CHAPTER I

INTRODUCTION

Let k be a finite field and F_k denote the set of all finite words with letters in k. F_k is a free monoid with identity ε, called the *empty word.* Consider the special linear group of degree two over k, $\mathrm{SL}_2(k)$, consisting of 2×2 matrices over k of determinant one. It has been proved in [2] Lemma 2.1 that $\mathrm{SL}_2(k)$ generated as a monoid by the set of matrices

$$S = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} : \alpha \in k \right\}.$$

We can view S as k and thus every word $w = \alpha_1 \dots \alpha_l \in F_k$ is corresponding to the product

$$\begin{bmatrix} 0 & 1 \\ -1 & \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ -1 & \alpha_l \end{bmatrix} \in \mathrm{SL}_2(k).$$

This gives rise to an onto homomorphism of monoids

$$\pi : F_k \longrightarrow \mathrm{SL}_2(k).$$

We define an equivalence relation $\sim$ on $k^2 \smallsetminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, the set of nonzero vectors, by

$$\begin{bmatrix} s \\ t \end{bmatrix} \sim \begin{bmatrix} u \\ v \end{bmatrix} \Leftrightarrow \begin{bmatrix} s \\ t \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} \text{ for some } \lambda \in k^{\times}.$$

Its equivalence classes are the lines spanned by $\begin{bmatrix} 1 \\ x \end{bmatrix}$, $x \in k$, and the line spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, called the *infinite line*, with the origin deleted. Then we usually represent these classes as vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$, $x \in k$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The set of all equivalence classes, denoted by $\mathbb{P}^1(k)$ and called the *projective* 1*-space*. The group $\mathrm{SL}_2(k)$ acts on $\mathbb{P}^1(k)$ by left multiplication. Bacher defined the subset $\mathcal{A}$ of F_k by

$$\mathcal{A} = \left\{ w \in F_k : \pi(w) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The sets $\mathcal{A}$ and $\mathcal{C} = F_k \setminus \mathcal{A}$ divide F_k into two disjoint pieces. This partition leads to an equivalence relation on F_k.

For $r \in k$, we define two disjoint subsets $\mathcal{A}_r$ and $\mathcal{C}_r$ of F_k by

$$\mathcal{A}_r = \left\{ w \in F_k : \pi(w) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ r \end{bmatrix} \right\}.$$

and $\mathcal{C}_r = F_k \setminus \mathcal{A}_r$. Hence $\mathcal{A} = \mathcal{A}_0$. In Chapter 2, we investigate arithmetic and combinatorial properties of the equivalence relation on F_k induced by the partition $\mathcal{A}_r$ and $\mathcal{C}_r$.

Let N be a positive integer. Another route to extend Bacher's work is to study the special linear group over $\mathbb{Z}/N\mathbb{Z}$, the ring of integers modulo N. We present this topic in Chapter 3. Write F_N for the set of all finite words with letters in $\mathbb{Z}/N\mathbb{Z}$. Consider the special linear group of degree two over $\mathbb{Z}/N\mathbb{Z}$, $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, consisting of 2×2 matrices over $\mathbb{Z}/N\mathbb{Z}$ of determinant one. Let

$$S' = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} : \alpha \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

We show that this set generates $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ as a monoid. Our proof is different from [2] Lemma 2.1. We use the basic fact that every closed subset of a finite group is a group. This result shows that every element of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ can be written in at least one way as a finite word with letters in S'.

We can also consider S' as $\mathbb{Z}/N\mathbb{Z}$ and hence every word $w = \alpha_1 \dots \alpha_l \in F_N$ is corresponding to the product

$$\begin{bmatrix} 0 & 1 \\ -1 & \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ -1 & \alpha_l \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

This yields an onto homomorphism of monoids

$$\pi : F_N \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

For $\mathbb{Z}/N\mathbb{Z}$, we define an equivalence relation $\sim'$ on $(\mathbb{Z}/N\mathbb{Z})^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ by

$$\begin{bmatrix} s \\ t \end{bmatrix} \sim' \begin{bmatrix} u \\ v \end{bmatrix} \Leftrightarrow \begin{bmatrix} s \\ t \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} \text{ for some } \lambda \in (\mathbb{Z}/N\mathbb{Z})^\times.$$

Here $(\mathbb{Z}/N\mathbb{Z})^\times$ denotes the unit group of the ring $\mathbb{Z}/N\mathbb{Z}$. The group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on the set of equivalence classes by left multiplication. Parallel to Bacher's, we set

$$\bar{\mathcal{A}} = \left\{ w \in F_N : \pi(w) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and $\bar{\mathcal{C}} = F_N \setminus \bar{\mathcal{A}}$. We study this partition of F_N in the last chapter.