n \in N$, we define a scalar multiplication on a basis $\tau(M \times N)$ as follows:

$$\alpha(m \otimes n) = m\alpha \otimes n = m \otimes \alpha n.$$

Extend the definition linearly, we obtain that

$$\alpha\left(\sum_{i=1}^{p}(m_i \otimes n_i)\right) = \sum_{i=1}^{p} \alpha(m_i \otimes n_i)$$

for all $\sum_{i=1}^{p}(m_i \otimes n_i) \in M \otimes_K N$.

Proposition 3.3.8. *Let M_K and ${}_KN$ be modules over a semifield K. Then $M \otimes_K N$ is a K-module. Moreover, if M_K and ${}_KN$ are vector spaces over a semifield K, then $M \otimes_K N$ is a vector space over K.*

Proof. This is straightforward. □

Example 3.3.9.

(i) Recall that $\mathbb{Q}_{\mathbb{Q}_0^+}$ and ${}_{\mathbb{Q}_0^+}\mathbb{R}$ are vector spaces over $\mathbb{Q}_0^+$. Then the tensor product $\mathbb{Q} \otimes_{\mathbb{Q}_0^+} \mathbb{R}$ is a vector space over $\mathbb{Q}_0^+$.

(ii) Let $n, m \in \mathbb{N}$. Since $\mathbb{Q}^n_{\mathbb{Q}_0^+}$ and ${}_{\mathbb{Q}_0^+}\mathbb{R}^m$ are vector spaces over $\mathbb{Q}_0^+$, the tensor product $\mathbb{Q}^n \otimes_{\mathbb{Q}_0^+} \mathbb{R}^m$ is a vector space over $\mathbb{Q}_0^+$.

A very important tool for studying tensor products of modules over rings is the universal mapping property of tensor products. How so?

For given modules M, N and A over a ring R, in order to define an R-module homomorphism from the tensor product $M \otimes_R N$ into A, it is enough to define a bilinear map from the product $M \times N$ into A.

This inspires us to achieve the version of the universal mapping property for tensor products of modules over semifields.

Definition 3.3.10. If M_K and ${}_KN$ are modules over a semifield K and A is an (additive) abelian group, then a *middle linear map* (*over* K) from $M \times N$ to A is a function $T : M \times N \to A$ such that for all $m, m' \in M, n, n' \in N$, and $\alpha \in K$,

(i) $T(m + m', n) = T(m, n) + T(m', n)$,

(ii) $T(m, n + n') = T(m, n) + T(m, n')$ and

(iii) $T(m\alpha, n) = T(m, \alpha n)$.

Example 3.3.11. Given modules M_K and ${}_KN$ over a semifield K, the mapping $B : M \times N \to M \otimes_K N$ given by $(m, n) \mapsto m \otimes n$ is a middle linear map.

Proof. Let $m, m' \in M, n, n' \in N$, and $\alpha \in K$. Then

$$B(m + m', n) = (m + m') \otimes n = m \otimes n + m' \otimes n = B(m, n) + B(m', n),$$

$$B(m, n+n') = m \otimes (n+n') = m \otimes n + m \otimes n' = B(m,n) + B(m,n') \text{ and}$$

$$B(m\alpha, n) = m\alpha \otimes n = m \otimes \alpha n = B(m, \alpha n).$$

Thus B is a middle linear map. □

Definition 3.3.12. The middle linear map B defined in Example 3.3.11 is called the *canonical middle linear map.*

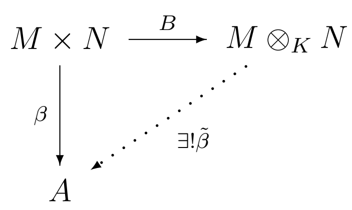
Theorem 3.3.13. *Let M_K and ${}_KN$ be modules over a semifield K. Then a function $\pi : \mathcal{FA}(M \times N) \to \mathcal{FA}(M \times N)/L$ defined by $\pi(x) = x + L$ for all $x \in \mathcal{FA}(M \times N)$ is an epimorphism of groups.*

Proof. This is straightforward. □

Definition 3.3.14. The function π in Theorem 3.3.13 is called the *canonical projection.*

Lemma 3.3.15. [2] *Let A and B be additive abelian group. If $f : A \to B$ is a group homomorphism and C is a subgroup of $\ker f$, then there is a unique group homomorphism $\hat{f} : A/C \to B$ such that $\hat{f}(a + C) = f(a)$ for all $a \in A$, $im\hat{f} = imf$ and $\ker \hat{f} = \ker f/C$. Moreover, $\hat{f}$ is an group isomorphism if and only if f is a group epimorphism and $C = \ker f$. In particular $A/\ker f \cong imf$.*

Theorem 3.3.16. *Let M_K and ${}_KN$ be modules over a semifield K and $B : M \times N \to M \otimes_K N$ the canonical middle linear map. For any module A over K and any middle linear map $\beta : M \times N \to A$, there exists a unique group homomorphism $\tilde{\beta} : M \otimes_K N \to A$ such that $\beta = \tilde{\beta} \circ B$, i.e., the following diagram commutes:*

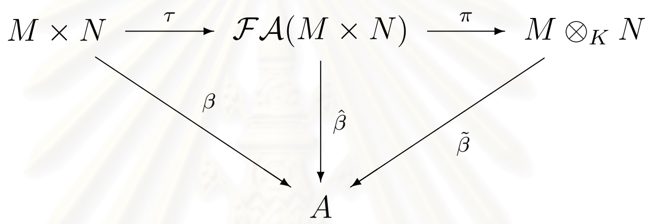


Proof. We consider $(A,+)$ as an abelian group. By Proposition 2.2.15, there exists a unique group homomorphism $\hat{\beta} : \mathcal{FA}(M \times N) \to A$ such that $\beta = \hat{\beta} \circ \tau$.

Let L be the subgroup defined in Definition 3.3.1. Then L is a subgroup of $\ker \hat{\beta}$ because β is a middle linear map and $\beta = \hat{\beta} \circ \tau$.

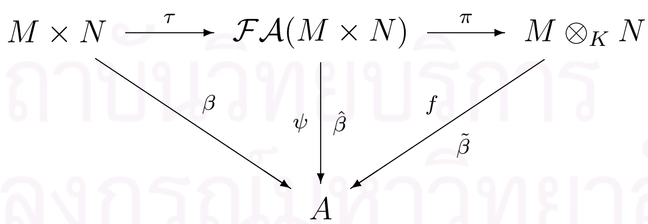
Let $\pi : \mathcal{FA}(M \times N) \to \mathcal{FA}(M \times N)/L$ be the cannonocal projection. Since L is a subgroup of $\ker \hat{\beta}$, by Lemma 3.3.15, there exists a unique group homomorphism $\tilde{\beta} : \mathcal{FA}(M \times N)/L \to A$ such that $\hat{\beta} = \tilde{\beta} \circ \pi$. Now we obtain a group homomorphism $\tilde{\beta} : M \otimes_K N \to A$. We must show that i) $\beta = \tilde{\beta} \circ B$ and ii) $\tilde{\beta}$ is unique.

i) Consider the diagram



For each $m \in M$ and $n \in N$, the canonical middle linear map $B : M \times N \to M \otimes_K N$ satisfies $B(m,n) = m \otimes n = \tau(m,n) + L = \pi(\tau(m,n)) = \pi \circ \tau(m,n)$. This shows that, in fact, $B = \pi \circ \tau$. Hence $\tilde{\beta} \circ B = \tilde{\beta} \circ (\pi \circ \tau) = (\tilde{\beta} \circ \pi) \circ \tau = \hat{\beta} \circ \tau = \beta$.

ii) Let $f : M \otimes_K N \to A$ be a group homomorphism such that $\beta = f \circ B$ and let $\psi = f \circ \pi$. Consider the diagram



Note that $\psi \circ \tau = (f \circ \pi) \circ \tau = f \circ (\pi \circ \tau) = f \circ B = \beta$. Then $\psi = \hat{\beta}$ because of the uniqueness of $\hat{\beta}$. Next, $f \circ \pi = \psi = \hat{\beta} = \tilde{\beta} \circ \pi$. Then, by the uniqueness of $\tilde{\beta}$, we obtain that $f = \tilde{\beta}$. □

Corollary 3.3.17. *If* $M_K, M'_K, {}_KN$ *and* ${}_KN'$ *are modules over a semifield* K*,* $f : M \to M'$ *and* $g : N \to N'$ *are right and left* K*-module homomorphisms, respectively, then there is a unique group homomorphism* $M \otimes_K N \to M' \otimes_K N'$ *such that* $m \otimes n \mapsto f(m) \otimes g(n)$ *for all* $m \in M$ *and* $n \in N$*.*

Proof. Define $h : M \times_K N \to M' \otimes_K N'$ by $h(m,n) = f(m) \otimes g(n)$ for all $m \in M$ and $n \in N$. From Theorem 3.3.16, it is enough to define a function h from $M \times N$ into $M' \otimes_K N'$. Next, we claim that h is a middle linear map. Since f and g are functions, we obtain that h is well-defined. Let $\alpha \in K, m, m'$ and $n, n' \in N$. Then

$$\begin{aligned} h(m+m',n) = f(m+m') \otimes g(n) &= (f(m)+f(m')) \otimes g(n) \\ &= f(m) \otimes g(n) + f(m') \otimes g(n) = h(m,n) + h(m',n). \end{aligned}$$

Similary, we also have $h(m, n+n') = h(m,n) + h(m,n')$. Next,

$$\begin{aligned} h(m\alpha,n) = f(m\alpha) \otimes g(n) = f(m)\alpha \otimes g(n) &= f(m) \otimes \alpha g(n) \\ &= f(m) \otimes g(\alpha n) = h(m,\alpha n). \end{aligned}$$

Therefore h is a middle linear map. By Theorem 3.3.16, there exists a unique group homomorphism $\tilde{h} : M \otimes_K N \to M' \otimes_K N'$ such that $\tilde{h}(m \otimes n) = \tilde{h} \circ B(m,n) = h(m,n) = f(m) \otimes g(n)$ for all $m \in M$ and $n \in N$. □

Note 3.3.18. The unique group homomorphism arised in Corollary 3.3.17 is denoted by $f \otimes g : M \otimes_K N \to M' \otimes_K N'$.

Proposition 3.3.19. *Let* $M_K, M'_K, M''_K, {}_KN, {}_KN'$ *and* ${}_KN''$ *be modules over a semifield* K*. If* $f : M \to M'$ *and* $\acute{f} : M' \to M''$ *are right* K*-module homomorphisms and* $g : N \to N'$ *and* $g' : N' \to N''$ *are left* K*-module homomorphisms, then*

$(f' \otimes g')(f \otimes g) := (f' \circ f) \otimes (g' \circ g) : M \otimes_K N \to M'' \otimes_K N''$ *is a group homomorphism.*

If f and g are right and left K-module isomorphisms, respectively, $f \otimes g$ is a group isomorphism with inverse $f^{-1} \otimes g^{-1}$.

Proof. This is straightforward. □

Theorem 3.3.20. *Let K and S be semifields and ${}_SA_K$, ${}_SA'_K$, ${}_KB$, ${}_KB'$, C_K, C'_K, ${}_KD_S$, ${}_KD'_S$ be modules as indicated.*

(i) *Then $A \otimes_K B$ is a left S-module such that $s(a \otimes b) = sa \otimes b$ for all $s \in S, a \in A$ and $b \in B$.*

(ii) *If $f : A \to A'$ is a homomorphism of $S - K$ bimodules and $g : B \to B'$ is a left K-module homomorphism, then the induced map $f \otimes g : A \otimes_K B \to A' \otimes_K B'$ is a left S-module homomorphism.*

(iii) *Then $C \otimes_K D$ is a right S-module such that $(c \otimes d)s = c \otimes ds$ for all $s \in S, c \in C$ and $d \in D$.*

(iv) *If $f : C \to C'$ is a right K-module homomorphism and $g : D \to D'$ is a homomorphism of $K - S$ bimodules, then the induced map $f \otimes g : C \otimes_K D \to C' \otimes_K D'$ is a homomorphism of right S-modules.*

Proof. (i) Let $s \in S$ and $\phi_s : A \times B \to A \otimes_K B$ be a function defined by $(a, b) \mapsto sa \otimes b$. It is easy to verify that ϕ_s is a middle linear map over K. Therefore, by Theorem 3.3.16 there exists a unique group homomorphism $\tilde{\phi}_s : A \otimes_K B \to A \otimes_K B$ such that $\tilde{\phi}_s \circ B(a, b) = \phi_s(a \otimes b) = sa \otimes b$ for each $a \in A$ and $b \in B$. For each element $u = \sum_{i=1}^{p} a_i \otimes b_i \in A \otimes_K B$, define su to be the element $\tilde{\phi}_s(u) = \sum_{i=1}^{p} \tilde{\phi}_s(a_i \otimes b_i) = \sum_{i=1}^{p} sa_i \otimes b_i$. Since $\tilde{\phi}_s$ is a group homomorphism, this action of s is well-defined (that is, independent of how u is written as a sum of elements in a basis).

Next, we show that $A \otimes_K B$ is a left S-module. Let $k, k_1, k_2 \in K$ and $u, u_1, u_2 \in A \otimes_K B$. Thus $u = \sum_{i=1}^{r} a_i \otimes b_i$, $u_1 = \sum_{i=1}^{p} a_i' \otimes b_i'$, and $u_2 = \sum_{i=1}^{q} a_i'' \otimes b_i''$ where $a_i, a_i', a_i'' \in A$ and $b_i, b_i', b_i'' \in B$. We can see that

$$\begin{aligned} s(u_1 + u_2) = \tilde{\phi}_s\left(\sum_{i=1}^{p} a_i' \otimes b_i' + \sum_{i=1}^{q} a_i'' \otimes b_i''\right) = \tilde{\phi}_s\left(\sum_{i=1}^{p+q} a_i' \otimes b_i'\right) = \sum_{i=1}^{p+q} \tilde{\phi}_s(a_i' \otimes b_i') \\ = \sum_{i=1}^{p+q} sa_i' \otimes b_i' = \sum_{i=1}^{p} sa_i' \otimes b_i' + \sum_{i=1}^{q} sa_i'' \otimes b_i'' = su_1 + su_2, \end{aligned}$$

where $a_i' = a_{i-p}''$ and $b_i' = b_{i-p}''$ for $i = p+1, \ldots, p+q$,

$$\begin{aligned} (s_1 + s_2)u &= \tilde{\phi}_{s_1+s_2}\left(\sum_{i=1}^{r} a_i \otimes b_i\right) \\ &= \sum_{i=1}^{r} \tilde{\phi}_{s_1+s_2}(a_i \otimes b_i) \\ &= \sum_{i=1}^{r} (s_1 + s_2)(a_i \otimes b_i) \\ &= \sum_{i=1}^{r} (s_1 a_i + s_2 a_i) \otimes b_i \\ &= \sum_{i=1}^{r} s_1 a_i \otimes b_i + \sum_{i=1}^{r} s_2 a_i \otimes b_i \\ &= \sum_{i=1}^{r} \tilde{\phi}_{s_1}(a_i \otimes b_i) + \sum_{i=1}^{r} \tilde{\phi}_{s_2}(a_i \otimes b_i) \\ &= \tilde{\phi}_{s_1}\left(\sum_{i=1}^{r} a_i \otimes b_i\right) + \tilde{\phi}_{s_2}\left(\sum_{i=1}^{r} a_i \otimes b_i\right) \\ &= s_1 u + s_2 u \end{aligned}$$

and

$$(s_1 s_2)u = \tilde{\phi}_{s_1 s_2}\left(\sum_{i=1}^{r} a_i \otimes b_i\right)$$

$$= \sum_{i=1}^{r} \tilde{\phi}_{s_1 s_2}(a_i \otimes b_i)$$

$$= \sum_{i=1}^{r} (s_1 s_2) a_i \otimes b_i$$

$$= \sum_{i=1}^{r} s_1(s_2 a_i) \otimes b_i$$

$$= \sum_{i=1}^{r} \tilde{\phi}_{s_1}(s_2 a_i \otimes b_i)$$

$$= \tilde{\phi}_{s_1}\left(\sum_{i=1}^{r} s_2 a_i \otimes b_i)\right)$$

$$= \tilde{\phi}_{s_1}\left(\sum_{i=1}^{r} \tilde{\phi}_{s_2}(a_i \otimes b_i)\right)$$

$$= \tilde{\phi}_{s_1}\left(\tilde{\phi}_{s_2}\left(\sum_{i=1}^{r} a_i \otimes b_i)\right)\right) = s_1(s_2 u).$$

Hence $A \otimes_K B$ is a left S-module.

(ii) Let $f : A \to A'$ be a homomorphism of $S - K$ bimodules and $g : B \to B'$ be a left K-module homomorphism. From (i), we obtain that $A \otimes_K B$ and $A' \otimes_K B'$ are left S-modules. By Corollary 3.3.17, there exists a unique group homomorphism $h : A \otimes_K B \to A' \otimes_K B'$ such that $a \otimes b \mapsto f(a) \otimes g(b)$ for all $a \in A$ and $b \in B$. So we claim that, for each $s \in S$ and $\sum_{i=1}^{r} a_i \otimes b_i \in A \otimes_K B$,

$$h\left(s\left(\sum_{i=1}^{r} a_i \otimes b_i\right)\right) = s\left(h\left(\sum_{i=1}^{r} a_i \otimes b_i\right)\right).$$

Let $s \in S$ and $\sum_{i=1}^{r} a_i \otimes b_i \in A \otimes_K B$. By (i) we have $s\left(\sum_{i=1}^{r} a_i \otimes b_i\right) = \sum_{i=1}^{r} s a_i \otimes b_i$. Thus

$$h\left(s\left(\sum_{i=1}^{r} a_i \otimes b_i\right)\right) = h\left(\sum_{i=1}^{r} s a_i \otimes b_i\right)$$

$$= \sum_{i=1}^{r} h(sa_i \otimes b_i)$$

$$= \sum_{i=1}^{r} f(sa_i) \otimes g(b_i)$$

$$= \sum_{i=1}^{r} sf(a_i) \otimes g(b_i)$$

$$= s\left(\sum_{i=1}^{r} f(a_i) \otimes g(b_i)\right)$$

$$= s\left(\sum_{i=1}^{r} h(a_i \otimes b_i)\right)$$

$$= s\left(h\left(\sum_{i=1}^{r} a_i \otimes b_i)\right)\right).$$

Hence h is a left S-module homomorphism.

(iii) The proof of (iii) is similar to (i).

(iv) The proof of (iv) is similar to (ii). □

Remark 3.3.21. Let K be a semifield . Every K-module M is a K-K bimodule from Note 3.1.2. In this case for every modules M and N over K, the tensor product $M \otimes_K N$ is also a K-K bimodule with

$$k(m \otimes n) = km \otimes n = mk \otimes n = m \otimes kn = m \otimes nk = (m \otimes n)k$$

for all $k \in K, m \in M$ and $n \in N$. Since K is a semifield, the tensor product of K-modules may be characterized by a useful variation of Theorem 3.3.20.

Definition 3.3.22. Let M, N and W be modules over a semifield K. A *bilinear* map (*over* K) from $M \times N$ to W is a function $T : M \times N \to W$ such that for all $m, m' \in M, n, n' \in N$, and $\alpha \in K$,

(i) $T(m + m', n) = T(m, n) + T(m', n)$,

(ii) $T(m, n + n') = T(m, n) + T(m, n')$ and

(iii) $T(m\alpha, n) = \alpha T(m, n) = T(m, \alpha n)$.

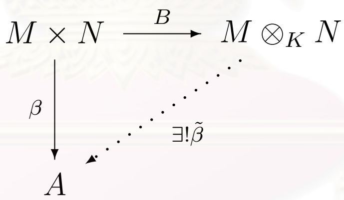
Note 3.3.23. The conditions (i) and (ii) in Definition 3.3.22 are the same conditions (i) and (ii) in Definition 3.3.10. For a semifield K, the condition (iii) in Definition 3.3.22 clearly implies the condition (iii) in Definition 3.3.10, whence every bilinear map is a middle linear map.

Example 3.3.24. If M and N are modules over a semifield K, so is $M \otimes_K N$ and the canonical middle linear map $B : M \times N \to M \otimes_K N$ is easily seen to be bilinear.

Definition 3.3.25. The bilinear map B in Example 3.3.24 is called the *canonical bilinear map* (*over* K).

Theorem 3.3.26. ***The Universal Mapping Property of Tensor Products***

Let M, N and A be modules over a semifield K and $B : M \times N \to M \otimes_K N$ the canonical bilinear map. For any bilinear map $\beta : M \times N \to A$, there exists a unique K-module homomorphism $\tilde{\beta} : M \otimes_K N \to A$ such that $\beta = \tilde{\beta} \circ B$, i.e., the following diagram commutes:



Proof. By Theorem 3.3.16 there is a unique group homomorphism $\tilde{\beta} : M \otimes_K N \to A$ such that $\beta = \tilde{\beta} \circ B$. Now, it suffices to prove that

(i) $\tilde{\beta}(\alpha x) = \alpha\tilde{\beta}(x)$ for all $\alpha \in K$ and $x \in M \otimes_K N$ and

(ii) $\tilde{\beta}$ is unique as a K-module homomorphism.

Let $\alpha \in K$ and $x \in M \otimes_K N$. Then $x = \sum_{i=1}^{p} m_i \otimes n_i$ for some $m_i \in M$ and $n_i \in N$. Therefore

$$\tilde{\beta}(\alpha x) = \tilde{\beta}\left(\alpha \sum_{i=1}^{p}(m_i \otimes n_i)\right) = \sum_{i=1}^{p} \tilde{\beta}\left(\alpha(m_i \otimes n_i)\right) = \sum_{i=1}^{p} \tilde{\beta}(\alpha m_i \otimes n_i)$$

$$= \sum_{i=1}^{p} \tilde{\beta}(B(\alpha m_i, n_i)) = \sum_{i=1}^{p} \beta(\alpha m_i, n_i) = \sum_{i=1}^{p} \alpha\beta(m_i, n_i) = \alpha \sum_{i=1}^{p} \beta(m_i, n_i)$$

$$= \alpha \sum_{i=1}^{p} \tilde{\beta} \circ B(m_i, n_i) = \alpha \sum_{i=1}^{p} \tilde{\beta}(m_i \otimes n_i) = \alpha\tilde{\beta}\left(\sum_{i=1}^{p}(m_i \otimes n_i)\right) = \alpha\tilde{\beta}(x).$$

For a K-module homomorphism $f : M \otimes_K N \to A$ such that $\beta = f \circ B$, we obtain that f is a group homomorphism such that $\beta = f \circ B$. By the uniqueness of $\tilde{\beta}$, it follows that $\tilde{\beta} = f$. Hence $\tilde{\beta}$ is a unique K-module homomorphism such that $\beta = \tilde{\beta} \circ B$. □

Note 3.3.27. If K and S are semifields and M_K, ${}_K N_S$, ${}_S W$ are modules, then $M \otimes_K N$ is a right S-module and $N \otimes_K W$ is a left K-module by Theorem 3.3.20. Consequently, both $(M \otimes_K N) \otimes_S W$ and $M \otimes_K (N \otimes_S W)$ are well-defined abelian additive groups.

Theorem 3.3.28. *Let K and S be semifields and M_K, ${}_K N_S$, ${}_S W$ are modules, then there is a group isomorphism between $(M \otimes_K N) \otimes_S W$ and $M \otimes_K (N \otimes_S W)$.*

Proof. For each $v \in (M \otimes_K N) \otimes_S W$, we know that v is a finite sum $\sum_{i=1}^{p} u_i \otimes w_i$ where $u_i \in M \otimes_K N$ and $w_i \in W$. Since $u_i \in M \otimes_K N$, we obtain that u_i is a finite sum $\sum_{j=1}^{r_i} m_{ij} \otimes n_{ij}$ where $m_{ij} \in M$ and $n_{ij} \in N$. Thus

$$v = \sum_{i=1}^{p} u_i \otimes w_i = \sum_{i=1}^{p}\left(\sum_{j=1}^{r_i} m_{ij} \otimes n_{ij}\right) \otimes w_i = \sum_{i=1}^{p}\sum_{j=1}^{r_i}((m_{ij} \otimes n_{ij}) \otimes w_i).$$

Therefore, $(M \otimes_K N) \otimes_S W$ is generated by all elements of the form $(m \otimes n) \otimes w$ where $m \in M, n \in N$ and $w \in W$. Similarly, $M \otimes_K (N \otimes_S W)$ is generated by all $m \otimes (n \otimes w)$ where $m \in M, n \in N$ and $w \in W$.

Now, we define a function $f : (M \otimes_K N) \times W \to M \otimes_K (N \otimes_S W)$ by

$$f\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right) = \sum_{i=1}^{p} (m_i \otimes (n_i \otimes w))$$

for all $\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right) \in (M \otimes_K N) \times W$. Next, we show that f is a middle linear map over S. It is obvious that f is well-defined. Let $\alpha \in S, \sum_{i=1}^{p} m_i \otimes n_i, \sum_{i=1}^{p} m_i' \otimes n_i' \in M \otimes_K N$ and $w, w' \in W$. Thus

$$f\left(\sum_{i=1}^{p} m_i \otimes n_i + \sum_{i=1}^{p} m_i' \otimes n_i', w\right) = f\left(\sum_{i=1}^{p+q} m_i \otimes n_i, w\right) = \sum_{i=1}^{p+q} (m_i \otimes (n_i \otimes w))$$
$$= \sum_{i=1}^{p} (m_i \otimes (n_i \otimes w)) + \sum_{i=1}^{q} (m_i' \otimes (n_i' \otimes w)) = f\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right) + f\left(\sum_{i=1}^{q} m_i' \otimes n_i', w\right),$$

where $m_i = m_{i-p}'$ and $n_i = n_{i-p}'$ for $i = p+1, \ldots, p+q$,

$$\begin{aligned} f\left(\sum_{i=1}^{p} m_i \otimes n_i, w + w'\right) &= \sum_{i=1}^{p} (m_i \otimes (n_i \otimes (w + w'))) \\ &= \sum_{i=1}^{p} (m_i \otimes ((n_i \otimes w) + (n_i \otimes w'))) \\ &= \sum_{i=1}^{p} ((m_i \otimes (n_i \otimes w)) + (m_i \otimes (n_i \otimes w'))) \\ &= \sum_{i=1}^{p} (m_i \otimes (n_i \otimes w)) + \sum_{i=1}^{p} (m_i \otimes (n_i \otimes w')) \\ &= f\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right) + f\left(\sum_{i=1}^{p} m_i \otimes n_i, w'\right) \end{aligned}$$

and

$$\begin{aligned} f\left(\left(\sum_{i=1}^{p} m_i \otimes n_i\right)\alpha, w\right) = f\left(\sum_{i=1}^{p} m_i \otimes n_i\alpha, w\right) &= \sum_{i=1}^{p} (m_i \otimes (n_i\alpha \otimes w)) \\ &= \sum_{i=1}^{p} (m_i \otimes (n_i \otimes \alpha w)) = f\left(\sum_{i=1}^{p} m_i \otimes n_i, \alpha w\right). \end{aligned}$$

Therefore f is a middle linear map over S. By Theorem 3.3.16 there exists a group homomorphism

$$\phi : (M \otimes_K N) \otimes_S W \to M \otimes_K (N \otimes_S W)$$

with $\phi((m \otimes n) \otimes w) = \phi \circ B((m \otimes n), w) = f(m \otimes n, w) = m \otimes (n \otimes w)$ for all $m \in M, n \in N$ and $w \in W$.

Similary, a function $h : M \times (N \otimes_S W) \to (M \otimes_K N) \otimes_S W$ defined by

$$h\left(m, \sum_{i=1}^{p} n_i \otimes w_i\right) = \sum_{i=1}^{p} ((m \otimes n_i) \otimes w_i)$$

for all $\left(m, \sum_{i=1}^{p} n_i \otimes w_i\right) \in M \times (N \otimes_S W)$ is also a middle linear map over K which induces a group homomorphism

$$\psi : M \otimes_K (N \otimes_S W) \to (M \otimes_K N) \otimes_S W$$

such that $\psi(m \otimes (n \otimes w)) = \psi \circ B(m, (n \otimes w)) = h(m, n \otimes w) = (m \otimes n) \otimes w$ for all $m \in M, n \in N$ and $w \in W$.

For each $m \in M, n \in N$ and $w \in W, \psi \circ \phi((m \otimes n) \otimes w) = (m \otimes n) \otimes w$, whence $\psi \circ \phi$ is the identity function on $(M \otimes_K N) \otimes_S W$ and $\phi \circ \psi(m \otimes (n \otimes w)) = m \otimes (n \otimes w)$, whence $\phi \circ \psi$ is the identity function on $M \otimes_K (N \otimes_S W)$.

Therefore ϕ and ψ are group isomorphisms. □

Corollary 3.3.29. *Let M, N and W be modules over a semifield K. Then*

$$(M \otimes_K N) \otimes_K W \cong M \otimes_K (N \otimes_K W)$$

as a K-module isomorphism.

Proof. From Theorem 3.3.28, the function $f : (M \otimes_K N) \times W \to M \otimes_K (N \otimes_K W)$ defined by $\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right) \mapsto \sum_{i=1}^{p} (m_i \otimes (n_i \otimes w))$ is a middle linear map over K. Next, we show that f is a bilinear map over K. Let $\alpha \in K, \sum_{i=1}^{p} m_i \otimes n_i \in M \otimes_K N$ and $w \in W$. It remains to show that $f\left(\alpha\left(\sum_{i=1}^{p} m_i \otimes n_i\right), w\right) = \alpha f\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right)$

because $f\left(\left(\sum_{i=1}^{p} m_i \otimes n_i\right)\alpha, w\right) = f\left(\sum_{i=1}^{p} m_i \otimes n_i, \alpha w\right)$. We can see that

$$\begin{aligned} f\left(\alpha\left(\sum_{i=1}^{p} m_i \otimes n_i\right), w\right) &= f\left(\sum_{i=1}^{p} \alpha m_i \otimes n_i, w\right) \\ &= \sum_{i=1}^{p} (\alpha m_i \otimes (n_i \otimes w)) \\ &= \sum_{i=1}^{p} \alpha\,(m_i \otimes (n_i \otimes w)) \\ &= \alpha\left(\sum_{i=1}^{p} (m_i \otimes (n_i \otimes w))\right) \\ &= \alpha f\left(\sum_{i=1}^{p} m_i \otimes n_i, w\right). \end{aligned}$$

Therefore f is a bilinear map. By the universal mapping property of tensor products there exists a K-module homomorphism

$$\phi : (M \otimes_K N) \otimes_S W \to M \otimes_K (N \otimes_S W)$$

with $\phi((m \otimes n) \otimes w) = B \circ \phi((m \otimes n), w) = f(m \otimes n, w) = m \otimes (n \otimes w)$ for all $m \in M, n \in N$ and $w \in W$. From the proof of Theorem 3.3.28 we obtain that ϕ is a bijective function. Therefore ϕ is a K-module isomorphism.

Hence $(M \otimes_K N) \otimes_K W \cong M \otimes_K (N \otimes_K W)$ as a K-module isomorphism. □

Later, we shall identify $(M \otimes_K N) \otimes_K W$ as $M \otimes_K (N \otimes_K W)$ under K-module isomorphism in Corollary 3.3.29 and simply write $M \otimes_K N \otimes_K W$. It is now possible to define recursively the n-fold tensor product:

$$M_1 \otimes_K M_2 \otimes_K \cdots \otimes_K M_n$$

where K is a semifield and $M_1, \ldots, M_n$ are module over K. Such iterated tensor product may also be characterized in terms of n-linear maps over K which will be discussed in Section 3.4.

3.4. Multilinear Maps

Definition 3.4.1. Let $M_1, \dots, M_n$ and W be modules over the same semifield K. An n-*linear map* or a *multilinear map* (*over* K) from $M_1 \times \cdots \times M_n$ into W is a function $\mu : M_1 \times \cdots \times M_n \to W$ such that if $\forall j \in \{1, \dots, n\}$ $\forall m_1, \dots, m_n$ with $m_i \in M_i$ for all i $\forall m'_j \in M_j$ $\forall \alpha \in K$, then

(i) $\mu(m_1, \dots, m_{j-1}, m_j + m'_j, m_j, \dots, m_{j+1}, \dots, m_n) =$
$\mu(m_1, \dots, m_{j-1}, m_j, m_{j+1}, \dots, m_n) + \mu(m_1, \dots, m_{j-1}, m'_j, m_{j+1}, \dots, m_n)$ and

(ii) $\mu(m_1, \dots, m_{j-1}, \alpha m_j, m_j, \dots, m_{j+1}, \dots, m_n) =$
$\alpha\mu(m_1, \dots, m_{j-1}, m_j, m_{j+1}, \dots, m_n)$,

In the case $n = 2$, an n-linear map $\mu : M_1 \times M_2 \to W$ is a *bilinear map.*

Moreover, we use $\mathcal{L}(M_1, \dots, M_n; W)$ to denote the set of all n-linear maps from $M_1 \times \cdots \times M_n$ into W. For $n = 1$ we simply write $\mathcal{L}(M_1)$ instead of $\mathcal{L}(M_1, M_1)$.

Proposition 3.4.2. *Let $M_1, \dots, M_n$ and W be modules over a semifield K. Then $\mathcal{L}(M_1, \dots, M_n; W)$ becomes a module over K if we use the zero function for the zero element of $\mathcal{L}(M_1, \dots, M_n; W)$ and define the operations as follows:*

(i) $(f + g)(m_1, \dots, m_n) = f(m_1, \dots, m_n) + g(m_1, \dots, m_n)$, *and*

(ii) $(\alpha f)(m_1, \dots, m_n) = \alpha f(m_1, \dots, m_n)$

for all $\alpha \in K$, $f, g \in \mathcal{L}(M_1, \dots, M_n; W)$ and $m_i \in M_i$ for all i.

Moreover, if $M_1, \dots, M_n$ and W are vector spaces over K, then $\mathcal{L}(M_1, \dots, M_n; W)$ is a vector space over K.

Proof. This is straightforward. □

Multilinear maps are important parts in order to define tensor products of more than two modules over semifields. We need another version of the universal mapping property of tensor products. Although, tensor products are defined between two modules over a semifield, we can generalize this to tensor products of finite modules

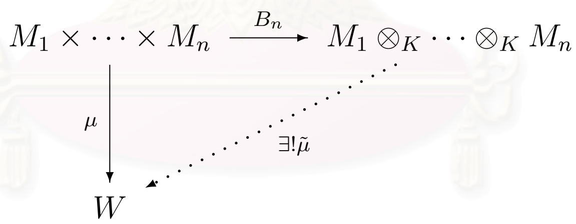
$M_1, \ldots, M_n$ over a same semifield K and write this as $M_1 \otimes_K M_2 \otimes_K \cdots \otimes_K M_n$ by dropping all parentheses because it is independent of those.

Example 3.4.3. Let $n \geq 2$ and $M_1, \ldots, M_n$ be modules over the semifield K. Then the function $B_n : M_1 \times \cdots \times M_n \to M_1 \otimes_K \cdots \otimes_K M_n$ defined by $(m_1, \ldots, m_n) \mapsto m_1 \otimes \cdots \otimes m_n$ is multilinear.

Proof. This is straightforward. $\square$

Definition 3.4.4. The multilinear map B_n defined in Example 3.4.3 is called the *canonical n-linear map* or *canonnical multilinear map.*

Theorem 3.4.5. *Let $n \geq 2$, $M_1, \ldots, M_n$ and W be modules over a semifield K. For any multilinear map $\mu : M_1 \times \ldots \times M_n \to W$, there exists a unique K-module homomorphism $\tilde{\mu} : M_1 \otimes_K \cdots \otimes_K M_n \to W$ such that $\mu = \tilde{\mu} \circ B_n$, i.e., the following diagram commutes:*

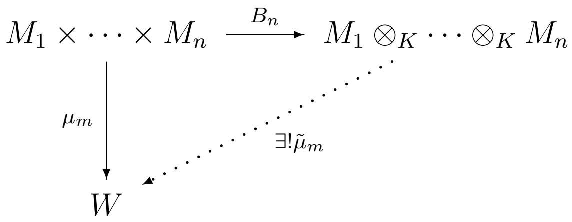


Proof. Let μ be a multilinear map from $M_1 \otimes_K \cdots \otimes_K M_n$ into W. We prove by induction. If $n = 2$, this is the universal mapping property of tensor products.

For each $m \in M_{n+1}$, let $\mu_m : M_1 \times \cdots \times M_n \to W$ be defined by

$$\mu_m(m_1, \ldots, m_n) = \mu(m_1, \ldots, m_n, m) \text{ for all } m_i \in M_i,\ 1 \leq i \leq n.$$

Then μ_m is multilinear. By the induction hypothesis, there exists a unique K-module homomorphism $\tilde{\mu}_m : M_1 \otimes_K \cdots \otimes_K M_n \to W$ such that $\mu_m = \tilde{\mu}_m \circ B_n$ (see the following diagram):



Let $\beta : (M_1 \otimes_K \cdots \otimes_K M_n) \times M_{n+1} \to W$ be defined by $\beta(x, m) = \tilde{\mu}_m(x)$ for all $x \in M_1 \otimes_K \cdots \otimes_K M_n$ and $m \in M_{n+1}$. Then β is well-defined because of the uniqueness of $\tilde{\mu}_m$. Next, we show that β is bilinear.

Let $\alpha \in K$, $x, y \in M_1 \otimes_K \cdots \otimes_K M_n$ and $u, v \in M_{n+1}$. Then

$$\beta(x + y, v) = \tilde{\mu}_v(x + y) = \tilde{\mu}_v(x) + \tilde{\mu}_v(y) = \beta(x, v) + \beta(y, v) \quad \text{and}$$

$$\beta(\alpha x, v) = \tilde{\mu}_v(\alpha x) = \alpha \tilde{\mu}_v(x).$$

Moreover, we have to show that

(i) $\beta(x, u + v) = \beta(x, u) + \beta(x, v)$, i.e., $\tilde{\mu}_{u+v} = \tilde{\mu}_u + \tilde{\mu}_v$ and

(ii) $\beta(x, \alpha v) = \alpha \beta(x, v)$, i.e., $\tilde{\mu}_{\alpha v} = \alpha \tilde{\mu}_v$.

Recall that $\tilde{\mu}_{u+v}$ is the unique K-module homomorphism such that $\mu_{u+v} = \tilde{\mu}_{u+v} \circ B_n$. Thus it is enough to show only that $(\tilde{\mu}_u + \tilde{\mu}_v) \circ B_n = \mu_{u+v}$. Let $m_i \in M_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}
((\tilde{\mu}_u + \tilde{\mu}_v) \circ B_n)(m_1, \ldots, m_n) &= (\tilde{\mu}_u \circ B_n)(m_1, \ldots, m_n) + (\tilde{\mu}_v \circ B_n)(m_1, \ldots, m_n) \\
&= \mu_u(m_1, \ldots, m_n) + \mu_v(m_1, \ldots, m_n) \\
&= \mu(m_1, \ldots, m_n, u) + \mu(m_1, \ldots, m_n, v) \\
&= \mu(m_1, \ldots, m_n, u + v) \\
&= \mu_{u+v}(m_1, \ldots, m_n).
\end{aligned}$$

Since $\tilde{\mu}_{\alpha v}$ is the unique K-module homomorphism such that $\mu_{\alpha v} = \tilde{\mu}_v \circ B_n$, we will show that $(\alpha\tilde{\mu}_v) \circ B_n = \mu_{\alpha v}$. Let $m_i \in M_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}((\alpha\tilde{\mu}_v) \circ B_n)(m_1, \ldots, m_n) &= \alpha\,(\tilde{\mu}_v \circ B_n)(m_1, \ldots, m_n) \\ &= \alpha\mu_v(m_1, \ldots, m_n) \\ &= \alpha\mu(m_1, \ldots, m_n, v) \\ &= \mu(m_1, \ldots, m_n, \alpha v) \\ &= \mu_{\alpha v}(m_1, \ldots, m_n).\end{aligned}$$

Therefore β is a bilinear.

For each $m_i \in M_i$ for all $i = 1, \ldots, n, n+1$. Consider the bilinear map $B : (M_1 \otimes_K \cdots \otimes_K M_n) \times M_{n+1} \to (M_1 \otimes_K \cdots \otimes_K M_n) \otimes_K M_{n+1}$. We can see that canonical

$$\begin{aligned}B(B_n(m_1, \ldots, m_n), m_{n+1}) &= B_n(m_1, \ldots, m_n) \otimes m_{n+1} \\ &= (m_1 \otimes \cdots \otimes m_n) \otimes m_{n+1} \\ &= m_1 \otimes \cdots \otimes m_n \otimes m_{n+1} \\ &= B_{n+1}(m_1, \ldots, m_n, m_{n+1}).\end{aligned}$$

Now, fix $i \in \{1, \ldots, n, n+1\}$ and let $\alpha \in K$, $m_i, m_i' \in M_i$ for ali i.

Case 1 $1 \leq i \leq n$. Then

$$\begin{aligned}&B_{n+1}(m_1, \ldots, m_i + m_i', \ldots, m_n, m_{n+1}) \\ &= B\left(B_n(m_1, \ldots, m_i + m_i', \ldots, m_n), m_{n+1}\right) \\ &= B(B_n(m_1, \ldots, m_i, \ldots, m_n) + B_n(m_1, \ldots, m_i', \ldots, m_n), m_{n+1})\end{aligned}$$

$$= B\left(B_n(m_1, \ldots, m_i, \ldots, m_n), m_{n+1}\right) + B\left(B_n(m_1, \ldots, m_i', \ldots, m_n), m_{n+1}\right)$$

$$= B_{n+1}(m_1, \ldots, m_i, \ldots, m_n, m_{n+1}) + B_{n+1}(m_1, \ldots, m_i', \ldots, m_n, m_{n+1})$$

and

$$\begin{aligned} B_{n+1}(m_1, \ldots, \alpha m_i, \ldots, m_n, m_{n+1}) &= B\left(B_n(m_1, \ldots, \alpha m_i, \ldots, m_n), m_{n+1}\right) \\ &= B\left(\alpha B_n(m_1, \ldots, m_i, \ldots, m_n), m_{n+1}\right) \\ &= \alpha B\left(B_n(m_1, \ldots, m_i, \ldots, m_n), m_{n+1}\right) \\ &= \alpha B_{n+1}(m_1, \ldots, m_i, \ldots, m_n, m_{n+1}). \end{aligned}$$

Case 2 $i = n+1$. Then

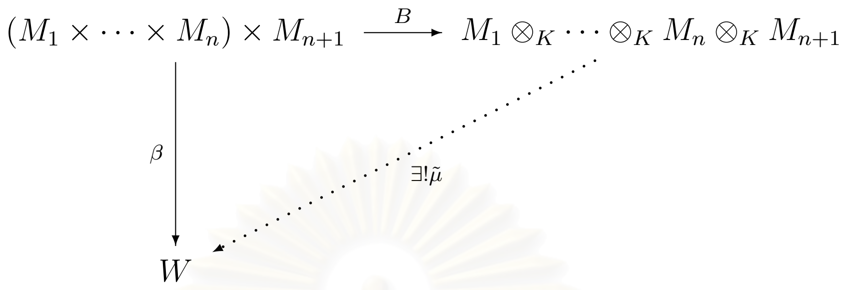
$$\begin{aligned} & B_{n+1}(m_1, \ldots, m_n, m_{n+1} + m_{n+1}') \\ &= B\left(B_n(m_1, \ldots, m_n), m_{n+1} + m_{n+1}'\right) \\ &= B\left(B_n(m_1, \ldots, m_n), m_{n+1}\right) + B\left(B_n(m_1, \ldots, m_n), m_{n+1}'\right) \\ &= B_{n+1}(m_1, \ldots, m_n, m_{n+1}) + B_{n+1}(m_1, \ldots, m_n', m_{n+1}) \end{aligned}$$

and

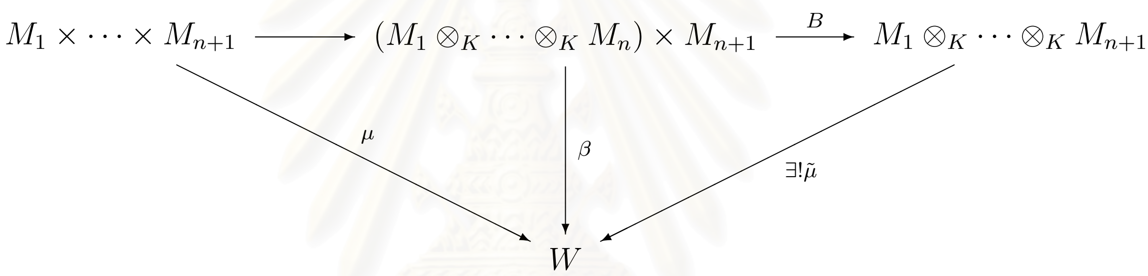
$$\begin{aligned} B_{n+1}(m_1, \ldots, m_n, \alpha m_{n+1}) &= B\left(B_n(m_1, \ldots, m_n), \alpha m_{n+1}\right) \\ &= \alpha B\left(B_n(m_1, \ldots, m_n), m_{n+1}\right) \\ &= \alpha B_{n+1}(m_1, \ldots, m_n, m_{n+1}). \end{aligned}$$

Therefore B_{n+1} is an $(n+1)$-linear map.

Now, we consider the case $n+1$ where $\mu : M_1 \times \cdots \times M_{n+1} \to W$ is a multilinear map. Since β and B are bilinear and by induction hypothesis, there exists a unique K-module homomorphism $\tilde{\mu} : M_1 \otimes_K \cdots \otimes_K M_n \otimes_K M_{n+1} \to W$ such that $\beta = \tilde{\mu} \circ B$.



Consider the following diagram.



Next, we show that i) $\mu = \tilde{\mu} \circ B_{n+1}$ and ii) $\tilde{\mu}$ is unique.

i) Let $m_i \in M_i$ for all $i = 1, \ldots, n+1$. Then

$$\begin{aligned}
(\tilde{\mu} \circ B_{n+1})(m_1, \ldots, m_n, m_{n+1}) &= \tilde{\mu}\left(B(B_n(m_1, \ldots, m_n), m_{n+1})\right) \\
&= \beta\left(B_n(m_1, \ldots, m_n), m_{n+1}\right) \\
&= \tilde{\mu}_{m_{n+1}}\left(B_n(m_1, \ldots, m_n)\right) \\
&= \mu_{m_{n+1}}(m_1, \ldots, m_n) \\
&= \mu(m_1, \ldots, m_n, m_{n+1}).
\end{aligned}$$

Thus $\mu = \tilde{\mu} \circ B_{n+1}$.

ii) Suppose that $\rho : M_1 \otimes_K \cdots \otimes_K M_n \otimes_K M_{n+1} \to W$ is a K-module homomorphism such that $\mu = \rho \circ B_{n+1}$. For each $v \in M_{n+1}$, let $\rho_v : M_1 \otimes_K \cdots \otimes_K M_n \to W$ be defined by $\rho_v(x) = \rho\,(B(x,v))$ for all $x \in M_1 \otimes_K \cdots \otimes_K M_n$. Thus ρ_v is a K-module homomorphism. Let $m_i \in M_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}(\rho_v \circ B_n)(m_1, \ldots, m_n) &= \rho\,(B(B_n(m_1, \ldots, m_n), v)) \\ &= \rho\,(B_{n+1}(m_1, \ldots, m_n, v)) \\ &= \mu(m_1, \ldots, m_n, v) \\ &= \mu_v(m_1, \ldots, m_n).\end{aligned}$$

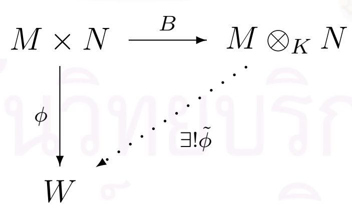
This shows that $\rho_v \circ B_n = \mu_v$. By the uniqueness of $\tilde{\mu}_v$ we obtain that $\rho_v = \tilde{\mu}_v$.

Next, we show that $\beta = \rho \circ B$. Let $x \in M_1 \otimes_K \cdots \otimes_K M_n$ and $v \in M_{n+1}$. Then $(\rho \circ B)(x,v) = \rho\,(B(x,v)) = \rho_v(x) = \tilde{\mu}_v(x) = \beta(x,v) = \tilde{\mu} \circ B(x,v)$. By the uniqueness of $\tilde{\mu}$, we have $\rho = \tilde{\mu}$. □

Theorem 3.4.6. *Let M, N and W be modules over a semifield K. Then*

$$\mathcal{L}(M, N; W) \cong \mathcal{L}(M \otimes_K N, W).$$

Proof. For each $\phi \in \mathcal{L}(M, N; W)$, by the universal mapping property of tensor products and the following diagram,



where B is the canonical bilinear map and ϕ is a bilinear map from $M \times N$ into W and the universal mapping property of tensor products, there exists a unique K-module homomorphism $\widetilde{\phi} : M \otimes_K N \to W$ such that $\phi = \widetilde{\phi} \circ B$. For this reason, we define $T : \mathcal{L}(M, N; W) \to \mathcal{L}(M \otimes_K N, W)$ by $\phi \mapsto \widetilde{\phi}$ for all $\phi \in \mathcal{L}(M, N; W)$. Then T is well-defined from the uniqueness of $\widetilde{\phi}$ for each $\phi \in \mathcal{L}(M, N; W)$.

Next, we show that T is a K-module homomorphism. Let $\alpha \in K$ and $\phi, \psi \in \mathcal{L}(M, N; W)$. Since $\phi, \psi \in \mathcal{L}(M, N; W)$, we obtain that $\alpha\phi, \phi + \psi \in \mathcal{L}(M, N; W)$. By the universal mapping property of tensor products, there exist K-module homomorphisms $\widetilde{\alpha\phi} : M \otimes_K N \to W$, $\widetilde{\phi + \psi} : M \otimes_K N \to W$, $\widetilde{\phi} : M \otimes_K N \to W$, and $\widetilde{\psi} : M \otimes_K N \to W$ such that $\alpha\phi = \widetilde{\alpha\phi} \circ B$, $\phi + \psi = \widetilde{\phi + \psi} \circ B$, $\phi = \widetilde{\phi} \circ B$ and $\psi = \widetilde{\psi} \circ B$. Then $\widetilde{\alpha\phi} \circ B = \alpha\phi = \alpha(\widetilde{\phi} \circ B) = \alpha\widetilde{\phi} \circ B$. For the uniqueness of $\widetilde{\alpha\phi}$ we have $\widetilde{\alpha\phi} = \alpha\widetilde{\phi}$. Thus $T(\alpha\phi) = \widetilde{\alpha\phi} = \alpha\widetilde{\phi} = \alpha T(\phi)$. Since $\widetilde{\phi + \psi} \circ B = \phi + \psi = \widetilde{\phi} \circ B + \widetilde{\psi} \circ B = (\widetilde{\phi} + \widetilde{\psi}) \circ B$ and the uniqueness of $\widetilde{\phi + \psi}$, we obtain that $\widetilde{\phi + \psi} = \widetilde{\phi} + \widetilde{\psi}$. Thus $T(\phi + \psi) = \widetilde{\phi + \psi} = \widetilde{\phi} + \widetilde{\psi} = T(\phi) + T(\psi)$. Therefore T is a K-module homomorphism.

In order to show that T is an injective function by showing that $\ker T = \{0\}$, let $\phi \in \ker T$. Then $\phi \in \mathcal{L}(M, N; W)$ and there exists $\widetilde{\phi} \in \mathcal{L}(M \otimes_K N, W)$ such that $\phi = \widetilde{\phi} \circ B$. Since $\phi \in \ker T$, we obtain that $0 = T(\phi) = \widetilde{\phi}$. Thus $\phi = \widetilde{\phi} \circ B = 0$. Therefore T is an injective function.

Finally, we show that T is a surjective function. Let $\widetilde{\phi} \in \mathcal{L}(M \otimes_K N, W)$. Let $\phi = \widetilde{\phi} \circ B$ and we claim that ϕ is a bilinear map. Since $\widetilde{\phi} : M \otimes_K N \to W$ and $B : M \times N \to M \otimes_K N$, we obtain that ϕ is a function from $M \times N$ into W. Let $\alpha \in K$, $m, m' \in M$ and $n, n' \in N$. Then

$$\begin{aligned}
\phi(m + m', n) &= \widetilde{\phi} \circ B(m + m', n) \\
&= \widetilde{\phi}(B(m + m', n)) \\
&= \widetilde{\phi}(B(m, n) + B(m', n)) \\
&= \widetilde{\phi}(B(m, n)) + \widetilde{\phi}(B(m', n)) \\
&= \phi(m, n) + \phi(m', n), \\
\phi(\alpha m, n) &= \widetilde{\phi} \circ B(\alpha m, n)
\end{aligned}$$

$$= \widetilde{\phi}\,(B(\alpha m, n))$$

$$= \widetilde{\phi}\,(\alpha B(m, n))$$

$$= \alpha\widetilde{\phi}\,(B(m, n))$$

$$= \alpha\phi(m, n)$$

and, similarly, we also have $\phi(m, n + n') = \phi(m, n) + \phi(m, n')$ and $\phi(m, \alpha n) = \alpha\phi(m, n)$. Thus ϕ is a bilinear map. Therefore $\phi \in \mathcal{L}(M, N; W)$. By the universal mapping property of tensor products, there exists a unique K-module homomorphism $\hat{\phi} : M \otimes_K N \to W$ such that $\phi = \hat{\phi} \circ B$. By the uniqueness of $\hat{\phi}$, we have $\hat{\phi} = \widetilde{\phi}$. Therefore $T(\phi) = \hat{\phi} = \widetilde{\phi}$.

Hence T is a K-module isomorphism from $\mathcal{L}(M, N; W)$ onto $\mathcal{L}(M \otimes_K N, W)$. Therefore $\mathcal{L}(M, N; W) \cong \mathcal{L}(M \otimes_K N, W)$. □

Theorem 3.4.7. *Let $M_1, \ldots, M_n$ and W be modules over a semifield K. Then*

$$\mathcal{L}(M_1, \ldots, M_n; W) \cong \mathcal{L}(M_1 \otimes_K \cdots \otimes_K M_n, W).$$

Proof. We obtain this theorem by applying Theorem 3.4.6 and induction. □

CHAPTER IV

TENSOR PRODUCTS OF VECTOR SPACES OVER SEMIFIELDS

In this chapter, we investigate tensor products of vector spaces over semifields satisfying a certain property. Definitions and theorems involving vector spaces over semifields satisfying a specific property are given in Section 4.1. Then, in Section 4.2, we discuss tensor products of vector spaces over such a semifield.

4.1. Vector Spaces over Semifields

Recall that, a system $(K, +, \cdot)$ is said to be a semifield if

(i) $(K, +)$ is a commutative semigroup with identity 0,

(ii) $(K \setminus \{0\}, \cdot)$ is an abelian group and $k \cdot 0 = 0 \cdot k = 0$ for all $k \in K$, and

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in K$,

and for a semifield K, a vector space V over K is an abelian additive group with identity 0, together with a function $K \times V \to V$ (the image of (k, v) being denoted by kv) such that for all $v, v_1, v_2 \in V$ and $k, k_1, k_2 \in K$,

(i) $k(v_1 + v_2) = kv_1 + kv_2$,

(ii) $(k_1 + k_2)v_1 = k_1v + k_2v$,

(iii) $(k_1k_2)v = k_1(k_2v)$, and

(iv) $1_K v = v$ where 1_K is the identity of $\big(K \setminus \{0\}, \cdot\big)$

Definition 4.1.1. Let V be a vector space over a semifield K. An element of V is called a *vector* of V. A vector $v \in V$ is a *linear combination* of $v_1, \ldots, v_n \in V$ if

$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_1, \ldots, \alpha_n \in K$. We denote $\alpha_1 v_1 + \cdots + \alpha_n v_n$ by $\sum_{i=1}^{n} \alpha_i v_i$.

Recall that, if V is a vector space over a field F and B is a basis of V, then each element of V can be written as a unique linear combination of elements of B. However, let K be a semifield and V be a vector space over K. If B is a basis of V, then there has not been proved yet that each element of V can be written as a unique linear combination of elements of B.

For this reason, we consider a particular semifield K which satisfies the following property:

$(*)$ For all $\alpha, \beta \in K$ there exists a $\gamma \in K$ such that $\alpha = \beta + \gamma$ or $\beta = \alpha + \gamma$.

Example 4.1.2.

(i) Every field is a semifield and satisfies the property $(*)$.

(ii) $\mathbb{Q}_0^+$ is a semifield satisfying the property $(*)$ but is not a field.

(iii) $\left(\mathbb{Q}_0^+, *, \cdot\right)$ and $(\mathbb{Z} \cup \{\varepsilon\}, \oplus, \odot)$ in Example 2.2.3 are semifields satisfying the property $(*)$ but are not fields.

(iv) $(\mathbb{Q}^+ \times \mathbb{Q}^+) \cup \{(0,0)\}$ is a semifield which does not satisfy the property $(*)$ is not a field since $(1,2) \neq (2,1) + (x,y)$ and $(2,1) \neq (1,2) + (x,y)$ for all $x, y \in \mathbb{Q}_0^+$.

Proposition 4.1.3. [5] *Let K be a semifield. If there exists $x \in K$ such that x has an additive inverse, then every element of K has an additive inverse and hence K is a field.*

Proposition 4.1.4. [3] *Let V be a nonzero vector space over a semifield K which is not a field, i.e., every nonzero vector has no additive inverse. If B is a basis of V, then every vector v of V can be written uniquely as $v = \sum_{i=1}^{n} \alpha_i \varepsilon_i b_i$ where $n \in \mathbb{N}$,*

$\alpha_i \in K$ and $\varepsilon_i b_i \in \{b_i, -b_i\}$ for all $b_i \in B$, that is, if $v = \sum_{i=1}^{n} \alpha_i \varepsilon_i b_i = \sum_{i=1}^{m} \beta_i \varepsilon_i b_i'$, then $m = n$ and by appropriate rearranging, $\alpha_i = \beta_i$ and $\varepsilon_i b_i = \varepsilon_i b_i'$ for all i.

In this chapter, as a result of Proposition 4.1.3, we let K denote a semifield satisfying the property $(*)$ such that K is not a field, i.e., every nonzero vector has no additive inverse. Moreover, let 1_K be the identity of K.

Theorem 4.1.5. *Let V be a vector space over a semifield K and $X \subseteq V$. Then X is linearly independent if and only if for all distinct elements $x_1, \ldots, x_n \in X$ and for all $\alpha_1, \ldots, \alpha_n \in K$, if $\alpha_1 \varepsilon_1 x_1 + \cdots + \alpha_n \varepsilon_n x_n = 0$, then $\alpha_i = 0$ for all i.*

Proof. First, assume that X is a linearly independent subset of V.

Case 1 If $X = \emptyset$, we are done.

Case 2 If $|X| = 1$ and $X \neq \{0\}$, then $X = \{a\}$ for some $a \in V \setminus \{0\}$. Thus, if $\alpha \varepsilon a = 0$, then $\alpha = 0$ since $\varepsilon a \neq 0$.

Case 3 Let $|X| > 1$ and $x \notin \langle X \backslash \{x\} \rangle$ for all $x \in X$. Without loss of generality, we suppose that there exist distinct $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in K$ with $\alpha_1 \neq 0$ such that $\alpha_1 \varepsilon_1 x_1 + \cdots + \alpha_n \varepsilon_n x_n = 0$. Then

$$\varepsilon_1 x_1 = -\left(\tfrac{\alpha_2}{\alpha_1}\varepsilon_2 x_2 + \tfrac{\alpha_3}{\alpha_1}\varepsilon_3 x_3 + \cdots + \tfrac{\alpha_n}{\alpha_1}\varepsilon_n x_n\right) \in \langle X \backslash \{x_1\} \rangle,$$

which is a contradiction. Therefore, for all distinct elements $x_1, \ldots, x_n \in X$ and for all $\alpha_1, \ldots, \alpha_n \in K$, if $\alpha_1 \varepsilon_1 x_1 + \cdots + \alpha_n \varepsilon_n x_n = 0$, then $\alpha_i = 0$ for all i.

Conversely, we assume that for all distinct elements $x_1, \ldots, x_n \in X$ and for all $\alpha_1, \ldots, \alpha_n \in K$, if $\alpha_1 \varepsilon_1 x_1 + \cdots + \alpha_n \varepsilon_n x_n = 0$, then $\alpha_i = 0$ for all i.

Case 1 If $X = \emptyset$, we are done.

Case 2 Let $|X| = 1$ and suppose that $X = \{0\}$. Then there exists $\alpha \in K \backslash \{0\}$ such that $\alpha \varepsilon 0 = 0$ which is a contradiction. Thus $X \neq \{0\}$.

Case3 Let $|X| > 1$ and we suppose that there exists $x \in X$ such that $x \in \langle X \backslash \{x\} \rangle$. Then there exist $\alpha_1, \ldots, \alpha_n \in K \setminus \{0\}$ and distinct $x_1, \ldots, x_n \in X \setminus \{x\}$ such that

$x = \alpha_1\varepsilon_1 x_1 + \cdots + \alpha_n\varepsilon_n x_n$. Therefore $\alpha_1\varepsilon_1 x_1 + \cdots + \alpha_n\varepsilon_n x_n + 1_K(-x) = 0$ which is a contradiction. Hence $\forall x \in X$, $x \notin \langle X \setminus \{x\} \rangle$.

Therefore X is linearly independent. □

Notation 4.1.6. In Proposition 4.1.4 we sometime write $\displaystyle\sum_{i=1}^{n} \beta_i b_i + \sum_{i=1}^{n} \gamma_i(-b_i)$ instead of $\displaystyle\sum_{i=1}^{n} \alpha_i\varepsilon_i b_i$. It means that, for each $i = 1, \ldots, n$ if $\varepsilon_i b_i = b_i$, then $\beta_i = \alpha_i$ and $\gamma_i = 0$ and if $\varepsilon_i b_i = -b_i$, then $\beta_i = 0$ and $\gamma_i = \alpha_i$.

Recall that, a subset X of a vector space V over a semifield is said to be a *basis* of V if X is a linearly independent set which spans V.

Theorem 4.1.7. [3] *Let A and B be finite subsets of a vector space V over a semifield. If they are bases of V, then $|A| = |B|$.*

Theorem 4.1.7 shows that if a vector space V over a semifield has two finite bases, then the two bases of V must have the same cardinality.

Now, we will extend this to the case that V has an infinite basis.

Theorem 4.1.8. *Let V be a vector space over a semifield which has an infinite basis X. Then every basis of V has the same cardinality as the cardinality of X.*

Proof. Let Y be another basis of a vector space V over a semifield K. First, we show that Y is infinite. Suppose on the contrary that Y were finite. Moreover, since Y generates V and every element of Y is a linear combination of a finite number of elements of X, there is a finite subset $\{x_1, \ldots, x_m\}$ of X which generates V. Since X is infinite, let

$$x \in X \setminus \{x_1, \ldots, x_m\}.$$

Then there exist $\alpha_i \in K$ such that $x = \alpha_1\varepsilon_i x_1 + \ldots + \alpha_m\varepsilon_m x_m$, which contradicts the linear independence of X. Therefore, Y is infinite. Hence every basis of V must be infinite.

Now, it remains to show that all bases of V have the same cardinality by applying Schroeder-Bernstein Theorem. Let $\mathfrak{P}(Y)$ be the set of all finite subsets of Y. Define a function $f : X \to \mathfrak{P}(Y)$ by $x \mapsto \{y_1, \ldots, y_n\}$, where $x = \alpha_1\varepsilon_1 y_1 + \cdots + \alpha_n\varepsilon_n y_n$ and $\alpha_i \neq 0$ for all i. Since Y is a basis, we obtain that f is well-defined. If im f were finite, then $\bigcup_{W \in \text{im } f} W$ would be a finite subset of Y that would generate $\langle X \rangle$ and hence V. This leads to a contradiction that a basis of V must be infinite according to the preceding paragraph. Hence im f is infinite.

Next, we show that $f^{-1}(W)$ is a finite subset of X for all $W \in \text{im } f \subseteq \mathfrak{P}(Y)$. Let $W \in \text{im } f \subseteq \mathfrak{P}(Y)$. Since W is finite and each $w \in W$ is a linear combination of a finite number of elements of X, there exists a finite subset U of X such that $\langle W \rangle \subseteq \langle U \rangle$. Let $x \in f^{-1}(W)$. Then $x \in \langle W \rangle \subseteq \langle U \rangle$ and x is a linear combination of elements of U. Since $x \in X$ and $U \subseteq X$, this contradicts the linear independence of X unless $x \in U$. Therefore, $f^{-1}(W) \subseteq U$, whence $f^{-1}(W)$ is finite.

For each $W \in \text{im } f$, order the elements of $f^{-1}(W)$, say $x_1, \ldots, x_m$, and define a function $g_W : f^{-1}(W) \to \text{im } f \times \mathbb{N}$ by $x_k \mapsto (W, k)$. Clearly, g_W is an injective function. Next, we show that the set of all $f^{-1}(W)$ where $W \in \text{im } f$ forms a partition of X. It is obvious that $\bigcup_{W \in \text{im } f} f^{-1}(W) = X$.

Let $W_1, W_2 \in \text{im } f$ such that $W_1 = \{a_1, \ldots, a_n\}$ and $W_2 = \{b_1, \ldots, b_n\}$. Suppose that $f^{-1}(W_1) \cap f^{-1}(W_2) \neq \emptyset$. Then $x = \alpha_1\varepsilon_1 a_1 + \cdots + \alpha_n\varepsilon_n a_n$ and $x = \beta_1\varepsilon_1 b_1 + \cdots + \beta_m\varepsilon_m b_m$ where $\alpha_i, \beta_i \in K$. Since Y is a basis of V and from Proposition 4.1.4, we can conclude that $n = m, \alpha_i = \beta_i$ and $\varepsilon_i a_i = \varepsilon_i b_i$. Then $W_1 = W_2$. Therefore $f^{-1}(W_1) = f^{-1}(W_2)$. Hence the set $f^{-1}(W)$, where $W \in \text{im } f$, forms a partition of X. Define a function $\phi : X \to \text{im } f \times \mathbb{N}$ by $x \mapsto g_W(x)$ where $x \in f^{-1}(W)$. Clearly, ϕ is an injective function. Then $|X| \leq |\text{im } f \times \mathbb{N}|$. Therefore, by Definition 2.2.4, Theorem 2.2.6 and Corollary 2.2.7, we obtain that

$$|X| \leq |\text{im } f \times \mathbb{N}| = |\text{im } f|\aleph_0 = |\text{im } f| \leq |\mathfrak{P}(Y)| = |Y|.$$

Interchanging X and Y in the preceding argument shows that $|Y| \leq |X|$. Therefore $|Y| = |X|$ by Schroeder-Bernstein Theorem. □

Corollary 4.1.9. *Let V be a vector space over a semifield which has a finite basis X. Then every basis of V has the same cardinality as the cardinality of X.*

Proof. We obtain that every basis of V must be finite from the proof of Theorem 4.1.8 and then must have the same cardinality as X from Theorem 4.1.7. Therefore, every basis of V has the same cardinality as the cardinality of X. □

Definition 4.1.10. Let V be a nonzero vector space over a semifield K. Then V is said to be *finite-dimensional* if V has a finite basis and V is said to be *infinite-dimensional* if V has an infinite basis. Moreover, if a basis of V has cardinality n, then we say that V is an *n-dimensional* vector space.

The *dimension* of V, denoted by $\dim V$ or $\dim_K V$, is the cardinality of a basis of V.

Example 4.1.11.

(i) $\dim \mathbb{Q} = 1$ since $\{1\}$ is a basis of $\mathbb{Q}$ over $\mathbb{Q}_0^+$.

(ii) Let $e_1, \ldots, e_n \in \mathbb{Q}^n$ be defined by

$$e_1 = (1, 0, \ldots, 0, 0),\ e_2 = (0, 1, \ldots, 0, 0),\ \ldots,\ e_n = (0, 0, \ldots, 0, 1).$$

Then $\{e_1, \ldots, e_n\}$ is a basis of the vector space $\mathbb{Q}^n$ over the field $\mathbb{Q}$. In fact, by the definition of a vector space over a semifield, we also obtain that $\{e_1, \ldots, e_n\}$ is a basis of the vector space $\mathbb{Q}^n$ over the semifield $\mathbb{Q}_0^+$, hence $\dim \mathbb{Q}^n = n$. Also, this fact is true if we replace $\mathbb{Q}$ by $\mathbb{R}$ and $\mathbb{Q}_0^+$ by $\mathbb{R}_0^+$.

Theorem 4.1.12. [3] *Let V be a vector space over a semifield and X a linearly independent nonempty subset of V. Then there exists a subset B of V such that $X \subseteq B$ and B is a basis of V.*

Corollary 4.1.13. [3] *Every vector space over a semifield has a basis.*

Theorem 4.1.14. [3] *Let V be a vector space over a semifield and X a subset of V such that X spans V. Then there exists a subset B of X such that B is a basis of V.*

Recall that, if V and W are vector spaces over a semifield K and T is a mapping from V into W, then T is said to be a linear transformation if for all $\alpha \in K$ and $v, v_1, v_2 \in V$,

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad \text{and} \quad T(\alpha v) = \alpha T(v).$$

Lemma 4.1.15. [3] *Let V and W be vector spaces over a semifield and $T : V \to W$ a linear transformation. If B is a subset of V which spans V, then $T(B)$ spans* $\operatorname{im} T$.

Theorem 4.1.16. [3] *Let V and W be vector spaces over a semifield and let $B = \{b_1, \ldots, b_n\}$ be a basis of V where $b_i \neq b_j$ for $i \neq j$. If $\{c_1, \ldots, c_n\}$ is a subset of W, then there exists a unique linear transformation $T : V \to W$ such that $T(b_i) = c_i$ for all $i \in \{1, \ldots, n\}$.*

Definition 4.1.17. Let T be a linear transformation from V into W. The *nullity* of T, denoted by $\operatorname{null} T$, is the dimension of $\ker T$. The *rank* of T, denoted by $\operatorname{rank} T$, is the dimension of $\operatorname{im} T$.

Definition 4.1.18. Let $n \in \mathbb{N}$ and V be a vector space over a semifield. A subset X of V is n*-independent* if $|X| \geq n$ and any n-vectors of X are linearly independent.

In [1], Guo, Y. Q. and Shum, K. P. studied an infinite n-independent subset of vector spaces over fields and some facts, independent subsets, proper subspaces and linear transformations. These are also true if we replace " vector spaces over fields " by " vector spaces over semifields ".

Theorem 4.1.19. *Let V be an n-dimensional vector space over a semifield and $s \in \mathbb{N}$. Then the following statements are equivalent.*

(i) *If V_i is a proper subspace of V for all $i = 1, 2, \ldots, s$, then $\bigcup_{i=1}^{s} V_i \subsetneq V$.*

(ii) *If $\{\alpha_{i1}, \ldots, \alpha_{ir}\}$ is a linearly independent subset of V with $1 \leq r \leq n-1$ and $i = 1, 2, \ldots, s$, then there exists an $\alpha \in V$ such that the extended set $\{\alpha, \alpha_{i1}, \ldots, \alpha_{ir}\}$ is still linearly indepdent for all $i = 1, 2, \ldots, s$.*

(iii) *There exists an infinite n-independent subset of V.*

(iv) *If $T_i, T_j \in \mathcal{L}(V)$ with $T_i \neq T_j$ whenever $i \neq j$ for all $i, j = 1, 2, \ldots, s$, then there exists an $\alpha \in V$ such that $T_i\alpha \neq T_j\alpha$.*

(v) *If $T_i, T_j \in \mathcal{L}(V)$ with $\operatorname{rank} T_i = \operatorname{rank} T_j = n$ and $T_i \neq T_j$ whenever $i \neq j$ for all $i, j = 1, 2, \ldots, s$, then there exists an $\alpha \in V$ such that $T_i\alpha \neq T_j\alpha$.*

(vi) *If $T_i, T_j \in \mathcal{L}(V)$ with $\operatorname{rank} T_i = \operatorname{rank} T_j = 1$ and $T_i \neq T_j$ whenever $i \neq j$ for all $i, j = 1, 2, \ldots, s$, then there exists an $\alpha \in V$ such that $T_i\alpha \neq T_j\alpha$.*

Proof. (i) $\Rightarrow$ (ii) For each $i = 1, 2, \ldots, s$ and $1 \leq r \leq n-1$, let $\{\alpha_{i1}, \ldots, \alpha_{ir}\}$ be linearly independent and $V_i = \langle \alpha_{i1}, \ldots, \alpha_{ir} \rangle$ be the subspace of V. It is easy to see that V_i is a proper subspace of V because $\dim V_i = r < n$ for all $i = 1, 2, \ldots, s$. By (i), we obtain that $\bigcup_{i=1}^{s} V_i \subsetneq V$. Then there exists an $\alpha \in V$ such that $\alpha \notin V_i$ for all $i = 1, \ldots, s$. Since $\alpha \notin V_i = \langle \alpha_{i1}, \ldots, \alpha_{ir} \rangle$ and $\{\alpha_{i1}, \ldots, \alpha_{ir}\}$ is linearly independent, we can conclude that the set $\{\alpha, \alpha_{i1}, \ldots, \alpha_{ir}\}$ is linearly independent, for any $i = 1, 2, \ldots, s$.

(ii) $\Rightarrow$ (iii) We proof by induction. First, let $X_1 = \{\alpha_1, \ldots, \alpha_m\}$ be an $n-$independent subset of V. This subset X always exists, for example, a basis of V. Then $|X_1| = m \geq n$ Let X_{1i} be a subset of X_1 consisting $n-1$ elements for $i = 1, 2, \ldots, \binom{m}{n-1}$. Since X_1 consists of n linearly independent vectors and X_{1i} is a set containing $n-1$ linearly independent vectors for all $i = 1, 2, \ldots, \binom{m}{n-1}$, by (ii) there exists $\alpha_{m+1} \in V$

such that the n-vector set $T_{1i} = X_{1i} \cup \{\alpha_{m+1}\}$ is still linearly independent for all $i = 1, 2, \ldots, \binom{m}{n-1}$.

Let $X_2 = X_1 \cup \{\alpha_{m+1}\}$. It is obvious that X_2 is still a subset of V in which any n vectors are linearly independent and $|X_2| = |X_1| + 1$. Continue this process, we obtain a chain of subsets X_i of V in which any n vectors of X_i are linearly independent, say the chain

$$\emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_i \subsetneq \cdots .$$

Clearly, $\bigcup_{i=1}^{\infty} X_i$ is an infinite set in which any n vectors are linearly independent.

(iii) $\Rightarrow$ (iv) Let X be an infinite n-independent subset of V. Assume that $T_i, T_j \in \mathcal{L}(V)$ such that $T_i \neq T_j$ if $i \neq j$, for all $i, j = 1, 2, \ldots, s$. let

$$V_{ij} = \big\{\alpha \in V \mid T_i\alpha = T_j\alpha\big\}.$$

Then, for all $i, j = 1, 2, \ldots, s$ with $i \neq j$, we can see that $0 < \dim V_{ij} \leq n - 1$. So that V_{ij} is a proper subspace of V. Then $|V_{ij} \cap X| \leq n - 1$. Let m be a number of subspaces V_{ij}. Then we have

$$\left|\left(\bigcup_{\substack{i,j=1\\ i\neq j}}^{s} V_{ij}\right) \cap X\right| = \left|\bigcup_{\substack{i,j=1\\ i\neq j}}^{s} (V_{ij} \cap X)\right| \leq \sum_{\substack{i,j=1\\ i\neq j}}^{s} |V_{ij} \cap X| \leq m(n-1).$$

This leads to $X \setminus \bigcup_{\substack{i,j=1\\ i\neq j}}^{s} V_{ij} \neq \emptyset$. Hence, there exists $\alpha \in X \setminus \bigcup_{\substack{i,j=1\\ i\neq j}}^{s} V_{ij}$ such that $\alpha \notin V_{ij}$ for all i, j. This implies that $(T_i - T_j)(\alpha) \neq 0$ for all $i \neq j$, $i, j = 1, 2, \ldots, s$. Consequently, $T_i(\alpha) \neq T_j(\alpha)$ for all $i \neq j$, $i, j = 1, 2, \ldots, s$. This proves (iv).

(iv) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi) are obvious.

(v) $\Rightarrow$ (i) Let V_i be a proper subspace of V for all $i = 1, 2, \ldots, s$. We show that $\bigcup_{i=1}^{s} V_i \subsetneq V$. Without loss of generality, we may assume that $V_i \nsubseteq V_j$ where $i \neq j$ and $i, j = 1, 2, \ldots, s$. For $i, j = 1, 2, \ldots, s$, let $\{\alpha_{i1}\alpha_{i2}, \ldots, \alpha_{ir_i}\}$ be a basis of V_i, where $1 \leq r_i = \dim V_i \leq n-1$. We extend this basis of V_i to a basis of V, say $\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir_i}, \alpha_{ir_i+1}, \ldots, \alpha_{in}\}$.

Now, we construct the linear transformation T_i on V by

$$T_i : \begin{cases} \alpha_{ij} \longmapsto \alpha_{ij} & \text{if } j = 1, 2, \ldots, r_i \\ \alpha_{ij} \longmapsto \alpha_{ij} + \alpha_{i,j-1} & \text{if } j = r_i + 1, \ldots, n. \end{cases} \tag{4.1.20}$$

Clearly, rank $T_i = n$ and $T_i \neq I$ (the identity transformation) for all $i, j = 1, \ldots, s$. Since $T(V_i) = V_i$, we obtain that $T_i \neq T_j$, where $i \neq j$, and $i, j = 1, 2, \ldots, s$. Therefore $\{T_0 = I, T_1, T_2, \cdots, T_s\}$ is a set of linear transformations satisfying the condition in (v). By the assumption, there exists $\alpha \in V$ such that $T_i\alpha \neq T_j\alpha$ where $i \neq j$ and $i, j = 0, 1, \ldots, s$. In particular, $T_i(\alpha) \neq T_0(\alpha) = I(\alpha) = \alpha$, for all $i = 1, \ldots, s$. By the definition of (4.1.20), $T(V_i) = V_i$ is a subspace of V under T_i. Thus $\alpha \notin V_i$ for all $i = 1, \ldots, s$. Hence $\bigcup_{i=1}^{s} V_i \subsetneq V$.

(vi) $\Rightarrow$ (i) Let V_i be a proper subspace of V for all $i = 1, 2, \ldots, s$. Without loss of generality, we assume that $V_i \nsubseteq V_j$, and $\dim V_i = n-1$ where $i \neq j$ and $i, j = 1, 2, \ldots, s$.

Now, for any $i \in \{1, 2, \ldots, s\}$, let $\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i,n-1}\}$ be a basis of V_i. We extend this basis of V_i to a basis of V, say $\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i,n-1}, \alpha_{i,n}\}$. Now we construct the linear transformations on V as follow:

$${}_1T_i : \begin{cases} \alpha_{ij} \longmapsto 0 & \text{if } j = 1, 2, \ldots, n-1, \quad \text{and} \\ \alpha_{in} \longmapsto \alpha_{in} \end{cases} \tag{4.1.21}$$

$${}_2T_i : \begin{cases} \alpha_{ij} \longmapsto 0 & \text{if } j = 1, 2, \ldots, n-1, \quad \text{and} \\ \alpha_{in} \longmapsto \alpha_{in} + \alpha_{i,n-1}. \end{cases} \tag{4.1.22}$$

Clearly, rank ${}_1T_i =$ rank ${}_2T_i = 1$. If $\alpha_{in} = \alpha_{jn}$ where $i \neq j$, and $i, j = 1, 2, \ldots, s$, then $V_i = V_j$ where $i \neq j$ and $i, j = 1, 2, \ldots, s$ so that $V_i = V_j$. This leads to a contradiction. So ${}_kT_i \neq {}_lT_j$ where $k, l = 1, 2$, and $i, j = 1, \ldots, s$ with $(k, i) \neq (l, j)$. Thus there are $2s$ linear transformations ${}_1T_1, {}_1T_2, \ldots, {}_1T_s, {}_2T_1, {}_2T_2, \ldots, {}_2T_s$ satisfying the condition in (vi). By the assumption we see that there exists $\alpha \in V$ such that ${}_kT_i \neq {}_lT_j$ where $k, l = 1, 2$, and $i, j = 1, \ldots, s$ with $(k, i) \neq (l, j)$. In particular, ${}_1T_i(\alpha) \neq {}_2T_i(\alpha)$ for all $i = 1, \ldots, s$. By (4.1.21) and (4.1.22), we can see that $V_i = \ker {}_1T_i = \ker {}_2T_i$ for all $i = 1, \ldots, s$. Thus $\alpha \notin V_i$. Hence $\bigcup_{i=1}^{s} V_i \subsetneq V$. □

4.2. Tensor Products of Vector Spaces over Semifields

One major different points from a vector space over any semifield and a vector space over a semifield satisfying the property $(*)$ is the existence of a basis. Although tensor products of vector spaces over any semifields were studied in Section 3.3, it is more benefit to learn whether there are other results regarding tensor products of vector spaces over a semifield satisfying the property $(*)$ by means of bases.

Theorem 4.2.1. [3] *Let V and W be finite-dimensional vector spaces over the same semifield and $T : V \to W$ a linear transformation. If dim $V =$ dim W, then T is injective if and only if T is surjective.*

Definition 4.2.2. [3] Let K be a semifield and F_K a field containing a subsemifield K. A linear transformation from a vector space V over K into F_K is called a *linear function*. Moreover, let $V^* = \mathcal{L}(V, F_K)$ and $V^{**} = (V^*)^*$. We call V^* the *dual space* of V and V^{**} the *double dual* of V.

Remark 4.2.3. [3] If V is a finite-dimensional vector space over a semifield, then dim $V =$ dim $V^* =$ dim V^{**}.

Theorem 4.2.4. [3] *Let V be a finite-dimensional vector space over a semifield, dim $V = n$, and $B = \{b_1, \ldots, b_n\}$ a basis of V. For each $i \in \{1, 2, \ldots, n\}$, let $f_i \in V^*$ be such that*

$$f_i(b_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i \ . \end{cases}$$

Then the following statements hold.

(i) $\{f_1, \ldots, f_n\}$ *is a basis of V^* which is called the* dual basis *of B.*

(ii) *For all $f \in V^*, f = \sum_{i=1}^{n} f(b_i) f_i$.*

(iii) *For all $v \in V, v = \sum_{i=1}^{n} f_i(v) b_i$.*

Definition 4.2.5. Let V and W be vector spaces over a semifield. For each $v \in V$ and $w \in W$, we say that *εv and $\varepsilon' w$ have the same sign* if

(i) $\varepsilon v = v$ and $\varepsilon' w = w$ or

(ii) $\varepsilon v = -v$ and $\varepsilon' w = -w$.

Moreover, we say that *εv and $\varepsilon' w$ have the different sign* if εv and $\varepsilon' w$ does not have the same sign.

Proposition 4.2.6. *Let V and W be vector spaces over a semifield K and $\alpha, \beta \in K$. Let $v \in V$ and $w \in W$ be such that εv and $\tilde{\varepsilon} w$ have the same sign and $\hat{\varepsilon} v$ and $\bar{\varepsilon} w$ have the same sign. Then there exists $\gamma \in K$ such that $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \gamma(\varepsilon' v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \gamma(\varepsilon'' w)$ where $\varepsilon' v$ and $\varepsilon'' w$ have the same sign.*

Proof. There are four cases to be considered.

Case 1 Assume that $\alpha(\varepsilon v) = \alpha v$ and $\beta(\hat{\varepsilon} v) = \beta v$. Then $\alpha(\tilde{\varepsilon} w) = \alpha w$ and $\beta(\bar{\varepsilon} w) = \beta w$. Thus $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \alpha v + \beta v = (\alpha + \beta) v$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \alpha w + \beta w = (\alpha + \beta) w$. Then we choose $\gamma = \alpha + \beta$.

Case 2 Assume that $\alpha(\varepsilon v) = \alpha(-v)$ and $\beta(\hat{\varepsilon} v) = \beta(-v)$. Then $\alpha(\tilde{\varepsilon} w) = \alpha(-w)$ and $\beta(\bar{\varepsilon} w) = \beta(-w)$. Thus $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \alpha(-v) + \beta(-v) = (\alpha+\beta)(-v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \alpha(-w) + \beta(-w) = (\alpha+\beta)(-w)$. Then we choose $\gamma = \alpha + \beta$.

Case 3 Assume that $\alpha(\varepsilon v) = \alpha v$ and $\beta(\hat{\varepsilon} v) = \beta(-v)$. Then $\alpha(\tilde{\varepsilon} w) = \alpha w$ and $\beta(\bar{\varepsilon} w) = \beta(-w)$.

If $\alpha = \beta + \gamma$ for some $\gamma \in K$, then $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = (\beta+\gamma)v + \beta(-v) = \gamma v$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = (\beta+\gamma)w + \beta(-w) = \gamma w$.

If $\beta = \alpha + \gamma$ for some $\gamma \in K$, then $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \alpha v + \beta(-v) = \alpha v + (\alpha + \gamma)(-v) = \gamma(-v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \alpha w + \beta(-w) = \alpha w + (\alpha+\gamma)(-w) = \gamma(-w)$.

Case 4 Assume that $\alpha(\varepsilon v) = \alpha(-v)$ and $\beta(\hat{\varepsilon} v) = \beta v$. Then $\alpha(\tilde{\varepsilon} w) = \alpha(-w)$ and $\beta(\bar{\varepsilon} w) = \beta w$.

If $\alpha = \beta + \gamma$ for some $\gamma \in K$, then $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = (\beta+\gamma)(-v) + \beta v = \gamma(-v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = (\beta+\gamma)(-w) + \beta w = \gamma(-w)$.

If $\beta = \alpha + \gamma$ for some $\gamma \in K$, then $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \alpha(-v) + (\alpha+\gamma)v = \gamma(v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \alpha(-w) + (\alpha+\gamma)w = \gamma(w)$.

From Case 1 – Case 4, we can conclude that there exists $\gamma \in K$ such that $\alpha(\varepsilon v) + \beta(\hat{\varepsilon} v) = \gamma(\varepsilon' v)$ and $\alpha(\tilde{\varepsilon} w) + \beta(\bar{\varepsilon} w) = \gamma(\varepsilon'' w)$ where $\varepsilon' v$ and $\varepsilon'' w$ have the same sign. □

Lemma 4.2.7. *Let V and W be vector spaces over a semifield K, $\mathcal{B}$ a basis of W and $\mathcal{C} = \{w_1, \ldots, w_n\} \subseteq \mathcal{B}$. Let $\beta : V \times W \to V$ be defined as follows: for each $v \in V$ and $w \in W$ such that*

$$w = \sum_{i=1}^{n} \alpha_i \varepsilon_i w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b$$

where $\sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b$ is a finite sum with $\alpha_1, \ldots, \alpha_n, \alpha_b \in K$,

$$\beta(v, w) = \sum_{i=1}^{n} \alpha_i (\varepsilon_i v).$$

Then β is bilinear.

Proof. Let $v, v' \in V$, $w, w' \in W$ and $a \in K$. Then

$$w = \sum_{i=1}^{n} \alpha_i \varepsilon_i w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b \quad \text{and} \quad w' = \sum_{i=1}^{n} \beta_i \varepsilon_i' w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \beta_b \varepsilon_b' b$$

where $\alpha_1, \ldots, \alpha_n, \alpha_b, \beta_1, \ldots, \beta_n, \beta_b \in K$ for all $b \in \mathcal{B} \setminus \mathcal{C}$. Then β is well-defined because every vector in W can be written uniquely as a linear combination of elements in $\mathcal{B}$. It is easy to verify that $\beta(v, aw) = a\beta(v, w) = \beta(av, w)$ and $\beta(v + v', w) = \beta(v, w) + \beta(v', w)$.

Next, we show that $\beta(v, w + w') = \beta(v, w) + \beta(v, w')$. We can see that $\beta(v, w) = \sum_{i=1}^{n} \alpha_i(\varepsilon_i v)$ and $\beta(v, w') = \sum_{i=1}^{n} \beta_i(\varepsilon_i' v)$. For each i, since $\varepsilon_i w_i$ and $\varepsilon_i v$ have the same sign and $\varepsilon_i' w_i$ and $\varepsilon_i' v$ have the same sign, by Proposition 4.2.6 there exists $\gamma_i \in K$ such that $\alpha_i \varepsilon_i w_i + \beta_i \varepsilon_i' w_i = \gamma_i \varepsilon_i'' w_i$, $\alpha_i \varepsilon_i v + \beta_i \varepsilon_i' v = \gamma_i \varepsilon_i'' v$ and $\varepsilon_i'' w_i$ and $\varepsilon_i'' v$ have the same sign. It follows that

$$\begin{aligned}
\beta(v, w + w') &= \beta\left(v, \sum_{i=1}^{n} \alpha_i \varepsilon_i w_i + \sum_{i=1}^{n} \beta_i \varepsilon_i' w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \beta_b \varepsilon_b' b\right) \\
&= \beta\left(v, \sum_{i=1}^{n} (\alpha_i \varepsilon_i w_i + \beta_i \varepsilon_i' w_i) + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \beta_b \varepsilon_b' b\right) \\
&= \beta\left(v, \sum_{i=1}^{n} \gamma_i (\varepsilon_i'' w_i) + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \beta_b \varepsilon_b' b\right) \\
&= \sum_{i=1}^{n} \gamma_i (\varepsilon_i'' v)
\end{aligned}$$

and

$$\begin{aligned}
\beta(v, w) + \beta(v, w') &= \beta\left(v, \sum_{i=1}^{n} \alpha_i \varepsilon_i w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \alpha_b \varepsilon_b b\right) + \beta\left(v, \sum_{i=1}^{n} \beta_i \varepsilon_i' w_i + \sum_{b \in \mathcal{B} \setminus \mathcal{C}} \beta_b \varepsilon_i' b\right) \\
&= \sum_{i=1}^{n} \alpha_i (\varepsilon_i v) + \sum_{i=1}^{n} \beta_i (\varepsilon_i' v)
\end{aligned}$$

$$= \sum_{i=1}^{n} (\alpha_i(\varepsilon_i v) + \beta(\varepsilon_i' v))$$

$$= \sum_{i=1}^{n} \gamma_i(\varepsilon_i'' v).$$

Therefore β is bilinear. □

Theorem 4.2.8. *Let V and W be vector spaces over a semifield and let $v_1, \ldots, v_n \in V$, $w_1, \ldots, w_n \in W$. If $\{w_1, \ldots, w_n\}$ is linearly independent and $\sum_{i=1}^{n} v_i \otimes \varepsilon_i w_i = 0$, then $v_i = 0$ for all i.*

Proof. Let $\mathcal{B}$ be a basis of W containing $w_1, \ldots, w_n$. For each $i = 1, 2, \ldots, n$, define $\beta_i : V \times W \to V$ by for each $v \in V$ and $w \in \mathcal{B}$

$$\beta_i(v, w) = \begin{cases} v & \text{if } w = w_i, \\ -v & \text{if } w = -w_i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 4.2.7 by letting $\mathcal{C} = \{w_i\}$ that β is bilinear. By the universal mapping property of tensor products, there exists a unique linear transformation $\tilde{\beta} : V \otimes_K W \to V$ such that $\beta = \tilde{\beta} \circ B$ where $B : V \times W \to V \otimes_K W$ is the canonical bilinear map. From $\sum_{j=1}^{n} v_j \otimes \varepsilon_j w_j = 0$, we apply $\tilde{\beta}$. Then

$$0 = \tilde{\beta}\left(\sum_{j=1}^{n} v_j \otimes \varepsilon_j w_j\right)$$

$$= \sum_{j=1}^{n} \tilde{\beta}(v_j \otimes \varepsilon_j w_j)$$

$$= \sum_{j=1}^{n} (\tilde{\beta} \circ B)(v_j, \varepsilon_j w_j)$$

$$= \sum_{j=1}^{n} \beta(v_j, \varepsilon_j w_j)$$

$$= \varepsilon_i v_i$$

Thus $v_i = 0$ for all i. □

Corollary 4.2.9. *Let V and W be vector spaces over a semifield. For all $v \in V$ and $w \in W$, if $v \neq 0$ and $w \neq 0$ then $v \otimes w \neq 0$.*

Proof. This is straightforward. □

Corollary 4.2.10. *Let V and W be vector spaces over a semifield and $\mathcal{B}$ a basis of V and $\mathcal{C}$ a basis of W. If $v, v' \in \mathcal{B}$ and $w, w' \in \mathcal{C}$ are such that $v \neq v'$ or $w \neq w'$, then $v \otimes w \neq v' \otimes w'$.*

Proof. This follows from Corollary 4.2.9. □

Note 4.2.11. Let V and W be vector spaces over a semifield, $v \in V$ and $w \in W$. Then $\varepsilon(v \otimes w) = (\varepsilon v \otimes w) = (v \otimes \varepsilon w)$.

Proof. This follows from Remark 3.3.6 (ii). □

Theorem 4.2.12. *Let V and W be vector spaces over a semifield K, $\mathcal{B}$ a basis of V and $\mathcal{C}$ a basis of W. Then $\{v \otimes w \mid v \in \mathcal{B} \text{ and } w \in \mathcal{C}\}$ is a basis of $V \otimes_K W$.*

Proof. Let $\mathcal{D} = \{v \otimes w \mid v \in \mathcal{B} \text{ and } w \in \mathcal{C}\}$. We claim that $\mathcal{D}$ is linearly independent. First of all, let $v, v' \in \mathcal{B}$, $w \in \mathcal{C}$ and $a, b \in K$ be such that $v \neq v'$ and

$$a\bar{\varepsilon}(v \otimes w) + b\hat{\varepsilon}(v' \otimes w) = 0.$$

Then $0 = a\bar{\varepsilon}(v \otimes w) + b\hat{\varepsilon}(v' \otimes w) = (a\bar{\varepsilon}v \otimes w) + (b\hat{\varepsilon}v' \otimes w) = (a\bar{\varepsilon}v + b\hat{\varepsilon}v') \otimes w$. Thus, we obtain from Theorem 4.2.8 that $a\bar{\varepsilon}v + b\hat{\varepsilon}v' = 0$. Suppose that $a \neq 0$. Then

$\bar{\varepsilon}v = a^{-1}(-b\hat{\varepsilon}v')$ which is a contradiction because v, v' are distinct elements in the basis $\mathcal{B}$. Therefore $a = 0$ and then $b = 0$.

For this reason, in order to show that $\mathcal{D}$ is linearly independent it is enough to let $v_1, \dots, v_n \in \mathcal{B}$, $w_1, \dots, w_n \in \mathcal{C}$, $\alpha_1, \dots, \alpha_n \in K$ be such $w_1, \dots, w_n$ are all distinct and $\alpha_1\varepsilon_1(v_1 \otimes w_1) + \cdots + \alpha_n\varepsilon_n(v_n \otimes w_n) = 0$. Then $(\alpha_1\varepsilon_1 v_1 \otimes w_1) + \cdots + (\alpha_n\varepsilon_n v_n \otimes w_n) = 0$. Since $w_1, \dots, w_n$ are all distinct elements in the basis $\mathcal{C}$, we obtain that $\{w_1, \dots, w_n\}$ is linearly independent. So that $\alpha_i\varepsilon_i v_i = 0$ for all $i = 1, \dots, n$ from Theorem 4.2.8. Since v_i is an element of $\mathcal{B}$ for all i, we obtain that $\alpha_i = 0$. Hence $\mathcal{D}$ is linearly independent.

Next, we show that $\mathcal{D}$ spans $V \otimes_K W$. Let $x \in V \otimes_K W$. Then $x = \sum_{i=1}^{n} m_i(v_i \otimes w_i)$ where $m_i \in \mathbb{Z}$, $v_i \in V$ and $w_i \in W$ for all i. For each $v_i \in V$ and $w_i \in W$, there exist $\alpha_{i1}, \dots, \alpha_{ij_i}, \beta_{i1}, \dots, \beta_{ij_i}, \gamma_{i1}, \dots, \gamma_{ik_i}, \lambda_{i1}, \dots, \lambda_{ik_i} \in K$ and $b_1, \dots, b_{j_i} \in B$ and $c_1, \dots, c_{k_i} \in C$ such that

$$v_i = \alpha_{i1}b_1 + \cdots + \alpha_{ij_i}b_{j_i} + \beta_{i1}(-b_1) + \cdots + \beta_{ij_i}(-b_{j_i}) = \eta_{i1}(\varepsilon_1 b_1) + \cdots + \eta_{ij_i}(\varepsilon_{j_i} b_{j_i})$$

$$w_i = \gamma_{i1}c_1 + \cdots + \gamma_{ik_i}c_{k_i} + \lambda_{i1}(-c_1) + \cdots + \lambda_{ik_i}(-c_{ki}) = \zeta_{i1}(\varepsilon_1 c_1) + \cdots + \zeta_{ik_i}(\varepsilon_{k_i} c_{k_i})$$

where

$$\eta_{ij} = \begin{cases} \alpha_{ij} & \text{if } \varepsilon_j b_j = b_j, \\ \beta_{ij} & \text{if } \varepsilon_j b_j = -b_j, \end{cases} \qquad \text{and} \qquad \zeta_{ij} = \begin{cases} \gamma_{ij} & \text{if } \varepsilon_j c_j = c_j, \\ \lambda_{ij} & \text{if } \varepsilon_j c_j = -c_j. \end{cases}$$

Then

$$\begin{aligned} x &= \sum_{i=1}^{n} m_i \left(\sum_{l=1}^{j_i} \alpha_{il} b_l + \sum_{l=1}^{j_i} \beta_{il}(-b_l) \right) \otimes \left(\sum_{p=1}^{k_i} \gamma_{ip} c_p + \sum_{p=1}^{k_i} \lambda_{ip}(-c_p) \right) \\ &= \sum_{i=1}^{n} m_i \left(\sum_{l=1}^{j_i} \alpha_{il} b_l \otimes \sum_{p=1}^{k_i} \gamma_{ip} c_p \right) + \sum_{i=1}^{n} m_i \left(\sum_{l=1}^{j_i} \alpha_{il} b_l \otimes \sum_{p=1}^{k_i} \lambda_{ip}(-c_p) \right) \end{aligned}$$

$$+\sum_{i=1}^{n} m_i\left(\sum_{l=1}^{j_i}\beta_{il}(-b_l)\otimes\sum_{p=1}^{k_i}\gamma_{ip}c_p\right)+\sum_{i=1}^{n} m_i\left(\sum_{l=1}^{j_i}\beta_{il}(-b_l)\otimes\sum_{p=1}^{k_i}\lambda_{ip}(-c_p)\right)$$

$$=\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\alpha_{il}\gamma_{ip}(b_l\otimes c_p)+\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\alpha_{il}\lambda_{ip}(b_l\otimes -c_p)$$

$$+\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\beta_{il}\gamma_{ip}(-b_l\otimes c_p)+\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\beta_{il}\lambda_{ip}(-b_l\otimes -c_p)$$

$$=\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i}[m_i(\alpha_{il}b_l\otimes\gamma_{ip}c_p)+m_i(\alpha_{il}b_l\otimes\lambda_{ip}(-c_p))+m_i(\beta_{il}(-b_l)\otimes\gamma_{ip}c_p)$$

$$+m_i(\beta_{il}(-b_l)\otimes\lambda_{ip}(-cp))].$$

Fix i, l, p and consider

$$m_i(\alpha_{il}b_l\otimes\gamma_{ip}c_p)+m_i(\alpha_{il}b_l\otimes\lambda_{ip}(-c_p))$$
$$+m_i(\beta_{il}(-b_l)\otimes\gamma_{ip}c_p)+m_i(\beta_{il}(-b_l)\otimes\lambda_{ip}(-cp)). \quad (4.2.13)$$

Case 1 If $\alpha_{il}\neq 0$ and $\gamma_{ip}\neq 0$, then $\beta_{il}=\lambda_{ip}=0$. Thus (4.2.13) is $m_i(\alpha_{il}b_l\otimes\gamma_{ip}c_p)$.

Case 2 If $\alpha_{il}\neq 0$ and $\lambda_{ip}\neq 0$, then $\beta_{il}=\gamma_{ip}=0$. Thus (4.2.13) is $m_i\left(\alpha_{il}b_l\otimes\lambda_{ip}(-c_p)\right)$.

Case 3 If $\beta_{il}\neq 0$ and $\gamma_{ip}\neq 0$, then $\alpha_{il}=\lambda_{ip}=0$. Thus (4.2.13) is $m_i\left(\beta_{il}(-b_l)\otimes\gamma_{ip}c_p\right)$.

Case 4 If $\beta_{il}\neq 0$ and $\lambda_{ip}\neq 0$, then $\alpha_{il}=\gamma_{ip}=0$. Thus (4.2.13) is $m_i\left(\beta_{il}(-b_l)\otimes\lambda_{ip}(-c_p)\right)$.

Therefore (4.2.13) is $m_i\left(\eta_{il}(\varepsilon_l b_l)\otimes\zeta_{ip}(\varepsilon_p c_p)\right)$. Hence

$$x=\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i}\left[m_i\left(\eta_{il}(\varepsilon_l b_l)\otimes\zeta_{ip}(\varepsilon_p c_p)\right)\right]$$

$$=\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\eta_{il}\zeta_{il}(\varepsilon_l b_l)\otimes(\varepsilon_p c_p)$$

$$=\sum_{i=1}^{n}\sum_{l=1}^{j_i}\sum_{p=1}^{k_i} m_i\eta_{il}\zeta_{il}\varepsilon_l\varepsilon_p(b_l\otimes c_p)$$

which is a finite sum. This shows that $x \in \langle \mathcal{D} \rangle$. Hence $\{v \otimes w \mid v \in \mathcal{B} \text{ and } w \in \mathcal{C}\}$ is a basis of $V \otimes_K W$. □

Example 4.2.14. Let V be a vector space over a semifield K of dimension 2 with basis $\mathcal{B} = \{v_1, v_2\}$. Then the following set is a basis of $V \otimes_K V$:

$$\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}.$$

Example 4.2.15.

(i) $\{1 \otimes 1\}$ is a basis of $\mathbb{Q} \otimes_{\mathbb{Q}_0^+} \mathbb{R}$.

(ii) $\mathbb{Q}^n$ and $\mathbb{R}^m$ are vector spaces over $\mathbb{Q}_0^+$ with basis $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_m\}$, respectively, then $\{e_i \otimes e'_j | i = 1, \ldots, n, \text{ and } j = 1, \ldots, m\}$ is a basis of $\mathbb{Q}^n \otimes_{\mathbb{Q}_0^+} \mathbb{R}^m$.

Corollary 4.2.16. *Let V and W be finite-dimensional vector spaces over a semifield K. Then $V \otimes_K W$ is also finite-dimensional and*

$$dim\ V \otimes_K W = (dim\ V)(dim\ W).$$

Proof. This follows from Theorem 4.2.12. □

Note 4.2.17. Let V and W be finite-dimensional vector spaces over a semifield K. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V with dual basis $\mathcal{B}' = \{\phi_1, \ldots, \phi_n\}$ and $\mathcal{C} = \{w_1, \ldots, w_m\}$ a basis of W with dual basis $\mathcal{C}' = \{\psi_1, \ldots, \psi_n\}$. Then

$$\{\phi_i \otimes \psi_j \mid i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\}$$

is a basis of $V^* \otimes_K W^*$.

Theorem 4.2.18. *Let V and W be finite-dimensional vector spaces over a semifield K. Then*

$$V^* \otimes_K W^* \cong (V \otimes_K W)^*$$

via the isomorphism $\tau : V^* \otimes_K W^* \to (V \otimes_K W)^*$ defined by

$$\tau(\phi \otimes \psi)(v \otimes w) = \phi(v)\psi(w) \textit{ for all } \phi \in V^*, \ \psi \in W^*, \ v \in V \textit{ and } w \in W.$$

Proof. In order to define a linear transformation from $V^* \otimes_K W^*$ to $(V \otimes_K W)^*$, we need to define a bilinear map $\beta : V^* \times W^* \to (V \otimes_K W)^*$ satisfying the following diagram:

$$\begin{array}{ccc} V^* \times W^* & \xrightarrow{B} & V^* \otimes_K W^* \\ \big\downarrow{\scriptstyle \beta} & & \\ (V \otimes_K W)^* & & \end{array}$$

where B is the canonical bilinear map. Note that $\beta(\phi, \psi)$ must be an element of $(V \otimes_K W)^*$ for each $\phi \in V^*$ and $\psi \in W^*$.

Let $\phi \in V^*$ and $\psi \in W^*$. An element of $(V \otimes_K W)^*$ can be derived from the universal mapping property of tensor products by considering the diagram below:

$$\begin{array}{ccc} V \times W & \xrightarrow{\overline{B}} & V \otimes_K W \\ \big\downarrow{\scriptstyle f_{\phi,\psi}} & & \\ F_K & & \end{array}$$

where $\overline{B}$ is the canonical bilinear map and $f_{\phi,\psi}$ is a bilinear map defined by

$$f_{\phi,\psi}(v, w) = \phi(v)\psi(w) \text{ for all } v \in V \text{ and } w \in W.$$

Then there exists a unique linear transformation $\tilde{f}_{\phi,\psi} : V \otimes_K W \to F_K$ such that $f_{\phi,\psi} = \tilde{f}_{\phi,\psi} \circ \overline{B}$. Consequently, $\tilde{f}_{\phi,\psi} \in (V \otimes_K W)^*$ and $\tilde{f}_{\phi,\psi}(v \otimes w) = (\tilde{f}_{\phi,\psi} \circ \overline{B})(v, w) = f_{\phi,\psi}(v, w) = \phi(v)\psi(w)$ for all $v \in V$ and $w \in W$.

Now we define $\beta : V^* \times W^* \to (V \otimes_K W)^*$ by $\beta(\phi, \psi) = \tilde{f}_{\phi,\psi}$ for all $\phi \in V^*$ and $\psi \in W^*$. By the uniqueness of $\tilde{f}_{\phi,\psi}$, we obtain that β is well-defined.

Next, we show that β is bilinear. Let $a, b \in K$, $\phi, \varphi \in V^*$, $\psi, \xi \in W^*$, $v \in V$ and $w \in W$. Then

$$
\begin{aligned}
\beta(a\phi + b\varphi, \psi)(v \otimes w) &= \tilde{f}_{a\phi+b\varphi,\psi}(v \otimes w) \\
&= f_{a\phi+b\varphi,\psi}(v, w) \\
&= (a\phi + b\varphi)(v)\psi(w) \\
&= a\phi(v)\psi(w) + b\varphi(v)\psi(w) \\
&= af_{\phi,\psi}(v, w) + bf_{\varphi,\psi}(v, w) \\
&= a\tilde{f}_{\phi,\psi} \circ \overline{B}(v, w) + b\tilde{f}_{\varphi,\psi} \circ \overline{B}(v, w) \\
&= a\tilde{f}_{\phi,\psi}(v \otimes w) + b\tilde{f}_{\varphi,\psi}(v \otimes w) \\
&= a\beta(\phi, \psi)(v \otimes w) + b\beta(\varphi, \psi)(v \otimes w).
\end{aligned}
$$

Similary, we also obtain $\beta(\phi, a\psi + b\xi) = a\beta(\phi, \psi) + b\beta(\phi, \xi)$. Therefore, by the universal mapping property of tensor products, there exists a unique linear transformation $\tau : V^* \otimes_K W^* \to (V \otimes_K W)^*$ such that $\beta = \tau \circ B$. Moreover, $\tau(\phi \otimes \psi)(v \otimes w) = \tilde{f}_{\phi,\psi}(v \otimes w) = \phi(v)\psi(w)$ for all $\phi \in V^*$, $\psi \in W^*$, $v \in V$ and $w \in W$.

Next, we will show that τ is an isomorphism. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V with dual basis $\mathcal{B}' = \{\phi_1, \ldots, \phi_n\}$ and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be a basis of W with dual basis $\mathcal{C}' = \{\psi_1, \ldots, \psi_m\}$. Then

$$
\begin{aligned}
\tau(\phi_i \otimes \psi_j)(v_k \otimes w_l) &= \phi_i(v_k)\psi_j(w_l) \\
&= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{if } i \neq k \text{ or } j \neq l. \end{cases}
\end{aligned}
$$

This shows that $\{\tau(\phi_i \otimes \psi_j) \mid i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\}$ forms the dual basis of the basis $\{v_k \otimes w_l \mid i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\}$ for $V \otimes_K W$. Thus τ maps the basis $\{\phi_i \otimes \psi_j \mid i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\}$ of $V^* \otimes_K W^*$ to the basis $\{\tau(\phi_i \otimes \psi_j) \mid i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m\}$ of $(V \otimes_K W)^*$. By Theorem 4.1.15 and Theorem 4.1.16, it follows that τ is surjective. Furthermore, τ is injective from Theorem 4.2.1. Hence τ is an isomorphism. Therefore $V^* \otimes_K W^* \cong (V \otimes_K W)^*$ □

Remark 4.2.19. Let V and W be finite-dimensional vector spaces over a semifield K. Then

$$V^* \otimes_K W^* \cong (V \otimes_K W)^* \cong \mathcal{L}(V, W; F_K).$$

Proof. We obtain from Theorem 4.2.18 that $V^* \otimes_K W^* \cong (V \otimes_K W)^*$ and from Theorem 3.4.6 that $(V \otimes_K W)^* = \mathcal{L}(V, W; F_K)$. As a result, $V^* \otimes_K W^* \cong (V \otimes_K W)^* \cong \mathcal{L}(V, W; F_K)$. □

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