CHAPTER IV

SOME GRAPHS THAT ARE NOT SUPER VERTEX-MAGIC

Our purpose in this chapter is to investigate some families of graphs that do not admit super vertex-magic total labelings. Some known results in [6] are stated with brief proof or without proof, so we show the proofs here.

Definition 4.1. The *Petersen graph* is the graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

Theorem 4.2. *The Petersen graph is not super vertex-magic.*

Proof. Assume that the Petersen graph is super vertex-magic.

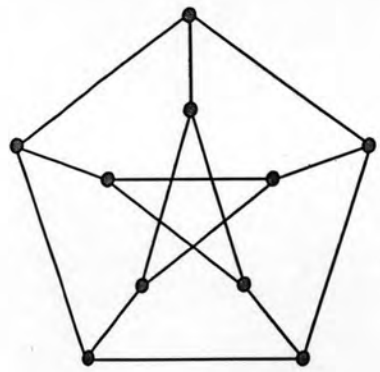

Figure 4.1 : The Petersen graph

Since the Petersen graph has 10 vertices and 15 edges, by Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(25)(26)}{10}-\frac{11}{2}=\frac{119}{2},$$

which is not an integer, a contradiction.

Hence the Petersen graph is not super vertex-magic. $\square$

Definition 4.3. A *wheel graph* with $n+1$ vertices, $n \geq 4$ is an integer, denoted by W_n is defined as $K_1 \vee C_n$.

Example 4.4. The wheel graph W_6 has 7 vertices and 12 edges.

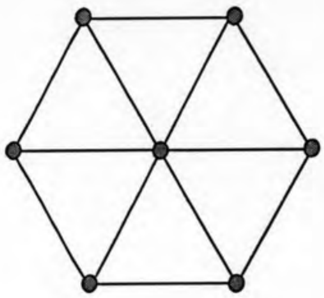

Figure 4.2 : The wheel graph W_6

Note that a wheel graph W_n has $v = n+1$ and $e = 2n$.

Theorem 4.5. ([6]) *If G is a wheel graph W_n where $n \geq 4$ is an integer, then G is not a super vertex-magic graph.*

Proof. Let G be a wheel graph W_n and $n \geq 4$ is an integer.
Assume that G is a super vertex-magic graph with the magic constant h.
Since $v = n+1$ and $e = 2n$, by Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(3n+1)(3n+2)}{n+1} - \frac{n+2}{2}$$
$$= \frac{9n^2+9n+2}{n+1} - \frac{n+2}{2} = \frac{2(9n^2+9n+2)}{2(n+1)} - \frac{(n+2)(n+1)}{2(n+1)}$$
$$= \frac{18n^2+18n+4}{2n+2} - \frac{n^2+3n+2}{2n+2} = \frac{17n^2+15n+2}{2n+2}.$$

Case 1 : $n = 2k$ for some integer $k \geq 2$.

$$h = \frac{17(2k)^2+15(2k)+2}{2(2k)+2} = \frac{68k^2+30k+2}{4k+2}$$
$$= \frac{(68k^2+34k)-(4k+2)+4}{4k+2} = 17k-1+\frac{4}{4k+2}.$$

Since $k \geq 2$, $4k+2 \geq 10$, thus $\frac{4}{4k+2}$ is not an integer, a contradiction.

Case 2 : $n = 2k-1$ for some integer $k \geq 3$.

$$h = \frac{17(2k-1)^2 + 15(2k-1) + 2}{2(2k-1)+2} = \frac{17(4k^2 - 4k + 1) + 30k - 15 + 2}{4k - 2 + 2}$$
$$= \frac{68k^2 - 68k + 17 + 30k - 15 + 2}{4k} = \frac{68k^2 - 38k + 4}{4k}$$
$$= \frac{68k^2}{4k} - \frac{36k}{4k} - \frac{2k-4}{4k} = 17k - 9 - \frac{2k-4}{4k}.$$

Since $k \geq 3$, $4k > 2k-4 > 0$, thus $\frac{2k-4}{4k}$ is not an integer, a contradiction. Hence G is not a super vertex-magic graph. □

Definition 4.6. A *ladder graph* with $2n$ vertices, $n \geq 3$ is an integer, denoted by L_n is defined as $P_2 \times P_n$.

Example 4.7. The ladder graph L_4 has 8 vertices and 10 edges.

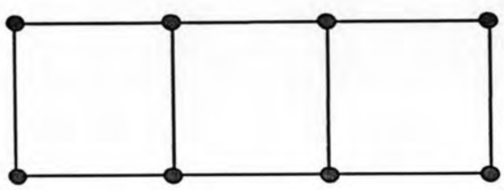

Figure 4.3 : The ladder graph L_4

Note that a ladder graph L_n has $v = 2n$ and $e = 3n-2$.

Theorem 4.8. ([6]) *If G is a ladder graph* L_n *where* $n \geq 3$ *is an integer, then G is not a super vertex-magic graph.*

Proof. Let G be a ladder graph L_n and $n \geq 3$ is an integer.
Assume that G is a super vertex-magic graph with the magic constant *h*.
Since $v = 2n$ and $e = 3n-2$, by Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(5n-2)(5n-1)}{2n} - \frac{2n+1}{2}$$
$$= \frac{25n^2-15n+2}{2n} - \frac{2n^2+n}{2n} = \frac{23n^2-16n+2}{2n}.$$

Case 1 : $n = 2k$ for some integer $k \geq 2$.

$$h = \frac{23(2k)^2-16(2k)+2}{2(2k)} = \frac{92k^2-32k+2}{4k}$$
$$= \frac{92k^2}{4k} - \frac{32k}{4k} + \frac{2}{4k} = 46k-16+\frac{1}{2k}.$$

Since $k \geq 2$, $\frac{1}{2k}$ is not an integer, a contradiction.

Case 2 : $n = 2k-1$ for some integer $k \geq 2$.

$$h = \frac{23(2k-1)^2-16(2k-1)+2}{2(2k-1)} = \frac{23(4k^2-4k+1)-32k+16+2}{4k-2}$$
$$= \frac{92k^2-92k+23-32k+16+2}{4k-2} = \frac{92k^2-124k+41}{4k-2}$$
$$= \frac{(92k^2-46k)-(76k-38)-(2k-3)}{4k-2} = 23k-19-\frac{2k-3}{4k-2}.$$

Since $k \geq 2$, $4k-2 > 2k-3 > 0$, thus $\frac{2k-3}{4k-2}$ is not an integer, a contradiction.

Hence G is not a super vertex-magic graph. □

Definition 4.9. A *fan graph* with $n+1$ vertices, $n \geq 3$ is an integer, denoted by F_n is defined as $K_1 \vee P_n$.

Example 4.10. The fan graph F_4 has 5 vertices and 7 edges.

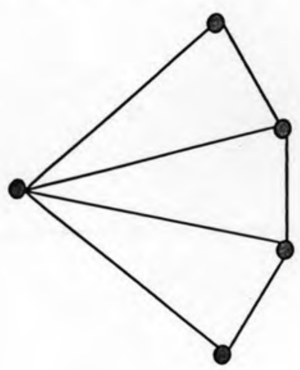

Figure 4.4 : The fan graph F_4

Note that a fan graph F_n has $v=n+1$ and $e=2n-1$.

Theorem 4.11. ([6]) *If G is a fan graph F_n where $n\geq 3$ is an integer, then G is not a super vertex-magic graph.*

Proof. Let G be a fan graph F_n and $n\geq 3$ is an integer.

Assume that G is a super vertex-magic graph with the magic constant h..

Since $v=n+1$ and $e=2n-1$, by Theorem 1.2.1,

$$h=\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(3n)(3n+1)}{n+1}-\frac{n+2}{2}=\frac{9n^2+3n}{n+1}-\frac{n+2}{2}$$
$$=\frac{2(9n^2+3n)}{2(n+1)}-\frac{(n+2)(n+1)}{2n+2}=\frac{18n^2+6n}{2n+2}-\frac{n^2+3n+2}{2n+2}=\frac{17n^2+3n-2}{2n+2}.$$

Case 1 : $n=3$ $(G\cong F_3)$.

When $n=3$, $h=20$. We will show that F_3 is not a super vertex-magic graph.

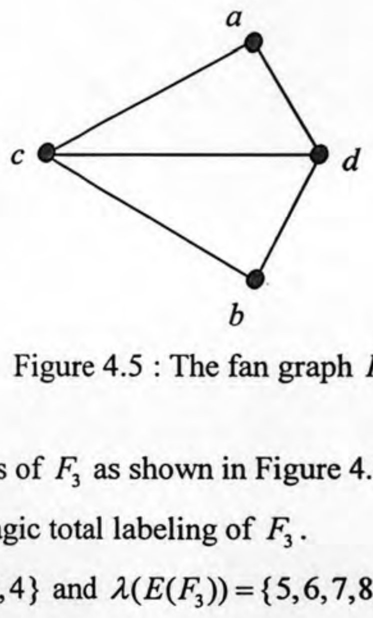

Figure 4.5 : The fan graph F_3

Let *a*, *b*, *c*, and *d* be vertices of F_3 as shown in Figure 4.5.

Let λ be a super vertex-magic total labeling of F_3.

We have $\lambda(V(F_3))=\{1,2,3,4\}$ and $\lambda(E(F_3))=\{5,6,7,8,9\}$.

Since $h=20$, the sum at each vertex must be equal to 20.

Now we will consider the value $\lambda(a)$.

If $\lambda(a)=1$, we have $\lambda(a)+\lambda(ac)+\lambda(ad)\leq 18$, a contradiction.

If $\lambda(a)=2$, we have $\lambda(a)+\lambda(ac)+\lambda(ad)=19$, a contradiction.

Therefore $\lambda(a)=3$ or $\lambda(a)=4$. Similarly, we have $\lambda(b)=3$ or $\lambda(b)=4$.

Case 1.1 : $\lambda(a)=3$. We have $\lambda(a)+\lambda(ac)+\lambda(ad)=20$.

Thus $\lambda(ac)=8$ and $\lambda(ad)=9$, or $\lambda(ac)=9$ and $\lambda(ad)=8$.

Now $\lambda(bc),\lambda(bd)\in\{5,6,7\}$ and $\lambda(b)=4$, thus $\lambda(b)+\lambda(bc)+\lambda(bd)\leq 17$, a contradiction.

Case 1.2 : $\lambda(a)=4$. We have $\lambda(a)+\lambda(ac)+\lambda(ad)=20$.

Thus $\lambda(ac)=7$ and $\lambda(ad)=9$, or $\lambda(ac)=9$ and $\lambda(ad)=7$.

Now $\lambda(bc),\lambda(bd)\in\{5,6,8\}$ and $\lambda(b)=4$, we have $\lambda(b)+\lambda(bc)+\lambda(bd)\leq 18$, a contradiction.

Thus vertex a cannot be labeled with any element of $\{1,2,3,4\}$.

Hence F_3 is not a super vertex-magic graph.

Case 2 : $n=11$ $(G\cong F_{11})$.

When $n=11$, $h=87$. We will show that F_{11} is not a super vertex-magic graph.

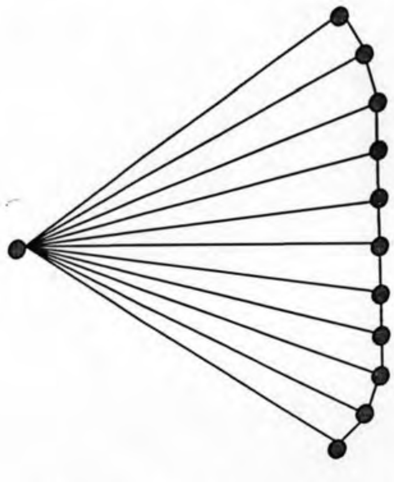

Figure 4.6 : The fan graph F_{11}

Since $2e=22<37=\sqrt{1369}=\sqrt{10(12)^2-6(12)+1}=\sqrt{10v^2-6v+1}$, by Theorem 1.2.7, the maximum degree of F_{11} is at most 6, a contradiction.

Case 3 : $n=2k$ for some integer $k\geq 2$.

$$h=\frac{17(2k)^2+3(2k)-2}{2(2k)+2}=\frac{68k^2+6k-2}{4k+2}$$

$$=\frac{(68k^2+34k)-(28k+14)+12}{4k+2}=17k-7+\frac{12}{4k+2}.$$

Since $k \geq 2$, case $k=2$, $4k+2=10$, we have $\frac{12}{4k+2}=\frac{12}{10}$.

case $k \geq 3$, $4k+2>12$, $\frac{12}{4k+2}$ is not an integer, a contradiction.

Case 4 : $n=2k-1$ for some integer $k \geq 3$ and $k \neq 6$.

$$h=\frac{17(2k-1)^2+3(2k-1)-2}{2(2k-1)+2}=\frac{17(4k^2-4k+1)+6k-3-2}{4k-2+2}$$

$$=\frac{68k^2-68k+17+6k-3-2}{4k}=\frac{68k^2-62k+12}{4k}$$

$$=\frac{68k^2-60k-(2k-12)}{4k}=17k-15-\frac{2k-12}{4k}=17k-15-\frac{k-6}{2k}.$$

Since $k \geq 3$, $2k > k-6$.

Also $k \neq 6$, $\frac{k-6}{2k}$ is not an integer, a contradiction.

Hence G is not a super vertex-magic graph. □

Definition 4.12. A *friendship graph* with $2n+1$ vertices, $n \geq 3$ is an integer, denoted by f_n is the graph obtained by taking n copies of the cycle C_3 with a vertex in common.

Example 4.13. The friendship graph f_4 has 9 vertices and 12 edges.

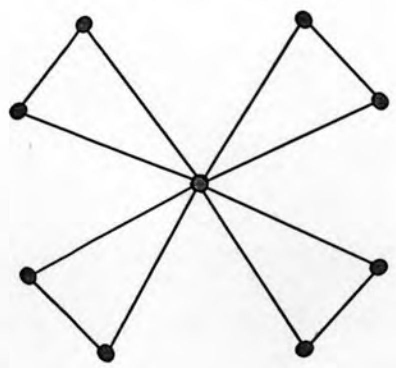

Figure 4.7 : The friendship graph f_4

Note that a friendship graph f_n has $v=2n+1$ and $e=3n$.

Theorem 4.14. ([6]) *If G is a friendship graph f_n where $n \geq 3$ is an integer, then G is not a super vertex-magic graph.*

Proof. Let G be a friendship graph f_n and $n \geq 3$ is an integer.
Assume that G is a super vertex-magic graph with the magic constant h.
Since $v = 2n+1$ and $e = 3n$, by Theorem 1.2.1,

$$\begin{aligned} h &= \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(5n+1)(5n+2)}{2n+1} - \frac{2n+2}{2} \\ &= \frac{25n^2+15n+2}{2n+1} - \frac{2n+2}{2} = \frac{2(25n^2+15n+2)}{2(2n+1)} - \frac{(2n+2)(2n+1)}{2(2n+1)} \\ &= \frac{50n^2+30n+4}{4n+2} - \frac{4n^2+6n+2}{4n+2} = \frac{46n^2+24n+2}{4n+2} \end{aligned}$$

Case 1 : $n = 2k$ for some integer $k \geq 2$.

$$\begin{aligned} h &= \frac{46(2k)^2+24(2k)+2}{4(2k)+2} = \frac{184k^2+48k+2}{8k+2} \\ &= \frac{(184k^2+46k)+(2k+2)}{8k+2} = 23k + \frac{2k+2}{8k+2}. \end{aligned}$$

Since $k \geq 2$, $8k+2 > 2k+2 > 0$, thus $\frac{2k+2}{8k+2}$ is not an integer, a contradiction.

Case 2 : $n = 2k-1$ for some integer $k \geq 2$.

$$\begin{aligned} h &= \frac{46(2k-1)^2+24(2k-1)+2}{4(2k-1)+2} = \frac{46(4k^2-4k+1)+48k-24+2}{8k-4+2} \\ &= \frac{184k^2-184k+46+48k-24+2}{8k-2} = \frac{184k^2-136k+24}{8k-2} \\ &= \frac{184k^2-136k+24}{8k-2} = \frac{(184k^2-46k)-(88k-22)-(2k-2)}{8k-2} \\ &= 23k-11-\frac{2k-2}{8k-2}. \end{aligned}$$

Since $k \geq 2$, $8k-2 > 2k-2 > 0$, thus $\frac{2k-2}{8k-2}$ is not an integer, a contradiction.

Hence G is not a super vertex-magic graph. □

Definition 4.15. A *prism graph* with $2n$ vertices, $n \geq 3$ is an integer, denoted by Pr_n is defined as $P_2 \times C_n$.

Example 4.16. The prism graph Pr_3 has 6 vertices and 9 edges.

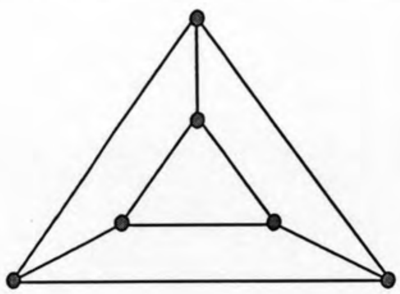

Figure 4.8 : The prism graph Pr_3

Note that a prism graph Pr_n has $v = 2n$ and $e = 3n$.

Theorem 4.17. *If G is a prism graph Pr_n where $n \geq 3$ and n is odd, then G is not a super vertex-magic graph.*

Proof. Let G be a prism graph Pr_n, odd $n \geq 3$.
Assume that G is a super vertex-magic graph with the magic constant h.
Since $v = 2n$ and $e = 3n$, by Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(5n)(5n+1)}{2n} - \frac{2n+1}{2}$$
$$= \frac{25n+5}{2} - \frac{2n+1}{2} = \frac{23n+4}{2}.$$

Since h is an integer, we have $\frac{23n+4}{2}$ is an integer.
Thus n is even, a contradiction.
Hence G is not a super vertex-magic graph. □

Definition 4.18. A *book graph* with $2n$ vertices, $n \geq 3$ is an integer, denoted by B_n is defined as $P_2 \times K_{1,n}$.

Example 4.19. The book graph B_3 has 8 vertices and 9 edges.

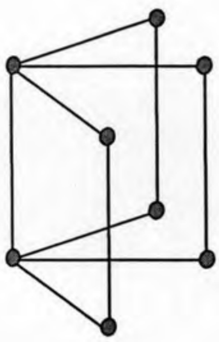

Figure 4.9 : The book graph B_3

Note that a book graph B_n has $v = 2n$ and $e = 3n - 3$.

Theorem 4.20. *If G is a book graph B_n where $n \geq 3$ is an integer, then G is not a super vertex-magic graph.*

Proof. Let *G* be a book graph B_n and $n \geq 3$.
Assume that *G* is a super vertex-magic graph with the magic constant *h*..
Since $v = 2n$ and $e = 3n - 3$, by Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(5n-3)(5n-2)}{2n} - \frac{2n+1}{2}$$

$$= \frac{25n^2 - 25n + 6}{2n} - \frac{2n+1}{2} = \frac{25n^2 - 25n + 6}{2n} - \frac{2n^2 + n}{2n} = \frac{23n^2 - 26n + 6}{2n}.$$

Case 1 : $n = 2k$ for some integer $k \geq 2$.

$$h = \frac{23(2k)^2 - 26(2k) + 6}{2(2k)} = \frac{92k^2 + 52k + 6}{4k}$$

$$= \frac{92k^2}{4k} + \frac{52k}{4k} + \frac{6}{4k} = 23k + 13 + \frac{6}{4k}.$$

Since $k \geq 2$, $\frac{6}{4k}$ is not an integer, a contradiction.

Case 2 : $n = 2k - 1$ for some integer $k \geq 2$.

$$h = \frac{23(2k-1)^2 - 26(2k-1) + 6}{2(2k-1)} = \frac{23(4k^2 - 4k + 1) - 52k + 26 + 6}{4k - 2}$$

$$= \frac{23(4k^2-4k+1)-52k+26+6}{4k-2} = \frac{92k^2-92k+23-52k+26+6}{4k-2}$$
$$= \frac{92k^2-144k+55}{4k-2} = \frac{(92k^2-46k)-(96k-48)-(2k-7)}{4k-2}$$
$$= 23k-24-\frac{2k-7}{4k-2}.$$

Since $k \geq 2$, $4k-2 > 2k-7$.

Thus $\frac{2k-7}{4k-2}$ is not an integer, a contradiction.

Hence G is not a super vertex-magic graph. □

Definition 4.21. Let $n \geq 6$ and n be even. A *crown graph* with n vertices, denoted by Cr_n, is a graph whose set of vertices is $V = \{v_0, v_1, ..., v_{n-1}\}$ and whose set of edges is $E = \{v_0 v_2, v_2 v_4, ..., v_{n-2} v_0\}$.

Example 4.22. The crown graph Cr_8 has 8 vertices and 12 edges.

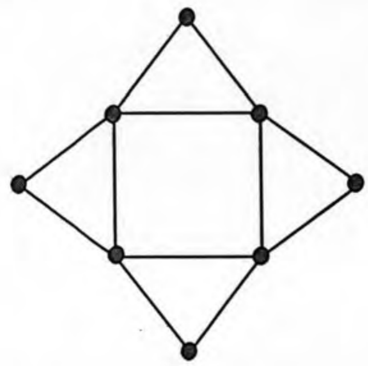

Figure 4.10 : The crown graph Cr_8

Note that a crown graph Cr_n has $v = n$ and $e = \frac{3}{2}n$.

Theorem 4.23. *If G is a crown graph Cr_n where $n \geq 6$ and n is even, then G is not a super vertex-magic graph.*

Proof. Let G be a crown graph Cr_n, even $n \geq 6$.

Assume that G is a super vertex-magic graph with the magic constant h..

Since $v = n$ and $e = \frac{3}{2}n$, by Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(\frac{5}{2}n)(\frac{5}{2}n+1)}{n} - \frac{n+1}{2}$$
$$= \frac{25}{4}n + \frac{5}{2} - \frac{n+1}{2} = \frac{23n+8}{4}.$$

Since $n \geq 6$ and n is even, thus $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$.

Case 1 : $n = 4k+2$ for some integer $k \geq 1$.

$$h = \frac{23(4k+2)+8}{4} = \frac{92k+54}{4} = 23k + \frac{54}{4}.$$

Thus h is not an integer, a contradiction.

Case 2 : $n = 4k$ for some integer $k \geq 2$.

$$h = \frac{23(4k)+8}{4} = 23k+2.$$

Thus h is an integer.

Now $v = n = 4k$ and $e = \frac{3}{2}n = \frac{3}{2}(4k) = 6k$.

The degrees of vertices of G are either 2 or 4.

Let $V_1(G)$ be the set of vertices of G with degree 2 and $V_2(G)$ be the set of vertices of G with degree 4. We have $|V_1(G)| = 2k$ and $|V_2(G)| = 2k$.

Let λ be a super vertex-magic total labeling of G and $a \in V_1(G)$.

Assume that b and c are 2 edges incident with vertex a.

We have $\lambda(a) \in \{1, 2, ..., 4k\}$ and $\lambda(b), \lambda(c) \in \{4k+1, 4k+2, ..., 10k\}$.

We will consider the value $\lambda(a)$.

Let m be any element in $\{1, 2, ..., 2k+1\}$ and $\lambda(a) = m$.

Since $m \leq 2k+1$, $\lambda(a) + \lambda(b) + \lambda(c) = m + \lambda(b) + \lambda(c) \leq 2k+1+\lambda(b)+\lambda(c)$

Also $h = 23k+2 = \lambda(a)+\lambda(b)+\lambda(c)$. Thus $23k+2 \leq 2k+1+\lambda(b)+\lambda(c)$ or $21k+1 \leq \lambda(b)+\lambda(c)$.

Since $\lambda(b)+\lambda(c) \leq (10k-1)+10k = 20k-1$, thus $21k+1 \leq 20k-1$ or $k \leq 2$, a contradiction.

Thus vertex a cannot be labeled with any element of $\{1, 2, ..., 2k+1\}$.

Therefore $\lambda(a) \in \{2k+2, 2k+3, ..., 4k\}$.

Thus $|\{2k+2, 2k+3, ..., 4k\}| = 2k-1$ and $|V_1(G)| = 2k$, a contradiction.

Hence G is not a super vertex-magic graph. □