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WORST-CASE PERFORMANCE ANALYSIS AND CONTROLLER SYNTHESIS FOR LUR'E SYSTEMS WITH TIME DELAYS AND PARAMETRIC UNCERTAINTIES

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ฐาปนา นามประดิษฐ์: การวิเคราะห์สมรรถนะกรณีเลวสุดและการสังเคราะห์ตัวควบคุมสำหรับระบบลูเรที่มีการประวิงเวลาและความไม่แน่นอนเชิงพารามิเตอร์.(WORST-CASE PERFORMANCE ANALYSIS AND CONTROLLER SYNTHESIS FOR LUR'E SYSTEMS WITH TIME DELAYS AND PARAMETRIC UNCERTAINTIES) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ศ. ดร.เดวิด บรรเจิดพงศ์ชัย, 92 หน้า.

เป็นที่ทราบกันดีว่า ปัญหาเสถียรภาพเป็นปัญหาพื้นฐานที่สุดปัญหาหนึ่ง ในการวิเคราะห์และออกแบบระบบควบคุม นับตั้งแต่มีการนำเสนอวิธีการแยกโดดเดี่ยวองค์ประกอบไม่เชิงเส้นออกจากระบบเชิงเส้นในรูปแบบของปัญหาลูเร มีงานวิจัยจำนวนมากศึกษาเกี่ยวกับเสถียรภาพสัมบูรณ์ ซึ่งเป็นปัญหาการลู่เข้าในวงกว้างเชิงเสถียรภาพสำหรับระบบลูเร และได้มีการนำไปใช้ในหลายบริบท วิทยานิพนธ์นี้นำเสนอการวิเคราะห์และสังเคราะห์ระบบลูเรที่มีการประวิงเวลาและความไม่แน่นอนแบบจำกัดขอบเขตนอร์ม ด้วยวิธีอสมการเมทริกซ์เชิงเส้น ในส่วนของการวิเคราะห์ครอบคลุมถึงปัญหาการวิเคราะห์เสถียรภาพสัมบูรณ์ และการวิเคราะห์สมรรถนะ $\mathcal{H}_\infty$ กรณีเลวสุด เราใช้วิธีตรงของเลียปูโนฟร่วมกับฟังก์ชันนัลเลียปูโนฟ-คราซอฟสกีที่มีการแบ่งช่วงการประวิงเวลาออกเป็นส่วนๆ ที่เท่ากัน และผนวกอินทิกรัลของความไม่เป็นเชิงเส้นเพื่อนำไปใช้ตรวจสอบเสถียรภาพ และคำนวณหาขอบเขตบนของสมรรถนะ $\mathcal{H}_\infty$ กรณีเลวสุด การวิเคราะห์ดังกล่าว อยู่ในรูปแบบปัญหาการหาค่าเหมาะที่สุด ภายใต้เงื่อนไขอสมการเมทริกซ์เชิงเส้น ซึ่งหาคำตอบได้อย่างมีประสิทธิภาพ เราใช้วิธีแบ่งครึ่งร่วมกันกับการหาค่าเหมาะที่สุดภายใต้เงื่อนไขอสมการเมทริกซ์เชิงเส้น เพื่อหาค่าเวลาประวิงสูงสุดที่ระบบยังคงมีเสถียรภาพ ในส่วนของการสังเคราะห์ ครอบคลุมถึงการออกแบบการรักษาเสถียรภาพ ด้วยการป้อนกลับสถานะและปัญหาการควบคุม $\mathcal{H}_\infty$ โดยการขยายผลจากการวิเคราะห์ ปัญหาการสังเคราะห์ตัวควบคุมทั้งสองอยู่ในรูปแบบของการหาค่าเหมาะที่สุดเชิงไม่คอนเวกซ์ เงื่อนไขการออกแบบเป็นอสมการเมทริกซ์เชิงเส้นคู่ เราพัฒนาขั้นตอนการหาค่าเหมาะที่สุดเชิงระบบพิกัด ซึ่งใช้หลักการสลับแก้ปัญหาการหาค่าเหมาะที่สุด ภายใต้เงื่อนไขอสมการเมทริกซ์เชิงเส้นสองปัญหา เพื่อหาตัวควบคุมป้อนกลับสถานะที่เป็นคำตอบ การทดลองเชิงตัวเลขและการออกแบบการควบคุมสำหรับเครื่องปฏิกรณ์แบบถังกวนที่มีการป้อนกลับเพื่อรีไซเคิล แสดงให้เห็นว่าการวิเคราะห์และสังเคราะห์ที่นำเสนอ ปรับปรุงผลลัพธ์ได้อย่างมีนัยยะสำคัญ เมื่อเปรียบเทียบกับผลตอบสนองของระบบวงเปิด

ภาควิชา ......วิศวกรรมไฟฟ้า...... ลายมือชื่อนิสิต ....
สาขาวิชา ......วิศวกรรมไฟฟ้า...... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....
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THAPANA NAMPRADIT : WORST-CASE PERFORMANCE ANALYSIS AND CONTROLLER SYNTHESIS FOR LUR'E SYSTEMS WITH TIME DELAYS AND PARAMETRIC UNCERTAINTIES. ADVISOR : PROF. DAVID BANJERDPONGCHAI, Ph.D., 92 pp.

It is well known that the stability is the most fundamental problem in analysis and design of control systems. Once the concept of the *nonlinear isolation method* is introduced in terms of Lur'e problem, the *absolute stability* which is the global asymptotically stability of Lur'e systems have been extensively studied and utilized in a number of different contexts. This dissertation presents the analysis and synthesis of Lur'e systems with uncertain time-invariant delays and norm-bounded uncertainties, using linear matrix inequalities (LMIs). The analysis part is devoted to the absolute stability and the worst-case $\mathcal{H}_\infty$ performance analysis problems. The direct Lyapunov method with a delay-partitioning Lyapunov-Krasovskii functional containing the integral of nonlinearities is applied for determining stability, and calculating an upper bound of the worst-case $\mathcal{H}_\infty$ performance. Both analyses are formulated as optimization problems involving LMIs which can be efficiently solved. A *bisection method associated with LMI optimization* is introduced to determine the maximum allowable time delay. The synthesis part involves the state-feedback stabilization and $\mathcal{H}_\infty$ control problems. Extending the proposed analyses, the controller design problems are first formulated as non-convex optimization problem involving bilinear matrix inequalities (BMIs). We develop algorithms based on *coordinate optimization*, which alternate between two LMI optimizations, to solve for the robust control problems. Numerical results from benchmark problems including the continuous stirred tank reactor (CSTR) with recycle stream show that the proposed analysis and synthesis give significant improvement on the results comparing to the open-loop response.

Department : Electrical Engineering	Student's Signature ....
Field of Study : Electrical Engineering	Advisor's Signature ....
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CONTENTS

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

LIST OF SYMBOLS

$\mathbf{R}$	The set of real numbers.
$\mathbf{R}_+$	The set of nonnegative numbers.
$\mathbf{R}^m$	The set of real m-vectors.
$\mathbf{R}^{m\times n}$	The vector space of $m \times n$ real matrices.
X^T	The transpose of a matrix $X \in \mathbf{R}^{m\times n}$.
I_m	The identity matrix of size m or the identity of linear operator. We omit the subscript when m can be determined from context.
X^{-1}	The inverse of X or the inverse of linear operator X, *i.e.*, $XX^{-1} = I$.
$\mathrm{diag}(X_1, \ldots, X_N)$	The block-diagonal matrix with $X_1, \ldots, X_N$ along the diagonal, *i.e.*, $\begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & X_N \end{bmatrix}$,
$X > 0\ (X \geq 0)$	The symmetric X is positive definite (semidefinite), *i.e.*, $X = X^T$ and $z^T X z > 0$ $(z^T X z \geq 0)$ for all $z \in \mathbf{R}^n$.
$X > Y\ (X \geq Y)$	The symmetric X and Y satisfy $X - Y > 0$ $(X - Y \geq 0)$.
$\mathcal{L}_2^n$	The space of square-integrable signal in $\mathbf{R}^n$. We omit the superscript when n can be determined from context.
$\|\cdot\|_2$	The $\mathcal{L}_2$ norm of a signal.
$:=$	"is defined as"
$\in$	"belongs to"
$\forall$	"for all"
$\blacksquare$	The end of the proof.

LIST OF ACRONYMS

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
SDP	Semidefinite Programming
LTI	Linear Time-invariant
TDS	Time-delay Systems
LSTD	Lur'e Systems with Time Delays
LKF	Lyapunov-Krasovskii functional
CSTR	Continuous Stirred Tank Reactor

CHAPTER I
INTRODUCTION

1.1 Motivation

Lur'e systems are nonlinear systems described by linear dynamic systems with feedback through sector-bounded nonlinearities. The nonlinearities with sector-bounded constraints represent the common input/output characteristics of physical devices such as saturating actuators or physical behaviors such as mechanical friction with dead zone. The original study of Lur'e systems has focused on the *absolute stability problem* (Lur'e and Postnikov, 1944), *i.e.*, justify whether the system is globally asymptotically stable without having much information about the nonlinear characteristics. Besides nonlinearity effects, time delays are frequently encountered in engineering applications such as chemical processes, communication networks and manufacturing systems (Gu et al., 2003; Zhong, 2006; Normey-Rico and Camacho, 2007). In most cases, time delays are detrimental to stability and performance of closed-loop systems. The existence of both nonlinearity and time delay may degrade the stability as well as performance of the control systems. Therefore, numerous problems on Lur'e systems with time delays (LSTD) have been widely investigated. The state-space equation of LSTD is described as follows.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + B_p p(t), \\ q(t) &= C_q x(t), \\ p(t) &= \phi(q(t)), \end{aligned} \tag{1.1}$$

where the initial condition is specified by $x(t) = \varphi(t), \forall t \in [-h, 0]$, $h \in \mathbf{R}_+$ is a time-invariant time delay in the state. The vector $x(t) \in \mathbf{R}^n$ is the state, $q(t)$ and $p(t) \in \mathbf{R}^{n_p}$ are the input and output of ϕ, which denotes a vector mapping of sector-bounded nonlinearities.

1.2 Previous Research

Absolute stability criteria of LSTD were proposed by Popov and Halanay (1962). It is well known that an effective method for determining stability of the system with time delays is the direct Lyapunov method. However, the method using Lyapunov function is applicable to a limited number of time-delay systems. Thereafter, Krasovskii (1963) was the first to adopt Lyapunov functional, instead of Lyapunov function, to the stability problem of linear time-delay systems.

The Lyapunov-Krasovskii functional (LKF) became a foundation for subsequent stability analysis tests for both time-delay systems and LSTD. The first delay-independent absolute stability criteria for LSTD (Somolinos, 1977; Verriest and Aggoune, 1998; Gan and Ge, 2001) were proposed in the algebraic form, but no analytical or numerical methods were given to determine the solution. Later, Bliman (2001), He and Wu (2003) established new delay-independent criteria. The stability conditions are expressed in terms of LMIs which are sufficient conditions for stability test regardless of time delays. In general, delay-independent criteria are conservative, because no conclusion can be made for the systems whose stability depends on the duration of time delay. Subsequent studies aim to reduce of conservatism of stability criteria and use maximum allowable time delay ($\bar{h}_{\max}$) as a measure of conservatism. The meaning of $\bar{h}_{\max}$ is the maximum time delay that ensures stability of LSTD. The study of delay-dependent criteria has focused on increasing $\bar{h}_{\max}$. There are extensive studies to develop delay-dependent criteria for (1.1) where the LKF has the following integral term.

$$\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)R\dot{x}(\xi)d\xi d\theta \tag{1.2}$$

Note that the time derivative of (1.2) gives the result as follows.

$$-\int_{t-h}^{t} \dot{x}^T(\xi)R\dot{x}(\xi)d\xi. \tag{1.3}$$

Nevertheless, the stability test could not be easily formulated in terms of LMIs.

In several stability tests of LTDS (Fridman, 2001; Wu et al., 2010), model transformation such as the Leibniz-Newton formula (Park, 1999) and the descriptor transformation approach (Fridman, 2001) are used to change point-wise delay systems to distributed delay systems. The model transformation aims to cancel the term (1.3) in the formulation. However, many model transformations introduce additional dynamics and require extra assumptions (Gu and Niculescu, 2000, 2001) which make the stability tests conservative. In addition, model transformation always produces the cross product terms between the system state and delayed state. The absolute stability criteria for the transformed systems usually employ bounding technique (Park, 1999; Moon et al., 2001; Yu et al., 2003) on the cross product terms. As a result, the bound of the cross product terms can lead to conservatism. Subsequently, He et al. (2005) proposed free weighting matrices (FWM) approach. Matrix variables are used to express the relationship between the terms in the Leibniz-Newton formula as a null summing term. It includes to the derivative of the LKF in order to eliminate the integral term (1.3). Although the FWM approach does not require the bounding technique, it introduces new slack variables apart from matrix variables in the LKF.

With regard to the computational cost, Gu (2001) and Han (2005) employed Jensen inequality to bound the integral term (1.3) as follows.

$$-\int_{t-h}^{t} \dot{x}(\xi) R \dot{x}(\xi) d\xi \leq -\left(\int_{t-h}^{t} \dot{x}(\xi) d\xi\right)^T \left(\frac{R}{h}\right) \left(\int_{t-h}^{t} \dot{x}(\xi) d\xi\right).$$

The stability condition is easily formulated in terms of LMIs without using model transformation and the bounding technique. The criteria employing Jensen inequality give the results as conservative as the criteria based on the FWM do (Zhang and Yu, 2008), but they use much less number of variables. Motivated by Han (2005), Xu and Feng improved stability criterion by applying Jensen inequality to augmented LKF (Xu and Feng, 2007). Their approach yields a less conservative result, but requires a large number of decision variables.

To further improve the delay-dependent criteria, let us consider the Lyapunov functional with a parameter varying Lyapunov matrix, *i.e.*, $R(\theta)$, as follows.

$$\int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(\xi) R(\theta) \dot{x}(\xi) d\xi d\theta. \tag{1.4}$$

Inspired by the piecewise linear discretization scheme proposed by Gu (1997) and its refinement (Gu, 2001; Gu et al., 2003), several researchers proposed novel methods to analyze the stability of systems involving time delays. They include discretization scheme of the delay (Gouaisbaut and Peaucelle, 2006), N-segmentation of delay length (Park et al., 2008; Wu et al., 2009), delay-decomposition approach (Han, 2008), and delay-dividing approach (Qiu et al., 2010). The main principle of these methods is to divide an interval $[-h, 0]$ into N partitions, *i.e.*, $[-h, -h+r]$, $[-h+r, -h+2r]$, . . ., $[-r, 0]$, where $r = h/N$, and to separately apply the LKF terms involving delay to each subinterval. For example, an integral (1.2) is divided into the following.

$$\begin{aligned} &\int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}^T(\xi) R_1 \dot{x}(\xi) d\xi d\theta + \int_{-2r}^{-r} \int_{t+\theta}^{t} \dot{x}^T(\xi) R_2 \dot{x}(\xi) d\xi d\theta + \dots \\ &\qquad + \int_{-h}^{-h+r} \int_{t+\theta}^{t} \dot{x}^T(\xi) R_N \dot{x}(\xi) d\xi d\theta, \end{aligned} \tag{1.5}$$

where $R_1, \dots, R_N$ are free matrix variables. Then, each integral term can be bounded by using Jensen inequality. As a result, the criterion uses more free variables and possibly leads to a less conservative result. Recently, in Briat (2011), it is shown that Jensen inequality is conservative if the integration domain is large, *i.e.*, the interval $[-h, 0]$ is large. By dividing the interval of integration, the integration domain can be made smaller so that the conservatism of Jensen inequality can be reduced. Moreover, in Briat (2011), it is proved that the Jensen's gap can be made arbitrarily small provided that the order of uniform fragmentation is chosen sufficiently large. Although conservatism of the criterion is reduced as N is increased, it requires a large number of Lyapunov matrix variables. Thus, it is logical to trade off between conservatism and computational complexity.

Among the criteria based on delay partitioning approach, Wu et al. (2009) and Qiu and Zhang (2011) proposed the absolute stability criteria for LSTD with time-invariant delay. Numerical examples show that these criteria yields much less conservative results comparing to the criterion in Han (2005). Nevertheless, there is a room for improvement because the LKF used in Wu et al. (2009) does not have the integral of nonlinearities. On the contrary, the LKF based on Qiu and Zhang (2011) utilize this integral, but their delay interval is only divided into two unequal subintervals, namely, $[-h, -h/3]$ and $[-h/3, 0]$. Recently, Nampradit and Banjerdpongchai (2014) fulfilled this gap by developing an improved absolute stability criterion for the systems with time-invariant delay. The criterion combines the delay partitioning approach (N equidistant fragments) with utilizing integral terms involving sector-bounded nonlinearities in the Lyapunov functional. The numerical results confirm that the criterion in Nampradit and Banjerdpongchai (2014) provides substantial improvement comparing to those in Wu et al. (2009) and Qiu and Zhang (2011) especially when the sector bound is comparatively large.

For the $\mathcal{H}_\infty$ control design for LSTD, there have been a few research works. The work by Lu et al. (2003) gives a delay-dependent approach to design the output feedback $\mathcal{H}_\infty$ control of LSTD. The descriptor model transformation (Fridman, 2001) and the bounding technique are used to develop the design formulation in terms of LMI. Although the descriptor transformation is equivalent to the original system, it is shown in Park (1999) that the chosen bounding technique is quite conservative. From the numerical results in Nampradit and Banjerdpongchai (2014), the criteria based on the delay-partitioning approach provide much less conservative results than that obtained from Yu et al. (2003), which utilizes a tight bounding technique. Thus the design in Lu et al. (2003) tends to give more conservative result comparing to the design using delay-partitioning techniques. In Wang and Zuo (2005), both state-feedback and output feedback are proposed to guarantee the desired level of worst-case $\mathcal{H}_\infty$ performance. However, the information of delay interval is not utilized in their control design. In particular, they consider the simple form of LKF, *i.e.*,

$$V(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\xi)Qx(\xi)\, d\xi, \tag{1.6}$$

where $P > 0$ and $Q > 0$. As a result, the design in Wang and Zuo (2005) tends to be conservative, and is limited to some applications. Therefore, it is interesting to fulfill the gaps of research work by extending the delay-partitioning approach in Nampradit and Banjerdpongchai (2014) to the state-feedback stabilization and state-feedback $\mathcal{H}_\infty$ control, which should reduce the conservatism comparing to existing design methods.

1.3 Research Objectives

The first objective of this dissertation is to improve absolute stability criteria of Lur'e systems with uncertain time-invariant delays. This objective is achieved by combining a delay partitioning approach (Gu, 1997, 2001; Gu et al., 2003; Gouaisbaut and Peaucelle, 2006; Han, 2008; Park et al., 2008; Wu et al., 2009; Qiu et al., 2010) with utilizing integral terms involving sector-bounded nonlinearities in the Lyapunov functional. The absolute stability criterion is formulated as an LMI feasibility problem using the Jensen inequality. The improvement of the criterion can be measured using the maximum allowable time delay, which is the maximum time delay that ensures absolute stability of the LSTD. The bisection method is applied to efficiently determine the maximum allowable time delay for a given system. In addition, we extend the derived criterion for the system with a certain form of norm-bounded uncertainty by eliminating some uncertain parameters (Xie, 1996).

The second objective of this dissertation is to develop an effective method to analyze the worst-case $\mathcal{H}_\infty$ performance of LSTD. Fortunately, combining the absolute stability analysis with robust performance bounds on the total output energy of the system results in worst-case performance analysis criteria (Boyd et al., 1994). The worst-case $\mathcal{H}_\infty$ performance for both nominal and uncertain systems are developed. The problem of calculating an upper bound of the worst-case performance for such systems can be cast as a linear objective minimization problem over LMIs. It should be emphasized that there is no work on how to compute the worst-case $\mathcal{H}_\infty$ performance for LSTD.

The third objective of this dissertation is to establish robust state-feedback stabilization for LSTD where time delays are uncertain and time-invariant. The feedback stabilization design is developed by extending the absolute stability analysis. In particular, applying the absolute stability criterion to the closed-loop system, when the feedback gain is considered as the design parameter, results in the BMI problems. The difficulties occur when both the analysis variables and the feedback gain are optimized simultaneously. An LMI-based technique is used to solve this problem. A local solution of the BMI feasibility problem is determined by alternating between two LMI feasibility optimizations.

The last objective of this dissertation is to develop a robust state-feedback $\mathcal{H}_\infty$ control design for LSTD. The $\mathcal{H}_\infty$ control is achieved by minimizing the upper bound of the $\mathcal{L}_2$ gain (*i.e.*, the worst-case $\mathcal{H}_\infty$ performance) of the closed-loop system, when the feedback gain is considered as the design parameter. The coupling in the analysis and synthesis parameters leads to the BMI minimization problem. However, an LMI-based iterative algorithm (El Ghaoui and Balakrishnan, 1994) is developed to solve such controller design problem.

1.4 Contributions

1. An approach to reduce conservatism of delay-dependent stability criteria of LSTD has been developed. The analysis approach combines the delay partitioning approach with the Lyapunov functional containing integral of sector-bounded nonlinearities. The stability criterion for the systems with time-varying norm-bounded uncertainty also derived. Both of stability tests are cast as LMI feasibility problem, which can be solved efficiently. The maximum allowable time delay is determined using the proposed bisection algorithm. In addition, two benchmark problems indicate that the new criteria give a significant improvement on computing the maximum allowable time delay especially when the sector bound is comparatively large.

2. A method to compute the worst-case $\mathcal{H}_\infty$ performance of LSTD is investigated. The worst-case performance analysis criterion combines the proposed absolute stability analysis with robust performance bounds on the total output energy. The worst-case performance criterion for the systems with time-varying norm-bounded uncertainty also derived. The computing of the worst-case $\mathcal{H}_\infty$ performance is formulated as LMI minimization problem, which can be efficiently solved. It should be emphasized that there is no work on how to compute the worst-case $\mathcal{H}_\infty$ performance for LSTD.

3. A synthesis procedure of the state-feedback stabilization for LSTD has been developed. The design method employs the new LKF with delay partitioning technique. In other words, the proposed absolute stability analysis is applied to the closed-loop system, when the feedback gain is treated as a design parameter. The iterative LMI approach, which alternates between two LMI feasibility problems, is used for solving the LMI stabilization problem. The interesting feature of this procedure is that there are some variables shared between two stages of the iteration. This improves the convergence of the algorithm.

4. A synthesis procedure of the state-feedback $\mathcal{H}_\infty$ control for LSTD has been investigated. The synthesis relies on the delay partitioning technique and the LKF containing integral of sector-bounded nonlinearities. The direct extension of the proposed worst-case $\mathcal{H}_\infty$ performance analysis to controller synthesis results in the minimization problem over BMIs. An LMI iterative algorithm, as a local optimization procedure, is used to solve such design problem. The shared variables between each iteration help improving the convergent rate, and the algorithm converges to a local optimum which depends on the starting points. This procedure is applied to several benchmark problems and is shown that it is conceptually simple and efficiently implemented.

1.5 Thesis Outline

The organization of this dissertation is as follows. In Chapter 2, we first briefly describe LMI and BMI, which will be used as a framework for analysis and design in this dissertation. Then, we will describe some notions of system operators and two important lemmas which are served as tools for developing both analysis and design proposed in the dissertation.

In Chapter 3, we describe the absolute stability criteria for both nominal LSTD and LSTD subject to norm-bounded uncertainty. The criterion based on the delay-partitioning LKF with the integral of sector-bounded nonlinearities is formulated as an optimization problem involving LMIs. The combination between bisection method and solving SDP is used to determine the maximum allowable time delay for the LSTD. This is followed by two numerical examples show the comparison of the maximum allowable time delay between the proposed criteria and other existing criteria.

In Chapter 4, we consider the worst-case performance criteria for both nominal and uncertain LSTD. By combining the absolute stability analysis in Chapter 3 and the robust performance bounds on the total output energy of the system, we formulate the worst-case $\mathcal{H}_\infty$ performance criteria in terms of LMIs. It is followed by the numerical results and compares the upper bounds of the worst-case $\mathcal{H}_\infty$ performance between the proposed criteria and other criteria.

In Chapter 5, the problem of the robust state-feedback stabilization and the robust state-feedback $\mathcal{H}_\infty$ control design of LSTD are presented. The stabilization problem is extended from the absolute stability analysis in Chapter 3, and the $\mathcal{H}_\infty$ control problem is developed from the worst-case performance analysis in Chapter 4. The direct consequence of optimizing both the analysis and synthesis variables leads to the design constraint involving BMIs. In this chapter, the coordinate optimization, which iterates between solving LMI problem are proposed for both problems. Lastly, the examples of controller design and the time-response simulations are presented.

In Chapter 6, we consider a continuous stirred tank reactor (CSTR) with recycle stream with uncertain reaction coefficient and nonlinear flow rate. A selected model of CSTR with recycle stream is treated as an LSTD. The state-feedback $\mathcal{H}_\infty$ control for such system is design via the proposed iterative LMI algorithm. The simulation of time-response subject to a sample disturbance is presented.

In the last chapter, the conclusions of the research are presented. Moreover, some suggestions on extensions and future research are proposed.

CHAPTER II

MATHEMATICAL PRELIMINARIES

This chapter shortly summarizes the key notations which we will use to present the main theoretical results in the dissertation. First, we briefly present the definitions of LMI and BMI, which will be a framework for the analysis and synthesis. Second, we will introduce some notions of the system operators, such as $\mathcal{L}_2$ stability and $\mathcal{L}_2$ gain. Lastly, we will summarize two lemmas which are helpful for developing both analysis and design proposed in the dissertation.

2.1 Linear Matrix Inequalities

Several problems in robust control theory can be formulated as convex optimizations involving linear matrix inequalities (LMIs) (Boyd et al., 1994; Vandenberghe and Boyd, 1996). An LMI has the following form.

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \geq 0, \tag{2.1}$$

where the symmetric matrices $F_i = F_i^T \in \mathbf{R}^{n\times n}, i = 0, 1, \ldots, m$ are given and $x \in \mathbf{R}^n$ is the decision variable. The inequality in (2.1) means that $F(x)$ is positive semidefinite. The LMI constraint is nonlinear and nonsmooth, but convex in x, *i.e.*, if $F(u) \geq 0$ and $F(v) \geq 0$, then $F(\alpha u+(1-\alpha)v) \geq 0, \forall \alpha \in [0, 1]$. We will often encounter problems in which the variables are matrices. For example, given matrices A, B, $Q = Q^T$ and $R = R^T > 0$, the matrix inequality

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0, \tag{2.2}$$

is an LMI over the matrix variable P, and it can be written out in the form (2.1) (Boyd et al., 1994). There are many standard LMI problems, but the class of problems which we will be working with is called the semidefinite programming (SDP). A semidefinite program is defined as a minimization of a linear objective subject to a constraint that an affine combination of symmetric matrices be positive semidefinite, *i.e.*,

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad F(x) \geq 0. \end{aligned} \tag{2.3}$$

where $c \in \mathbf{R}^m$ and F_i, $i = 0, 1, \ldots, m$ are given. A special case of semidefinite programs, which is also considered in this dissertation is the LMI feasibility problem:

$$\begin{array}{ll} \text{find} & x \\ \text{satisfying} & F(x) \geq 0. \end{array} \tag{2.4}$$

The semidefinite program is a convex optimization problem, and there exist polynomial time algorithms for solving such convex problem. Solutions algorithms include the cutting plan, the ellipsoid methods, and the interior point methods. Typically the first two algorithms converge very slowly in practice (Banjerdpongchai, 1997). The interior point methods have recently been developed to solve these semidefinite programming Nesterov and Nemirovski (1994); Vandenberghe and Boyd (1996); Gahinet and Nemirovski (1997). The methods do not require explicit analytic derivatives and they can be numerically solved in about $5 - 50$ iterations where each iteration is a least-squares problem (Vandenberghe and Boyd, 1996).

2.2 Bilinear Matrix Inequalities

An optimization problem involving bilinear matrix inequalities (BMIs) is as extension of the semidefinite program. The BMI has the form

$$F(x, y) := F_{00} + \sum_{i=1}^{m} x_i F_{i0} + \sum_{j=1}^{n} y_j F_{0j} + \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j F_{ij} \geq 0, \tag{2.5}$$

where $F_{ij} = F_{ij}^T \in \mathbf{R}^{p \times p}$, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$ are given and the decision variables $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$. Most of robust controller synthesis problems lead to the BMI problem (El Ghaoui and Balakrishnan, 1994; Goh et al., 1994, 1995). The BMI problems are NP-hard (Toker and Özbay, 1995), and the global optimization of the BMI problems is an ongoing research topic. However, there are many heuristic methods used to find local solution. In particular, there is a coordinate optimization (El Ghaoui and Balakrishnan, 1994) that utilizes efficient LMI tools to solve (2.5). For fixed x, finding y satisfying (2.5) is a SDP, and for fixed y, finding x satisfying (2.5) is another SDP.

2.3 Notions of System Operators

$\mathcal{L}_2^n$ is the Hilbert space of square-integrable signals defined over $\mathbf{R}_+$ with n-components; $\mathcal{L}_2^n$ is often abbreviated as $\mathcal{L}_2$. The symbol $\|\cdot\|_2$ stands for the $\mathcal{L}_2$ norm.

Definition 1 ***($\mathcal{L}_2$ Stability)*** *A causal operator $H : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be $\mathcal{L}_2$-stable if there* *exist $\gamma \geq 0$ and β such that*

$$\|Hw\| \leq \gamma\|w\| + \beta, \quad \forall w \in \mathcal{L}_2. \tag{2.6}$$

Definition 2 ***($\mathcal{L}_2$ gain)*** *The $\mathcal{L}_2$ gain of H is defined as the smallest γ such that (2.6) holds for some β.*

For a LTI system, the $\mathcal{L}_2$ gain is equivalent to the $\mathcal{H}_\infty$ norm. Note that this norm arises naturally in robust control application, such as the disturbance attenuation problem.

2.4 Linear Algebra

The following lemmas will be very useful in developing the absolute stability criteria, the worst-case performance analysis, and the controller designs presented in this dissertation. The first lemma helps converting an analysis criterion to an LMI constraint, and by the second lemma, we can deal with the analysis and synthesis of LSTD with a certain form of norm-bounded uncertainty.

Lemma 1 ***Jensen Inequality (Gu et al. (2003); Gu (2001))*** *For any constant matrix $M \in \mathbf{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to \mathbf{R}^m$ such that the integrations concerned are well defined, then*

$$\gamma \int_0^\gamma \omega^T(\beta)M\omega(\beta)d\beta \geq \left(\int_0^\gamma \omega(\beta)d\beta\right)^T M \left(\int_0^\gamma \omega(\beta)d\beta\right).$$

Proof: See Gu et al. (2003) Page 322. ∎

Lemma 2 ***(Xie (1996))****. Given matrices Q, H, E and R of appropriate dimensions and with Q and R symmetrical and $R > 0$, then*

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists some $\epsilon > 0$ such that

$$Q + \epsilon HH^T + \epsilon^{-1}E^T RE < 0.$$

Proof: See Xie (1996). ∎

2.5 Summary

This chapter presents some definitions and mathematical preliminaries which are necessary for developing both analysis and synthesis in this dissertation. Linear matrix inequalities and Bilinear matrix inequalities provide a framework for converting the analysis and design problems into optimization problems. The notions and lemmas of system operator are used to develop the conditions of the worst-case performance analysis. Finally, we state two lemmas which provide key steps of developing the analysis and design criteria in the dissertation.

CHAPTER III

ABSOLUTE STABILITY ANALYSIS

3.1 Introduction

In this chapter, we will develop a new absolute stability criterion for LSTD. The absolute stability problem can be formulated by using Lyapunov-Krasovskii theorem (Krasovskii, 1963) with the Lyapunov functional containing the integral of sector-bounded nonlinearities. The Lyapunov functional terms involving delay are partitioned to be associated with each equidistant fragment on the length of time delay (Gu, 1997, 2001; Gu et al., 2003). Employing the Jensen inequality (Gu et al., 2003; Gu, 2001) and $\mathcal{S}$-procedure (Yakubovich, 1977), the absolute stability criterion is formulated in terms of LMIs which can be efficiently solved using available LMI solvers. In addition, the stability criterion is extended to Lur'e systems subject to norm-bounded uncertainties by using the matrix eliminating lemma (Xie, 1996). A similar approach to the proposed criterion was proposed in Wu et al. (2009), but the major difference is that the criterion in Wu et al. (2009) has not utilized any information of sector-bounded nonlinearity. In this work, the bisection method is used to determine the maximum allowable time delays to ensure the stability of Lur'e systems in the presence of uncertain time-invariant delays.

The organization of this chapter is as follows. In §3.2, we state the considered absolute stability analysis problem. Then, in §3.3, we present the main contribution including the absolute stability criteria for both nominal and uncertain LSTD with the algorithm for calculating the maximum allowable time delay. Finally, the numerical examples show the comparison of the results between the proposed criteria and other existing criteria in §3.4.

3.2 Problem Statement

The state-space equation of LSTD is described as follows.

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + A_1 x(t-h) + B_p p(t), \\
q(t) &= C_q x(t), \\
p(t) &= \phi(q(t)),
\end{aligned}
\tag{3.1}
$$

with sector-bounded nonlinearities $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and the initial condition $x(t) = \varphi(t)$, $\forall t \in [-h, 0]$, where $h \in \mathbf{R}_+$ is a time-invariant time delay in the state. The vector $x(t) \in \mathbf{R}^n$ is the state, $q(t)$ and $p(t) \in \mathbf{R}^{n_p}$ are the input and output of ϕ, which is a vector mapping of sector-bounded nonlinearities belong to the set Φ characterized by memoryless, time-invariant nonlinearities satisfying certain sector conditions. In particular, given an input vector $\sigma := [\sigma_1, \ldots, \sigma_{n_p}]^T$, a lower bound vector $l := [l_1, \ldots, l_{n_p}]^T$ and an upper bound vector $m := [m_1, \ldots, m_{n_p}]^T$, with $l_i < m_i$ for all $i = 1, \ldots, n_p$, the set Φ can be described by

$$\begin{aligned}\Phi(l, m) :=& \{\phi : \mathbf{R}^{n_p} \to \mathbf{R}^{n_p} : \phi(\sigma) = \left[\phi_1(\sigma_1), \ldots, \phi_{n_p}(\sigma_{n_p})\right]^T, \\ & l_i\sigma_i^2 \leq \sigma_i\phi_i(\sigma_i) \leq m_i\sigma_i^2, \text{ for all } i = 1, \ldots, n_p\}.\end{aligned}$$

In addition, the (A, B_p) pair and (C_q, A) pair are assumed to be controllable and observable, respectively. By considering the LSTD described by (3.1) the absolute stability of the system is defined as follows.

Definition 3 *The system (3.1) is delay-dependently absolutely stable if there exists a positive bound of time delays, $\bar{h} \in \mathbf{R}_+$, such that the equilibrium $x = 0$ is globally uniformly asymptotically stable for any time-invariant delay $h \in (0, \bar{h}]$.*

Problem 1 *The absolute stability problem for (3.1) is to determine the maximum allowable time delay, $\bar{h}_{\max}$, that the system (3.1) is delay-dependently absolutely stable.*

We also consider LSTD with norm-bounded uncertainty described as follows.

$$\begin{aligned}\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) + [B_p + \Delta B_p(t)]p(t), \\ q(t) &= C_q x(t), \\ p(t) &= \phi(q(t)),\end{aligned} \tag{3.2}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$. The norm-bounded uncertainty is assumed to be of the form

$$\begin{aligned}\Delta A(t) &= DF(t)E_0, \\ \Delta A_1(t) &= DF(t)E_1, \\ \Delta B_p(t) &= DF(t)E_2,\end{aligned} \tag{3.3}$$

where D, E_0, E_1 and E_2 are known constant real matrices of appropriate dimensions, and represent the structure of uncertainty, and $F(t)$ is an unknown matrix function with *Lebesgue measurable elements* satisfying the constraint

$$F^T(t)F(t) \leq I. \tag{3.4}$$

Definition 4 *The system (3.2) is delay-dependently robustly absolutely stable if there exists a positive bound of time delays, $\bar{h} \in \mathbf{R}_+$, such that the equilibrium $x = 0$ is globally uniformly asymptotically stable for any time-invariant delay $h \in (0, \bar{h}]$ and all uncertain matrix function $F(t)$ satisfying (3.4).*

Problem 2 *The robust absolute stability problem for (3.2) is to determine the maximum allowable time delay, $\bar{h}_{\max}$, that the system (3.2) is delay-dependently robustly absolutely stable.*

3.3 Absolute Stability Criteria

In this section, we present Theorem 1 for absolute stability criterion of nominal LSTD, and Theorem 2 for robust absolute stability criterion of LSTD with norm-bounded uncertainty. Both criteria are in terms of LMIs for a given bound of time delay, and the maximum allowable time delay can be determined using the proposed bisection algorithm.

3.3.1 Nominal Systems

The following theorem provides a sufficient condition to guarantee absolute stability of the system (3.1) for a given bound of time delay.

Theorem 1 *For a given $\bar{h} > 0$, the Lur'e systems with time delays (3.1) is delay-dependently absolutely stable for any constant time delay $h \in (0, \bar{h}]$, if there exist symmetric matrices $P > 0$, $Q_k > 0$ and $R_k > 0$ for all $k = 1, \ldots, N$, diagonal matrices $\Lambda \geq 0$ and $T \geq 0$ satisfying*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & 0 \\ * & * & \Psi_{33} \end{bmatrix} < 0, \tag{3.5}$$

where

$$\Psi_{11} = \begin{bmatrix} \begin{pmatrix} PA + A^T P \\ +Q_1 - R_1 \end{pmatrix} & PA_1 & \begin{pmatrix} PB_p + C_q^T T \\ +A^T C_q^T \Lambda \end{pmatrix} \\ * & -Q_N - R_N & A_1^T C_q^T \Lambda \\ * & * & \begin{pmatrix} B_p^T C_q^T \Lambda \\ +\Lambda C_q B_p - 2T \end{pmatrix} \end{bmatrix},$$

$$\Psi_{12} = \begin{bmatrix} R_1 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \end{bmatrix} + \begin{bmatrix} 0_{n\times(N-2)n} & 0 \\ 0_{n\times(N-2)n} & R_N \\ 0_{n\times(N-2)n} & 0 \end{bmatrix}, \quad \Psi_{13} = \begin{bmatrix} \frac{\bar{h}}{N} A^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} A_1^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} B_p^T \sum_{k=1}^N R_k \end{bmatrix},$$

$$\Psi_{22} = \begin{bmatrix} \Gamma_1 & R_2 & & \\ * & \Gamma_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & \Gamma_{N-1} \end{bmatrix}, \quad \Psi_{33} = -\sum_{k=1}^{N} R_k,$$

with $\Gamma_k := -Q_k + Q_{k+1} - R_k - R_{k+1}$.

Proof: Assume that $A + A_1$ is Hurwitz, and (A, B_p, C_q) is the minimal realization. Consider the LKF candidate of the form

$$V(x_t) := V_1 + V_2 + V_3 + V_4, \tag{3.6}$$

with

$$\begin{aligned} V_1 &= x^T(t)Px(t), \\ V_2 &= 2\sum_{i=1}^{n_p} \lambda_i \int_0^{q_i} \phi_i(\sigma)d\sigma, \\ V_3 &= \sum_{k=1}^{N} \int_{t-kr}^{t-(k-1)r} x^T(\xi)Q_k x(\xi)d\xi, \\ V_4 &= r\sum_{k=1}^{N} \int_{-kr}^{-(k-1)r} \int_{t+\theta}^{t} \dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi d\theta, \end{aligned}$$

where P, $Q_1, \ldots, Q_N$, and $R_1, \ldots, R_N$ are positive definite symmetric matrices of dimension $n \times n$, scalars $\lambda_1, \ldots, \lambda_{n_p}$ are non-negative, x_t denotes a piece of trajectory $x(t+\theta)$ for $-h \leq \theta \leq 0$, and $r = h/N$. We seek P, $Q_1, \ldots, Q_N$, $R_1, \ldots, R_N$, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n_p}) \geq 0$ such that $\dot{V}(x_t) < 0$ for all nonzero x_t satisfying (3.1) with a set of sector-bounded conditions

$$0 \leq q_i(t)\phi_i(q_i(t)) \leq q_i^2(t), \quad \forall i = 1, \ldots, n_p. \tag{3.7}$$

To verify that $\dot{V}(x_t) < 0$ under the set of constraints (3.7), we apply $\mathcal{S}$-procedure (Yakubovich, 1977) to establish the *sufficient condition* as follows.

$$\dot{V}(x_t) - \sum_{i=1}^{n_p} \tau_i q_i(t)(p_i(t) - q_i(t)) \leq 0, \tag{3.8}$$

where $\tau_1 \geq 0, \ldots, \tau_{n_p} \geq 0$. Note that for the case of single nonlinearity ($n_p = 1$), $\mathcal{S}$-procedure is lossless, and the condition (3.8) is not only sufficient but also necessary for $\dot{V}(x_t) < 0$ under the constraints (3.7). By defining $T = \mathrm{diag}(\tau_1, \ldots, \tau_{n_p})$, the inequality (3.8) can be written in vector-matrix notation as

$$\dot{V}(x_t) - 2p^T(t)Tp(t) + 2x^T(t)C_q^T Tp(t) \leq 0. \tag{3.9}$$

The derivative of each term in the LKF (3.6) with respect to time along the solution of (3.1) is given by

$$
\begin{aligned}
\dot{V}_1 &= x^T(t)\left[A^T P + PA\right]x(t) + 2x^T(t)PA_1 x(t-h) + 2x^T(t)PB_p p(t), \\
\dot{V}_2 &= 2x^T(t)A^T C_q^T \Lambda p(t) + 2x^T(t-h)A_1^T C_q^T \Lambda p(t) + p^T(t)\left[B_p^T C_q^T \Lambda + \Lambda C_q B_p\right]p(t), \\
\dot{V}_3 &= x^T(t)Q_1 x(t) - x^T(t-h)Q_N x(t-h) + \sum_{k=1}^{N-1}\left[x^T(t-kr)(-Q_k + Q_{k+1})x(t-kr)\right], \\
\dot{V}_4 &= \dot{x}^T(t)\sum_{k=1}^{N}\left(r^2 R_k\right)\dot{x}(t) - r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r}\dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi.
\end{aligned}
$$

Note that $\dot{V}_1$, $\dot{V}_2$, and $\dot{V}_3$ can be directly cast in terms of LMI. However, the first term of $\dot{V}_4$ is rewritten by using the state equation in (3.1) as

$$
\dot{x}^T(t)\sum_{k=1}^{N}\left(r^2 R_k\right)\dot{x}(t) = \mathbf{x}_1^T \begin{bmatrix} A^T \\ A_1^T \\ B_p^T \end{bmatrix} r^2 \sum_{k=1}^{N} R_k \begin{bmatrix} A^T \\ A_1^T \\ B_p^T \end{bmatrix}^T \mathbf{x}_1, \tag{3.10}
$$

where $\mathbf{x}_1 = \begin{bmatrix} x^T(t) & x^T(t-h) & p^T(t) \end{bmatrix}^T$. We employ Jensen inequality (Lemma 1) to bound the integral terms appeared in $\dot{V}_4$ as

$$
\begin{aligned}
-r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r}\dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi &\le \sum_{k=1}^{N}\left(\int_{t-kr}^{t-(k-1)r}\dot{x}(\xi)d\xi\right)^T(-R_k)\left(\int_{t-kr}^{t-(k-1)r}\dot{x}(\xi)d\xi\right), \\
&= \sum_{k=1}^{N}\left[x(t-(k-1)r) - x(t-kr)\right]^T(-R_k)\times \\
&\qquad \left[x(t-(k-1)r) - x(t-kr)\right], \\
&= 2\mathbf{x}_1^T \Psi_{12}\mathbf{x}_2 + \mathbf{x}_1^T \begin{bmatrix} -R_1 & 0 & 0 \\ * & -R_N & 0 \\ * & * & 0 \end{bmatrix}\mathbf{x}_1 \\
&\quad + \mathbf{x}_2^T \begin{bmatrix} -R_1 - R_2 & R_2 & & \\ * & -R_1 - R_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & -R_{N-1} - R_N \end{bmatrix}\mathbf{x}_2,
\end{aligned} \tag{3.11}
$$

where $\mathbf{x}_2 = \begin{bmatrix} x^T(t-r) & \cdots & x^T(t-h+r) \end{bmatrix}^T$, and the entries left blank are zero. Substituting $\dot{V}_1$, $\dot{V}_2$, $\dot{V}_3$ and $\dot{V}_4$ into inequality (3.9) and applying (3.10) and the upper bound (3.11) for $\dot{V}_4$, we obtain sufficient condition for $\dot{V}(x_t) < 0$ as follows.

$$\mathbf{x}^T \Psi(h) \mathbf{x} < 0, \tag{3.12}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, and

$$\Psi(h) = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} - \begin{bmatrix} \Psi_{13} \\ 0 \end{bmatrix} \Psi_{33}^{-1} \begin{bmatrix} \Psi_{13} \\ 0 \end{bmatrix}^T. \tag{3.13}$$

Inequality (3.12) holds for all $\mathbf{x} \neq \mathbf{0}$ if and only if the following matrix inequality is satisfied.

$$\Psi(h) < 0. \tag{3.14}$$

Lastly, applying Schur complement (Boyd et al., 1994, pp. 7–8) and substituting h with $\bar{h}$, the matrix inequality (3.14) is equivalent to (3.5).

If the matrix inequality (3.5) is satisfied, then $\dot{V}(x_t) < -\varepsilon \|x(t)\|^2$ for a sufficiently small ε, which ensures the asymptotic stability of the system (3.1). Up to this point, the sufficient condition (3.5) guarantees absolute stability of the system (3.1) only for the extreme case, *i.e.*, $h = \bar{h}$. To show that (3.5) also guarantees absolute stability of the system (3.1) for any constant time delay $h \in (0, \bar{h}]$. Let $h = \bar{h} - \Delta h$, where $0 < \Delta h < \bar{h}$. Obviously, h lies in the interval $(0, \bar{h})$. By substituting h with $\bar{h} - \Delta h$, and isolating all terms involving Δh, the matrix inequality (3.13) becomes

$$\Psi(h) = \Psi(\bar{h}) + \frac{(2\Delta h \bar{h} - \Delta h^2)}{N^2} \begin{bmatrix} \Psi_{13} \\ 0 \end{bmatrix} \Psi_{33}^{-1} \begin{bmatrix} \Psi_{13} \\ 0 \end{bmatrix}^T < 0. \tag{3.15}$$

Since $(2\Delta h \bar{h} - \Delta h^2)/N^2 > 0$ and Ψ_{33} is negative definite, the second term of $\Psi(h)$ is negative definite. Thus, if $\Psi(\bar{h}) < 0$ holds, $\Psi(h) < 0$ also holds. Theorem 1 guarantees absolute stability of the system (3.1) for any constant time delay $h \in (0, \bar{h}]$. This completes the proof.

■

Remark 1 *In Theorem 1, it is straightforward to handle general sector condition $\phi \in \Phi(l, m)$. By using* loop transformation *(Desoer and Vidyasagar, 1975), LSTD (3.1) with $\phi \in \Phi(l, m)$*

can be transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. *In particular, define*

$$\bar{\phi}_i(q_i(t)) := \frac{1}{m_i - l_i}\left[\phi_i(q_i(t)) - l_i q_i(t)\right].$$

It is easy to show that $0 \le \sigma_i \bar{\phi}_i(\sigma_i) \le \sigma_i^2$ *for all* $i = 1, \ldots, n_p$, i.e., $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. *Let* $L = \mathrm{diag}(l)$, $M = \mathrm{diag}(m)$, *and* $\bar{p}(t) = (M - L)^{-1}\big(p(t) - Lq(t)\big)$. *We then substitute* $p(t) = (M - L)\bar{p}(t) + Lq(t)$ *into (3.1), and obtain the equivalent LSTD system as follows.*

$$\begin{aligned}
\dot{x}(t) &= \bar{A}x(t) + A_1 x(t-h) + \bar{B}_p \bar{p}(t), \\
q(t) &= C_q x(t), \\
\bar{p}(t) &= \bar{\phi}(q(t)),
\end{aligned} \tag{3.16}$$

with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$, *where* $\bar{A} = A + B_p L C_q$ *and* $\bar{B}_p = B_p(M - L)$. *Note that the stability of LSTD (3.16) is equivalent to that of the original LSTD (3.1) with* $\phi \in \Phi(l, m)$, *so we can analyze stability properties of the latter system by considering the stability of the system (3.16).*

Remark 2 *For LSTD with multiple time delays,* i.e., $h_1, \ldots, h_m$, *we can easily generalize the LKF of the form (3.6) as follows*

$$\begin{aligned}
V(x_t) \;=\; & x^T(t)Px(t) \\
& + 2\sum_{i=1}^{n_p} \lambda_i \int_0^{q_i} \phi_i(\sigma)d\sigma \\
& + \sum_{i=1}^{m} \left(\sum_{k=1}^{N} \int_{t-kr_i}^{t-(k-1)r_i} x^T(\xi) Q_k^{(i)} x(\xi) d\xi \right) \\
& + \sum_{i=1}^{m} \left(r_i \sum_{k=1}^{N} \int_{-kr_i}^{-(k-1)r_i} \int_{t+\theta}^{t} \dot{x}^T(\xi) R_k^{(i)} \dot{x}(\xi) d\xi d\theta \right),
\end{aligned}$$

where $r_i = h_i/N$, *and* $Q_1^{(i)}, \ldots, Q_N^{(i)}$, $R_1^{(i)}, \ldots, R_N^{(i)}$ *for* $i = 1, \ldots, m$, *are free matrices of appropriate dimensions. By using the generalized LKF above, an absolute stability criterion for the LSTD with multiple time delays can be developed along the proof of Theorem 1.*

Remark 3 *The absolute stability criterion based on* N*-segmentation method proposed in Wu et al. (2009) is a special case of Theorem 1 when* $\Lambda = 0$ *and* $T = \tau I$ *for* $\tau \in \mathbf{R}_+$. *It should be noted that* Λ, *and* $\tau_2, \ldots, \tau_{n_p}$ *are additional free variables which can be potentially the key to establish the less conservative stability criterion.*

3.3.2 Uncertain Systems

This section presents a sufficient condition for the robust absolute stability of uncertain LSTD (3.2).

Theorem 2 *For a given $\bar{h} > 0$, the system (3.2) is delay-dependently robustly absolutely stable for any constant time delay $h \in (0, \bar{h}]$ if there exist positive symmetric matrices P, $Q_1, \ldots, Q_N$ and $R_1, \ldots, R_N$, diagonal matrices $\Lambda \geq 0$ and $T \geq 0$, a scalar $\epsilon > 0$ satisfying*

$$\begin{bmatrix} \tilde{\Psi}_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & 0 \\ * & * & \Psi_{33} & \Psi_{34} \\ * & * & * & \Psi_{44} \end{bmatrix} < 0, \tag{3.17}$$

where

$$\tilde{\Psi}_{11} = \Psi_{11} + \epsilon \begin{bmatrix} E_o \\ E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} E_0 & E_1 & E_2 \end{bmatrix}, \quad \Psi_{14} = \begin{bmatrix} PD \\ 0 \\ \Lambda C_q D \end{bmatrix},$$

$$\Psi_{34} = \frac{\bar{h}}{N} \left(\sum_{k=1}^{N} R_k \right) D, \quad \Psi_{44} = -\epsilon I,$$

and Ψ_{11}, Ψ_{12}, Ψ_{13}, Ψ_{22}, and Ψ_{33} are the same as defined in Theorem 1.

Proof: By applying Theorem 1 to the uncertain LSTD (3.2), the stability criterion consists of the following LMI

$$\Psi + \begin{bmatrix} \Psi_{14} \\ 0 \\ \Psi_{34} \end{bmatrix} F(t) \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix} F^T(t) \begin{bmatrix} \Psi_{14} \\ 0 \\ \Psi_{34} \end{bmatrix}^T < 0, \tag{3.18}$$

where Ψ is defined as

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & 0 \\ * & * & \Psi_{33} \end{bmatrix} < 0,$$

It follows from Lemma 2 that the matrix inequality (3.18) is true for all uncertain matrix $F(t)$ satisfying $F^T(t)F(t) \leq I$ if and only if there exists a scalar $\epsilon > 0$ such that

$$\Psi + \epsilon^{-1} \begin{bmatrix} \Psi_{14} \\ 0 \\ \Psi_{34} \end{bmatrix} \begin{bmatrix} \Psi_{14} \\ 0 \\ \Psi_{34} \end{bmatrix}^T + \epsilon \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix}^T < 0. \tag{3.19}$$

Absorbing the last term into Ψ and applying the *Schur complement* (Boyd et al., 1994), the matrix inequality (3.17) holds. This completes the proof. ■

Remark 4 *It is observed that the terms involving ϵ and ϵ^{-1} in the matrix inequality (3.19) are all positive. Then, the feasible set of (3.17) is smaller than that of (3.5), and $\bar{h}_{\max}$ obtained for the uncertain LSTD (3.2) should be less than that for the nominal LSTD (3.1).*

3.3.3 Computation

By fixing the bound on time delay $\bar{h}$, the condition (3.5) is LMI over matrix variables P, $Q_1, \ldots, Q_N$, $R_1, \ldots, R_N$, Λ and T. The feasibility of (3.5) can be justified by using efficient numerical techniques such as *interior-point method* (Nesterov and Nemirovski, 1994). We observe that the absolute stability criterion depends on the bound of time delay $\bar{h}$ in the interval $(0, \bar{h}_{\max}]$. From inequality (3.15), $\Psi(h_2) < 0$ implies $\Psi(h_1) < 0$ for every $\bar{h}_{\max} \geq h_2 \geq h_1 > 0$. Due to this fact, we can determine $\bar{h}_{\max}$ by applying the *bisection method* associated with solving the LMI feasibility problem with the LMI constraint (3.5). The algorithm is as follows.

Algorithm for determining $\bar{h}_{\max}$

Step 1: Choose h_a and h_b such that the LMI (3.5) is feasible for $\bar{h} = h_a$, but infeasible for $\bar{h} = h_b$.

Step 2: Estimate the $\bar{h}_{\max}$ as the mid-point between h_a and h_b as $h_c = (h_a + h_b)/2$.

Step 3: Solve LMI (3.5) for $\bar{h} = h_c$. If it is feasible, then $\bar{h}_{\max}$ lies between h_c and h_b, let $h_a = h_c$, else $\bar{h}_{\max}$ lies between h_a and h_c, let $h_b = h_c$.

Step 4: If the length of the search interval, $h_b - h_a$, is larger than the absolute tolerance, then go to Step 2, else return $\bar{h}_{\max} = h_a$ and terminate.

In general, the computational complexity for solving LMI problem depends on the number of decision variables and the dimension of matrices appeared in LMIs (Gahinet and Nemirovski, 1997). Table 3.1 shows the number of decision variables used in Theorem 1 and other existing delay-dependent stability criteria. Note that n, n_p, and N are the number of states,

Table 3.1: The number of decision variables by each criterion.

Criterion	Number of decision variables
Han (2005)	$1.5n^2 + 1.5n + 1$
Yu et al. (2003)	$3n^2 + 2n + 1$
He et al. (2005)	$3.5n^2 + 2.5n + 2nn_p + 0.5n_p^2 + 2.5n_p$
Xu and Feng (2007)	$4n^2 + 3n + 2n_p$
Qiu and Zhang (2011)	$8.5n^2 + 2.5n + nn_p + 1$
Wu et al. (2009)	$(N + 0.5)n^2 + (N + 0.5)n + 1$
Theorem 1	$(N + 0.5)n^2 + (N + 0.5)n + 2n_p$

nonlinearities, and delay partitions, respectively. Obviously, the number of decision variables used by Theorem 1 depends on the number of delay partitions. For $N < 8$, the number of decision variables used by the proposed criterion is less than that of Qiu and Zhang (2011). In addition, the difference between the number of decision variables used in Wu et al. (2009) and that used in Theorem 1 is $2n_p - 1$. Thus, our proposed criterion has computational complexity slightly more than that of Wu et al. (2009).

3.4 Numerical Examples

This section includes two numerical examples to demonstrate the effectiveness of the proposed absolute stability criteria. The maximum allowable time delays computed by using Theorem 1 and Theorem 2 are compared with those obtained from existing criteria based on the following methods.

- Model transformation and bounding technique (Yu et al., 2003),
- Jensen inequality (Han, 2005),
- Free weighting matrices approach (He et al., 2005),
- Augmented LKF and Jensen inequality (Xu and Feng, 2007),

- N-segmentation method and Jensen inequality (Wu et al., 2009),
- Asymmetric delay-dividing, and integral equality approach (Qiu and Zhang, 2011).

In this work, the LMI Lab (Gahinet et al., 1995) which employs the projective interior-point method (Gahinet and Nemirovski, 1997) is used for solving the LMI feasibility problem.

Example 1: Consider the system (3.2) with the following parameters.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \quad C_q = \begin{bmatrix} 0.6 & 0.8 \end{bmatrix},$$

$$\phi \in \Phi(0.35 - \delta l, 0.35 + \delta l), \quad \|\Delta A(t)\| \leq \alpha, \quad \|\Delta A_1(t)\| \leq \alpha, \quad \Delta B_p(t) = 0,$$

where δl characterizes the sector bound of nonlinearity and α represents magnitude of the uncertainty. This example is taken from Han (2005) with slight modifications. The uncertainty model $\Delta A(t)$ and $\Delta A_1(t)$ can be described by (3.3) with $D = \alpha I$, $E_0 = E_1 = I$, and $E_2 = 0$. We apply the loop transformation so that LSTD with $\phi \in \Phi(0.35 - \delta l, 0.35 + \delta l)$ is transformed to the equivalent LSTD with $\bar{\phi} \in \Phi(0,1)$, then apply the proposed absolute stability criteria to the equivalent system. For example, let $\delta l = 2.15$ and the nonlinearity ϕ belongs to the set $\Phi(-1.8, 2.5)$. We use the loop transformation to transform LSTD with ϕ to LSTD with $\bar{\phi} \in \Phi(0,1)$. The results of $\bar{h}_{\max}$ compared with those obtained from other criteria are shown in Table 3.2.

Table 3.2: Calculated maximum allowable time delay, $\bar{h}_{\max}$, for Example 1 when $\delta l = 2.15$.

Criterion	α			
	0.00	0.05	0.10	0.15
Yu et al. (2003)	1.0032	0.9261	0.8545	0.7889
Han (2005)	1.1371	1.0715	1.0109	0.9546
Wu et al. (2009), $N = 2$	1.3074	1.2251	1.1495	1.0796
Wu et al. (2009), $N = 3$	1.3411	1.2557	1.1771	1.1046
He et al. (2005)	1.4560	1.3522	1.2582	1.1723
Xu and Feng (2007)	1.4560	1.3522	1.2582	1.1723
Qiu and Zhang (2011)	1.6134	1.4919	1.3818	1.2813
Theorem 1/ Theorem 2, $N = 2$	1.6368	1.5125	1.3998	1.2971
Theorem 1/ Theorem 2, $N = 3$	1.6717	1.5433	1.4269	1.3209

As shown in Table 3.2, the criteria in He et al. (2005); Xu and Feng (2007); Qiu and Zhang (2011) which utilize the integral of nonlinearities in the LKF provide $\bar{h}_{\max}$ greater than

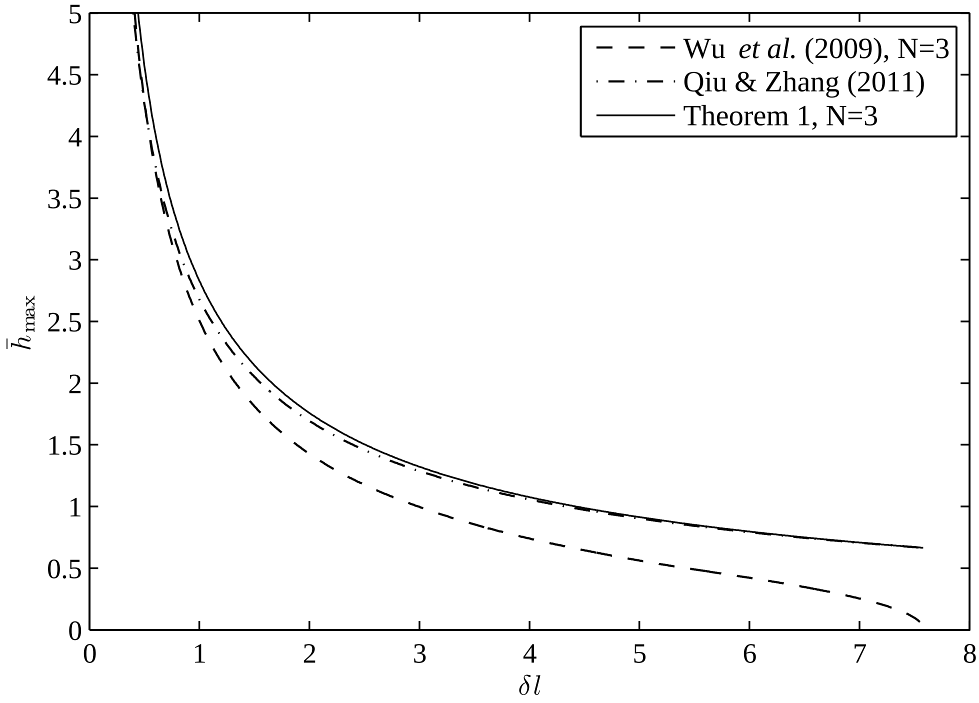

Figure 3.1: Calculated maximum allowable time delay vs. δl for Example 1.

those obtained from the criteria in Yu et al. (2003); Han (2005); Wu et al. (2009). Among eight criteria, Theorem 1 and 2 with $N = 3$ give the greatest $\bar{h}_{\max}$ for both nominal ($\alpha = 0$) and uncertain cases. Comparing to Wu et al. (2009) which is a special case of Theorem 1, for the nominal LSTD, Theorem 1 gives 24.6% increase of $\bar{h}_{\max}$ for the case $N = 3$, and yields 25.2% improvement of $\bar{h}_{\max}$ for the case $N = 2$. In addition, employing Theorem 1 with $N = 2$ to the nominal case achieves 1.5% increasing in $\bar{h}_{\max}$ comparing to Qiu and Zhang (2011), which adopts the asymmetric delay-dividing technique. Figure 3.1 depicts the value of $\bar{h}_{\max}$ obtained from three criteria when varying δl for the case of $\alpha = 0$. It is clearly seen that $\bar{h}_{\max}$ calculated from Theorem 1 is always greater than those obtained using previous criteria Wu et al. (2009) and Qiu and Zhang (2011). Thus, the proposed absolute stability criteria are less conservative than existing criteria.

Example 2: Consider the system (3.2) with the following parameters.

$$A = \begin{bmatrix} -2.0 & -1.0 \\ 0.5 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 1.0 \\ -0.1 & -0.8 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \phi \in \Phi\left(\begin{bmatrix} 0.25 - \delta l \\ 0.35 - \delta l \end{bmatrix}, \begin{bmatrix} 0.25 + \delta l \\ 0.35 + \delta l \end{bmatrix}\right),$$

$$\|\Delta A(t)\| \leq \alpha, \quad \|\Delta A_1(t)\| \leq \alpha \quad \Delta B_p(t) = 0.$$

This example is modified from the example given in Han (2005). Similar to the previous example, the norm-bounded uncertainty can be described by (3.3) with $D = \alpha I$, E_0 and E_1 are identity matrices of appropriate dimension, and $E_2 = 0$. We apply the loop transformation to change LSTD to the equivalent LSTD with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$, then apply the proposed absolute stability criteria to the equivalent system. For example, let $\delta l = 2.15$ and apply the loop transformation

$$\phi(q(t)) = 4.3\bar{\phi}(q(t)) - \mathrm{diag}(1.9, 1.8)q(t).$$

The calculated $\bar{h}_{\max}$ using Theorem 1 and 2 are compared with those obtained from other existing criteria. Table 3.3 shows the results.

Table 3.3: Calculated maximum allowable time delay, $\bar{h}_{\max}$, for Example 2 when $\delta l = 2.15$.

Criterion	α			
	0.00	0.05	0.10	0.15
Yu et al. (2003)	1.0280	0.8593	0.6903	0.4644
Han (2005)	1.5102	1.3516	1.2074	1.0357
Wu et al. (2009), $N = 2$	1.6174	1.4231	1.2493	1.0403
Wu et al. (2009), $N = 3$	1.6282	1.4312	1.2493	1.0420
He et al. (2005)	1.8642	1.6188	1.4070	1.2081
Xu and Feng (2007)	1.9033	1.6435	1.4276	1.2365
Qiu and Zhang (2011)	1.9270	1.6647	1.4268	1.1736
Theorem 1/ Theorem 2, $N = 2$	1.9650	1.6950	1.4519	1.2148
Theorem 1/ Theorem 2, $N = 3$	1.9836	1.7042	1.4558	1.2177

As expected, Theorem 1 and 2, with $N = 3$ provide the largest $\bar{h}_{\max}$ for all cases. The results for nominal system indicate that Theorem 1 gains 21.8% and 21.5% of improvement in $\bar{h}_{\max}$ comparing to Wu et al. (2009) for the case $N = 2$ and $N = 3$, respectively. Using Theorem 1, we can obtain $\bar{h}_{\max}$ about 2% larger than that computed from Qiu and Zhang (2011) in the nominal case. Next, we vary δ in the nominal LSTD and compute $\bar{h}_{\max}$. Figure 3.2 shows that our proposed criterion gives larger $\bar{h}_{\max}$ than those obtained from Wu et al. (2009) and Qiu and Zhang (2011), especially when the sector bounds are large. Moreover, for $\delta l \in [4.71, 6.35]$, the proposed criterion is the only criterion that can find the solution of $\bar{h}_{\max}$,

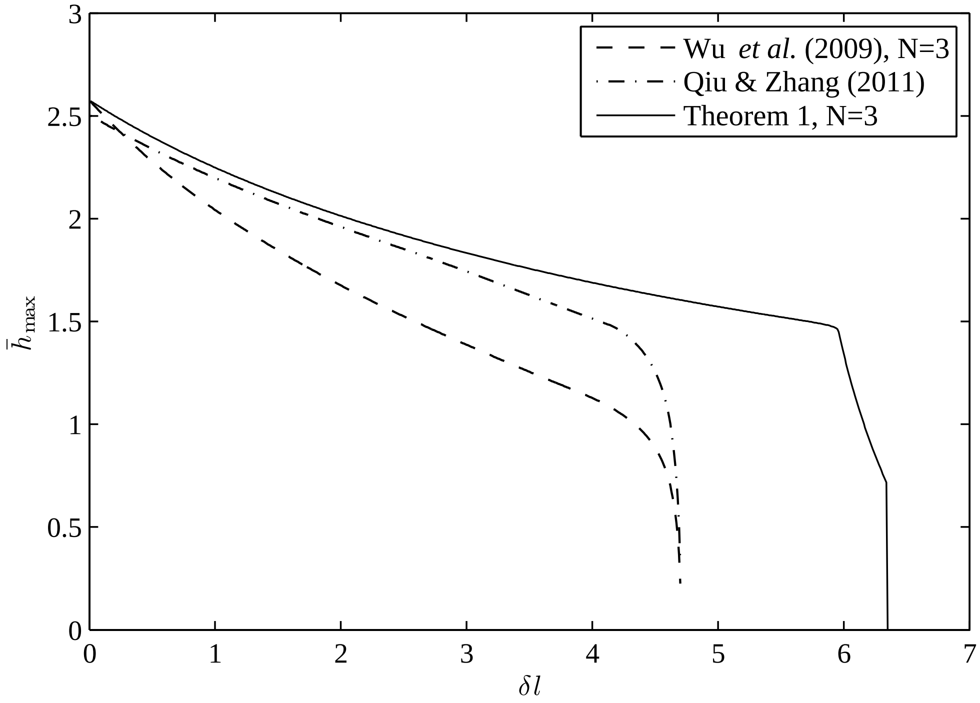

Figure 3.2: Calculated maximum allowable time delay vs. δl for Example 2.

where other methods cannot. These results confirm that the proposed absolute stability criteria have advantage over existing criteria.

In order to show the effectiveness of the proposed criterion more clearly, we compare average CPU time used by different criteria to compute $\bar{h}_{\max}$ for the nominal LSTD in Example 1 and 2. The results are given in Table 3.4. Note that the tests are performed on a 2.93 GHz Intel Core i3 processor with 2 GB of memory. From the data in Table 3.4, we observe that Theorem 1 spends time 30%–50% more than those of Wu et al. (2009). However, the proposed criteria can achieve much greater $\bar{h}_{\max}$ as reported in Table 3.2 and 3.3. In addition, Qiu & Zhang's criterion requires twice computational time of Theorem 1, but provides $\bar{h}_{\max}$ smaller than that of Theorem 1.

3.5 Discussion

The criterion in Yu et al. (2003) applies a model transformation and bounding technique, which leads to quite conservative results as appeared from the smallest $\bar{h}_{\max}$ in both Example 1 and 2. In fact, the delay-dependent criteria with Jensen inequality (Han, 2005) and free weighting matrices approach (He et al., 2005) can produce the same results and cannot decrease the

Table 3.4: Computational time used by different criteria for the nominal LSTD.

Criterion	Example 1		Example 2	
	#Vars.	CPU time (ms)	#Vars.	CPU time (ms)
Yu et al. (2003)	16	9	16	9
Han (2005)	10	9	10	10
Wu et al. (2009), $N = 2$	16	32	16	32
He et al. (2005)	36	41	46	38
Xu and Feng (2007)	24	80	26	86
Theorem 1, $N = 2$	17	51	19	43
Wu et al. (2009), $N = 3$	22	90	22	72
Theorem 1, $N = 3$	23	122	25	108
Qiu and Zhang (2011)	45	240	50	222

conservatism (Zhang and Yu, 2008). However, the difference in $\bar{h}_{\max}$ obtained from these two criteria is occurred, because the integral of nonlinearities is employed in the LKF with FWM criterion (He et al., 2005). An attempt to reduce the conservatism by utilizing Jensen inequality to the augmented LKF with the integral of nonlinearities (Xu and Feng, 2007) shows a slight increase of $\bar{h}_{\max}$ for Example 2 comparing to the FWM criterion (He et al., 2005). For the delay partitioning based criteria, Wu et al. (2009) give some improvements comparing to Han (2005), which is a special case of the criterion in Wu et al. (2009). Nonetheless, the results obtained from Wu et al. (2009) are still conservative than other delay partitioning based criteria, particularly Qiu and Zhang (2011), Theorem 1 and Theorem 2. Both Qiu and Zhang (2011) and our proposed criteria employ the integral of nonlinearities in the LKF, and thus a greater value of $\bar{h}_{\max}$ can be obtained. Moreover, the criteria in Wu et al. (2009) are the special case of our proposed criteria as mentioned in Remark 3. In Qiu and Zhang (2011), the delay interval $[-h, 0]$ is divided into $[-h, -h/3]$ and $[-h/3, 0]$, and the resulting criterion is formulated in terms of LMIs by using integral equality, which in fact is another form of FWM approach (He et al., 2005). In other words, the criterion proposed in Qiu and Zhang (2011) uses delay partitioning with $N = 2$, and applies FWM approach instead of using Jensen inequality. Nevertheless, the numerical results indicate that Theorem 1 and Theorem 2 give $\bar{h}_{\max}$ greater than those provided by the criteria in Qiu and Zhang (2011). In view of criterion in Wu et al. (2009) and our proposed criteria, it is obvious that the criteria with $N = 3$ is less conservative than that with $N = 2$.

In view of numerical results, exploiting the integral of the sector-bounded nonlinearities in the LKF can further increase the maximum allowable time delays better than other criteria. The delay partitioning approach can diminish the conservatism of stability criteria, and by increasing the number of partition, N, the computed $\bar{h}_{\max}$ approaches the actual bound. When choosing N for certain applications, it has to be traded off between the desired conservatism and the computational time. When the sector bound is large, which is harmful to the stability of the system, it is clear to see that the proposed criteria give a larger value of $\bar{h}_{\max}$. Thus, the proposed stability criteria using the LKF with integral of nonlinearities significantly reduce the conservatism.

Further research should be done to consider the nonuniform fragmentation schemes, since they can reduce the conservatism in certain cases (Briat, 2011). In general, there is no common nonuniform partitioning which is optimal for all systems. Some attempts have been made for the case of two partitions by considering the length of the first partition as the variable to be optimized (Gao et al., 2008; Kazemy and Farrokhi, 2013), but there is no systematic procedure to determine optimal delay partitions.

3.6 Summary

In this chapter, we propose the improved criterion of the delay-dependent absolute stability for Lur'e systems with uncertain time-invariant delays. The criterion is derived based on the Lyapunov-Krasovskii functional with the integral of sector-bounded nonlinearities. The length of time delay is uniformly divided into the number of fragments so that the Lyapunov functional terms involving delay are partitioned to be associated with each fragment. The absolute stability criterion is derived from time derivative of the new Lyapunov-Krasovskii functional, then the Jensen inequality and $\mathcal{S}$-procedure are applied to formulate the sufficient condition in terms of LMIs. Furthermore, the criterion for LSTD subject to norm-bounded uncertainty is developed by using the eliminating lemma.

The partition of the time delay has advantage to reduce the gap of Jensen inequality, and utilizing the integral of sector-bounded nonlinearities can improve the conservatism of the stability criteria. Therefore, the combination of these approaches effectively provides the less conservative absolute stability criteria. Numerical examples show that the proposed criteria can enlarge the maximum allowable time delay comparing to other existing criteria especially when the sector bound is large.

CHAPTER IV

WORST-CASE $\mathcal{H}_\infty$ PERFORMANCE ANALYSIS

4.1 Introduction

This chapter presents a new worst-case $\mathcal{H}_\infty$ performance criterion for LSTD. The worst-case $\mathcal{H}_\infty$ performance is defined by $\mathcal{L}_2$-gain of nonlinear systems which represents the ratio between output energy and input energy. Basically, the worst-case $\mathcal{H}_\infty$ performance analysis problem can be formulated by combining the absolute stability analysis in the previous chapter and robust performance bounds on the total output energy of the system (Boyd et al., 1994). In this chapter, we develop the sufficient conditions for calculating an upper bound of the worst-case performance of both nominal and uncertain systems. The problem can be cast as a minimizing linear objective under LMI constraints which is a standard LMI problem and can be efficiently solved. To the best of our knowledge, there is no work on how to compute the worst-case $\mathcal{H}_\infty$ performance for LSTD. We develop performance analysis criteria by applying n-segmentation technique in Wu et al. (2009) and integral-equality approach in Qiu and Zhang (2011). In order to illustrate the effectiveness of the proposed criterion, the comparison between the proposed technique and other existing criteria is made on two numerical examples.

The organization of this chapter is as follows. In §4.2, we state the definition of the worst-case $\mathcal{H}_\infty$ performance and the worst-case performance problems. Then, in §4.3, we present the theorems for computing an upper of the worst-case performance for both nominal and uncertain LSTD. Lastly, the numerical examples show the comparison of the results between the proposed criteria and some existing criteria in §4.4.

4.2 Problem Statement

We consider Lur'e systems with uncertain time-invariant state delay described as follows.

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_1 x(t-h) + B_p p(t) + B_w w(t), \\
q(t) &= C_q x(t), \\
z(t) &= C_z x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{4.1}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and zero initial condition $x(t) = 0$, $\forall t \in [-h, 0]$. A scalar $h \in \mathbf{R}_+$ is a time-invariant time delay in the state, $x(t) \in \mathbf{R}^n$ is the state variable, $w(t) \in \mathbf{R}^{n_w}$ is the disturbance input which belongs to $\mathcal{L}_2$, $z(t) \in \mathbf{R}^{n_z}$ is the performance output, $q(t) \in \mathbf{R}^{n_p}$, and $p(t) \in \mathbf{R}^{n_p}$ are the input/output of a vector mapping of sector-bounded nonlinearities denoted by ϕ. The vector function ϕ belongs to the set Φ characterized by memoryless, time-invariant nonlinearities satisfying certain sector conditions. In particular, given an input vector $\sigma := [\sigma_1, \ldots, \sigma_{n_p}]^T$, a lower bound vector $l := [l_1, \ldots, l_{n_p}]^T$ and an upper bound vector $m := [m_1, \ldots, m_{n_p}]^T$, with $l_i < m_i$ for all $i = 1, \ldots, n_p$, the set Φ can be described by

$$\begin{aligned}\Phi(l,m) :=& \{\phi : \mathbf{R}^{n_p} \to \mathbf{R}^{n_p} : \phi(\sigma) = \left[\phi_1(\sigma_1), \ldots, \phi_{n_p}(\sigma_{n_p})\right]^T, \\ & l_i\sigma_i^2 \leq \sigma_i\phi_i(\sigma_i) \leq m_i\sigma_i^2, \text{ for all } i = 1, \ldots, n_p\}.\end{aligned}$$

In addition, the pairs (A, B_p) and (C_q, A) are assumed to be controllable and observable, respectively. Next, the definitions of $\mathcal{L}_2$-stability and the worst-case $\mathcal{H}_\infty$ performance for the system (4.1) are introduced.

Definition 5 *A causal operator $H : \mathbf{R}^n \to \mathbf{R}^n$ is said to be $\mathcal{L}_2$-stable if there exist $\gamma \geq 0$ and β such that*

$$\|Hw\| \leq \gamma\|w\| + \beta, \quad \forall w \in \mathcal{L}_2.$$

Definition 6 *Assume that the system (4.1) is $\mathcal{L}_2$-stable with finite gain and zero bias. The worst-case $\mathcal{H}_\infty$ performance of the system (4.1) is defined by its $\mathcal{L}_2$-gain described as follows.*

$$\mathcal{J}_\infty := \sup_{w \in \mathcal{L}_2, w \neq 0} \frac{\|z\|}{\|w\|}, \tag{4.2}$$

where the supremum is taken over all nonzero output trajectories of the system (4.1) under zero initial condition.

While the actual value of $\mathcal{J}_\infty$ is difficult to compute, its upper bound can be calculated from the following minimization problem.

$$\begin{aligned} &\text{minimize} && \gamma_\infty^2 \\ &\text{subject to} && \dot{V}(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) \leq 0, \end{aligned}$$

where $V(x_t)$ denotes the Lyapunov functional candidate. Therefore, the worst-case performance analysis problem can be stated as follows.

Problem 3 *The worst-case $\mathcal{H}_\infty$ performance analysis problem for LSTD (4.1) is to determine an upper bound of $\mathcal{J}_\infty$ of the system (4.1) for any time-invariant time delay $h \in (0, \bar{h}]$, i.e., for a given $\bar{h}$, determine $\gamma_\infty \in \mathbf{R}_+$ such that $\mathcal{J}_\infty \leq \gamma_\infty$.*

In this chapter, we also consider the LSTD with norm-bounded uncertainty described by the following equations.

$$\begin{aligned}
\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) \\
&\quad + [B_p + \Delta B_p(t)]p(t) + B_w w(t), \\
q(t) &= C_q x(t), \\
z(t) &= C_z x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{4.3}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and the initial condition $x(t) = 0$, $\forall t \in [-h, 0]$. The uncertain variables $\Delta A(t)$, $\Delta A_1(t)$, and $\Delta B_p(t)$ are time-varying, but norm-bounded. The uncertainties are assumed to be of the form.

$$\begin{aligned}
\Delta A(t) &= DF(t)E_0, \\
\Delta A_1(t) &= DF(t)E_1, \\
\Delta B_p(t) &= DF(t)E_2,
\end{aligned} \tag{4.4}$$

where D, E_0, E_1, and E_2 are known constant real matrices of appropriate dimensions, and represent the structure of uncertainties, and $F(t)$ is an unknown matrix function with *Lebesgue measurable elements* satisfying the constraint

$$F^T(t)F(t) \leq I. \tag{4.5}$$

Similar to the problem for nominal systems (4.1), the worst-case $\mathcal{H}_\infty$ performance for uncertain LSTD (4.3) can be stated as follows.

Problem 4 *The worst-case $\mathcal{H}_\infty$ performance analysis problem for LSTD (4.3) is to determine an upper bound of $\mathcal{J}_\infty$ of the system (4.3) for any time-invariant time delay $h \in (0, \bar{h}]$ and all uncertain matrix function $F(t)$ satisfying (4.5), i.e., for a given $\bar{h}$, determine $\gamma_\infty \in \mathbf{R}_+$ such that $\mathcal{J}_\infty \leq \gamma_\infty$.*

4.3 Worst-case $\mathcal{H}_\infty$ Performance Criteria

In this section, we present the sufficient conditions for computing an upper bound of the worst-case $\mathcal{H}_\infty$ performance for both nominal and uncertain LSTD. The criteria are express in terms of LMIs, and an upper bound γ_∞ can be computed from the linear objective minimization over LMI constraints.

4.3.1 Nominal Systems

The following Theorem provides a sufficient condition to compute an upper bound of $\mathcal{J}_\infty$ of the system (4.1) for a given bound of time delay.

Theorem 3 *For a given $\bar{h} \in \mathbf{R}_+$, an upper bound of the worst-case $\mathcal{H}_\infty$ performance of LSTD (4.1) for any time-invariant time delay $h \in (0, \bar{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (4.6) over symmetric matrices $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \dots, N$, and diagonal matrices $\Lambda \geq 0$, $T \geq 0$.*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & \Psi_{24} \\ * & * & \Psi_{33} & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} \leq 0, \tag{4.6}$$

where

$$\Psi_{11} = \begin{bmatrix} \begin{pmatrix} PA + A^T P + Q_1 \\ -R_1 + C_z^T C_z \end{pmatrix} & PA_1 & \begin{pmatrix} PB_p + C_q^T T \\ +A^T C_q^T \Lambda \end{pmatrix} \\ * & -Q_N - R_N & A_1^T C_q^T \Lambda \\ * & * & \begin{pmatrix} B_p^T C_q^T \Lambda \\ +\Lambda C_q B_p - 2T \end{pmatrix} \end{bmatrix},$$

$$\Psi_{12} = \begin{bmatrix} PB_w \\ 0 \\ \Lambda C_q B_w \end{bmatrix}, \quad \Psi_{13} = \begin{bmatrix} R_1 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \end{bmatrix} + \begin{bmatrix} 0_{n\times(N-2)n} & 0 \\ 0_{n\times(N-2)n} & R_N \\ 0_{n\times(N-2)n} & 0 \end{bmatrix},$$

$$\Psi_{14} = \begin{bmatrix} \frac{\bar{h}}{N} A^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} A_1^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} B_p^T \sum_{k=1}^N R_k \end{bmatrix}, \quad \Psi_{22} = -\gamma_\infty^2 I, \quad \Psi_{24} = \frac{\bar{h}}{N} B_w^T \sum_{k=1}^N R_k,$$

$$\Psi_{33} = \begin{bmatrix} \Gamma_1 & R_2 & & \\ * & \Gamma_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & \Gamma_{N-1} \end{bmatrix}, \ \Psi_{44} = -\sum_{k=1}^N R_k,$$

with $\Gamma_k := -Q_k + Q_{k+1} - R_k - R_{k+1}$.

Proof: Assume that $A + A_1$ is Hurwitz, and system matrices (A, B_p, C_q) are minimal realization. Consider the LKF candidate of the form

$$V(x_t) := V_1 + V_2 + V_3 + V_4, \tag{4.7}$$

with

$$\begin{aligned}
V_1 &= x^T(t)Px(t), \\
V_2 &= 2\sum_{i=1}^{n_p} \lambda_i \int_0^{q_i} \phi_i(\sigma)d\sigma, \\
V_3 &= \sum_{k=1}^{N} \int_{t-kr}^{t-(k-1)r} x^T(\xi)Q_k x(\xi)d\xi, \\
V_4 &= r\sum_{k=1}^{N} \int_{-kr}^{-(k-1)r} \int_{t+\theta}^{t} \dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi d\theta,
\end{aligned}$$

where P, $Q_1, \ldots, Q_N$, and $R_1, \ldots, R_N$ are positive definite symmetric matrices of dimension $n \times n$, scalars $\lambda_1, \ldots, \lambda_{n_p}$ are non-negative, x_t denotes a piece of trajectory $x(t+\theta)$ for $-h \leq \theta \leq 0$, and $r = h/N$. If there exists an LKF of the form (4.7) and $\gamma_\infty \in \mathbf{R}_+$ such that

$$\dot{V}(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) \leq 0, \tag{4.8}$$

for all x_t satisfying the system equations (4.1), then $\mathcal{J}_\infty \leq \gamma_\infty$. Therefore, we seek P, $Q_1, \ldots, Q_N$, $R_1, \ldots, R_N$, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n_p})$ such that the constraint (4.8) is satisfied for all nonzero x_t satisfying (4.1) with a set of sector-bounded conditions:

$$0 \leq q_i(t)\phi_i(q_i(t)) \leq q_i^2(t), \quad \forall i = 1, \ldots, n_p. \tag{4.9}$$

To verify (4.8) under the set of constraints (4.9), we apply $\mathcal{S}$-procedure Yakubovich (1977) to establish the *sufficient condition* as follows.

$$\dot{V}(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) - \sum_{i=1}^{n_p} \tau_i q_i(t)(p_i(t) - q_i(t)) \leq 0, \tag{4.10}$$

where $\tau_1 \geq 0, \ldots, \tau_{n_p} \geq 0$. Note that for the case of single nonlinearity ($n_p = 1$), $\mathcal{S}$-procedure is lossless, and the condition (4.10) is not only sufficient but also necessary for (4.8). By defining $T = \mathrm{diag}(\tau_1, \ldots, \tau_{n_p})$, the inequality (4.10) can be written in vector-matrix notation as follows.

$$\dot{V}(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) - 2p^T(t)Tp(t) + 2x^T(t)C_q^T Tp(t) \leq 0. \tag{4.11}$$

The derivative of each term in the LKF (4.7) with respect to time along the solution of (4.1) is given by

$$
\begin{aligned}
\dot{V}_1 &= x^T(t)\left[A^TP+PA\right]x(t)+2x^T(t)PA_1x(t-h)+2x^T(t)PB_pp(t)+2x^T(t)PB_ww(t),\\
\dot{V}_2 &= 2x^T(t)A^TC_q^T\Lambda p(t)+2x^T(t-h)A_1^TC_q^T\Lambda p(t)+p^T(t)\left[B_p^TC_q^T\Lambda+\Lambda C_qB_p\right]p(t)\\
&\quad +\Lambda C_qB_ww(t),\\
\dot{V}_3 &= x^T(t)Q_1x(t)-x^T(t-h)Q_Nx(t-h)+\sum_{k=1}^{N-1}\left[x^T(t-kr)(-Q_k+Q_{k+1})x(t-kr)\right],\\
\dot{V}_4 &= \dot{x}^T(t)\sum_{k=1}^{N}\left(r^2R_k\right)\dot{x}(t)-r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r}\dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi.
\end{aligned}
$$

Note that $\dot{V}_4$ cannot be formulated in terms of LMI. Therefore, we rewrite the first term by using the state equation in (4.1) as the following.

$$
\dot{x}^T(t)\sum_{k=1}^{N}\left(r^2R_k\right)\dot{x}(t)=\mathbf{x}_1^T\begin{bmatrix}A^T\\A_1^T\\B_p^T\\B_w^T\end{bmatrix}r^2\sum_{k=1}^{N}R_k\begin{bmatrix}A^T\\A_1^T\\B_p^T\\B_w^T\end{bmatrix}^T\mathbf{x}_1, \tag{4.12}
$$

where $\mathbf{x}_1=\begin{bmatrix}x^T(t) & x^T(t-h) & p^T(t) & w^T(t)\end{bmatrix}^T$, and employ Jensen inequality (Lemma 1) to bound the integral terms appeared in $\dot{V}_4$.

$$
\begin{aligned}
-r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r}\dot{x}^T(\xi)R_k\dot{x}(\xi)d\xi &\le \sum_{k=1}^{N}\left(\int_{t-kr}^{t-(k-1)r}\dot{x}(\xi)d\xi\right)^T(-R_k)\left(\int_{t-kr}^{t-(k-1)r}\dot{x}(\xi)d\xi\right),\\
&= \sum_{k=1}^{N}\left[x(t-(k-1)r)-x(t-kr)\right]^T(-R_k)\times\\
&\qquad \left[x(t-(k-1)r)-x(t-kr)\right],\\
&= 2\mathbf{x}_1^T\Psi_{13}\mathbf{x}_2+\mathbf{x}_1^T\begin{bmatrix}-R_1 & 0 & 0\\ * & -R_N & 0\\ * & * & 0\end{bmatrix}\mathbf{x}_1\\
&\quad +\mathbf{x}_2^T\begin{bmatrix}-R_1-R_2 & R_2 & & \\ * & -R_1-R_2 & \ddots & \\ & \ddots & \ddots & R_{N-1}\\ & & * & -R_{N-1}-R_N\end{bmatrix}\mathbf{x}_2,
\end{aligned}
\tag{4.13}
$$

where $\mathbf{x}_2 = \begin{bmatrix} x^T(t-r) & \cdots & x^T(t-h+r) \end{bmatrix}^T$, and the entries left blank are zero. Substituting $\dot{V}_1$, $\dot{V}_2$, $\dot{V}_3$ and $\dot{V}_4$ into inequality (4.11) and applying (4.12) and the upper bound (4.13) for $\dot{V}_4$, the sufficient condition for (4.11) is given as follows.

$$\mathbf{x}^T\Psi(h)\mathbf{x} \leq 0, \tag{4.14}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, and

$$\Psi(h) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & 0 \\ * & * & \Psi_{33} \end{bmatrix} - \begin{bmatrix} \Psi_{14} \\ \Psi_{24} \\ 0 \end{bmatrix} \Psi_{44}^{-1} \begin{bmatrix} \Psi_{14} \\ \Psi_{24} \\ 0 \end{bmatrix}^T. \tag{4.15}$$

Inequality (4.14) holds for all $\mathbf{x} \neq \mathbf{0}$ if and only if the following matrix inequality is satisfied.

$$\Psi(h) \leq 0. \tag{4.16}$$

Lastly, applying Schur complement (Boyd et al., 1994, pp. 7–8) and substituting h with $\bar{h}$, the matrix inequality (4.16) is equivalent to (4.6).

It is important to note that the sufficient condition (4.6) guarantees the upper bound γ_∞ for the case of $h = \bar{h}$. Next, we show that (4.6) also guarantees the same γ_∞ for the system (4.1) for any time-invariant time delay $h \in (0, \bar{h}]$. Let $h = \bar{h} - \Delta h$, where $0 < \Delta h < \bar{h}$. Clearly, h lies in the interval $(0, \bar{h})$. By substituting h with $\bar{h} - \Delta h$, and isolating all terms involving Δh, the matrix inequality (4.15) becomes

$$\Psi(h) = \Psi(\bar{h}) + \frac{(2\Delta h\bar{h} - \Delta h^2)}{N^2} \begin{bmatrix} \Psi_{14} \\ \Psi_{24} \\ 0 \end{bmatrix} \Psi_{44}^{-1} \begin{bmatrix} \Psi_{14} \\ \Psi_{24} \\ 0 \end{bmatrix}^T \leq 0. \tag{4.17}$$

We observe that $(2\Delta h\bar{h} - \Delta h^2)/N^2 > 0$ and $\Psi_{44} < 0$. Then, the isolated terms are negative. Thus, if $\Psi(\bar{h}) \leq 0$ holds, $\Psi(h) \leq 0$ also holds. In other words, the matrix inequality $\Psi(\bar{h}) \leq 0$ implies (4.17), and Theorem 3 guarantees the upper bound γ_∞ of the system (4.1) for any time-invariant time delay $h \in (0, \bar{h}]$. This completes the proof. ■

It is appeared that the condition (4.6) is LMI over matrix variables P, $Q_1, \ldots, Q_N$, $R_1, \ldots, R_N$, Λ, and T for a given $\bar{h} \in \mathbf{R}_+$. Hence, the problem of minimizing γ_∞^2 subject to (4.6) can be cast as a minimization problem under LMI constraints.

4.3.2 Uncertain Systems

Next, we will derive a sufficient condition to compute an upper bound of $\mathcal{J}_\infty$ of the system (4.3) for a given bound of time delay.

Theorem 4 *For a given $\bar{h} \in \mathbf{R}_+$, an upper bound of the worst-case $\mathcal{H}_\infty$ performance of the systems (4.3) for any time-invariant time delay $h \in (0, \bar{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (4.18) over symmetric matrices $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \ldots, N$, and diagonal matrices $\Lambda \geq 0$, $T \geq 0$, and a scalar variable $\epsilon > 0$.*

$$\begin{bmatrix} \tilde{\Psi}_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} \\ * & \Psi_{22} & 0 & \Psi_{24} & 0 \\ * & * & \Psi_{33} & 0 & 0 \\ * & * & * & \Psi_{44} & \Psi_{45} \\ * & * & * & * & \Psi_{55} \end{bmatrix} \leq 0, \tag{4.18}$$

where

$$\tilde{\Psi}_{11} = \Psi_{11} + \epsilon \begin{bmatrix} E_0 \\ E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} E_0 & E_1 & E_2 \end{bmatrix}, \quad \Psi_{15} = \begin{bmatrix} PD \\ 0 \\ \Lambda C_q D \end{bmatrix},$$

$$\Psi_{45} = \frac{\bar{h}}{N} \left(\sum_{k=1}^{N} R_k \right) D, \quad \Psi_{55} = -\epsilon I,$$

and Ψ_{11}, Ψ_{12}, Ψ_{13}, Ψ_{14}, Ψ_{22}, Ψ_{24}, Ψ_{33}, and Ψ_{44} are the same as defined in Theorem 3.

Proof: By applying Theorem 3 to uncertain systems (4.3), the worst-case $\mathcal{H}_\infty$ performance criteria consist of the following LMI.

$$\Psi + \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix} F(t) \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix} F^T(t) \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix}^T \leq 0, \tag{4.19}$$

where Ψ is defined as

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & \Psi_{24} \\ * & * & \Psi_{33} & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} \leq 0,$$

It follows from Lemma 2 that the matrix inequality (4.19) is true for all uncertain matrix $F(t)$ satisfying $F^T(t)F(t) \leq I$ if and only if there exists a scalar $\epsilon > 0$ such that

$$\Psi + \epsilon^{-1} \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix} \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix}^T + \epsilon \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix}^T \leq 0. \tag{4.20}$$

Absorbing the last term into Ψ and applying the *Schur complement* Boyd et al. (1994), the matrix inequality (4.18) holds. This completes the proof. ■

Remark 5 *It is observed that the terms involving ϵ and ϵ^{-1} in the matrix inequality (4.20) are all positive. Then, the feasible set for (4.18) is smaller than that of (4.6), and the γ_∞ obtained for the uncertain LSTD (4.3) should be greater than that for the nominal LSTD.*

4.4 Numerical Examples

To the best of our knowledge, there is no work on how to compute the worst-case $\mathcal{H}_\infty$ performance for LSTD. In order to illustrate the effectiveness of the proposed criteria, we develop performance analysis criteria along with n-segmentation technique in Wu et al. (2009) and integral-equality approach in Qiu and Zhang (2011). The conservatism of each criterion is compared using two numerical examples. Note that the LMI Lab (Gahinet et al., 1995) which employs the projective interior-point method (Gahinet and Nemirovski, 1997), is used for solving the LMI minimization problems.

Example 3: Consider the systems (4.3) with the following parameters.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0.6 & 0.8 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi \in \Phi(0.35 - \delta l, 0.35 + \delta l), \quad \bar{h} = 1,$$

$$\|\Delta A(t)\| \leq \alpha, \quad \|\Delta A_1(t)\| \leq \alpha, \quad \Delta B_p(t) = 0,$$

where δl characterizes the sector bound of nonlinearity and α represents magnitude of the uncertainty. The system data is taken from Han (2005) with slight modifications. The uncertainty model $\Delta A(t)$, $\Delta A_1(t)$, and $\Delta B_p(t)$ can be described by (4.4) with $D = \alpha I$, $E_0 = E_1 = I$, and $E_2 = 0$. Loop transformation is applied so that LSTD with $\phi \in \Phi(0.35 - \delta l, 0.35 + \delta l)$ is transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(0, 1)$. We compute γ_∞ using Theorem 3 and Theorem 4 with different sector bound and uncertainty.

We compare the results of γ_∞ with that computed by other two delay partitioning methods based on the work of Wu et al. (2009) and Qiu and Zhang (2011). In Wu et al. (2009); Qiu and Zhang (2011), there are only absolute stability tests of LSTD, so we extend their methods to analyze the worst-case $\mathcal{H}_\infty$ performance. Note that the LKF used in Wu et al. (2009) does not have the integral of nonlinearities, whereas the LKF based on Qiu and Zhang (2011) and our proposed criteria utilize this integral. In Wu *et al.*'s and the proposed criteria, the delay interval is partitioned into N equidistant fragments. However, in Qiu and Zhang (2011), the delay interval is divided into two unequal subintervals, $[-h, -h/3]$ and $[-h/3, 0]$. Figure 4.1 shows γ_∞ versus δl in Example 3. It is observed that γ_∞ is increased as the sector bound δl is increased. Likewise, γ_∞ grows up as the uncertainty α is enlarged. Using Theorem 3 and Theorem 4 always gives smaller γ_∞ when compared with those obtained from other criteria, especially for large δl. Moreover, the proposed criteria can compute γ_∞ for a wider sector bound δl.

Example 4: Consider the systems (4.3) with the following parameters.

$$A = \begin{bmatrix} -2.0 & -1.0 \\ 0.5 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 1.0 \\ -0.1 & -0.8 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{h} = 1,$$

$$\phi \in \Phi\left(\begin{bmatrix} 0.25 - \delta l \\ 0.35 - \delta l \end{bmatrix}, \begin{bmatrix} 0.25 + \delta l \\ 0.35 + \delta l \end{bmatrix}\right), \quad \|\Delta A(t)\| \leq \alpha, \quad \|\Delta A_1(t)\| \leq \alpha, \quad \Delta B_p(t) = 0.$$

This example is modified from the system given in Han (2005). Similar to the previous example,

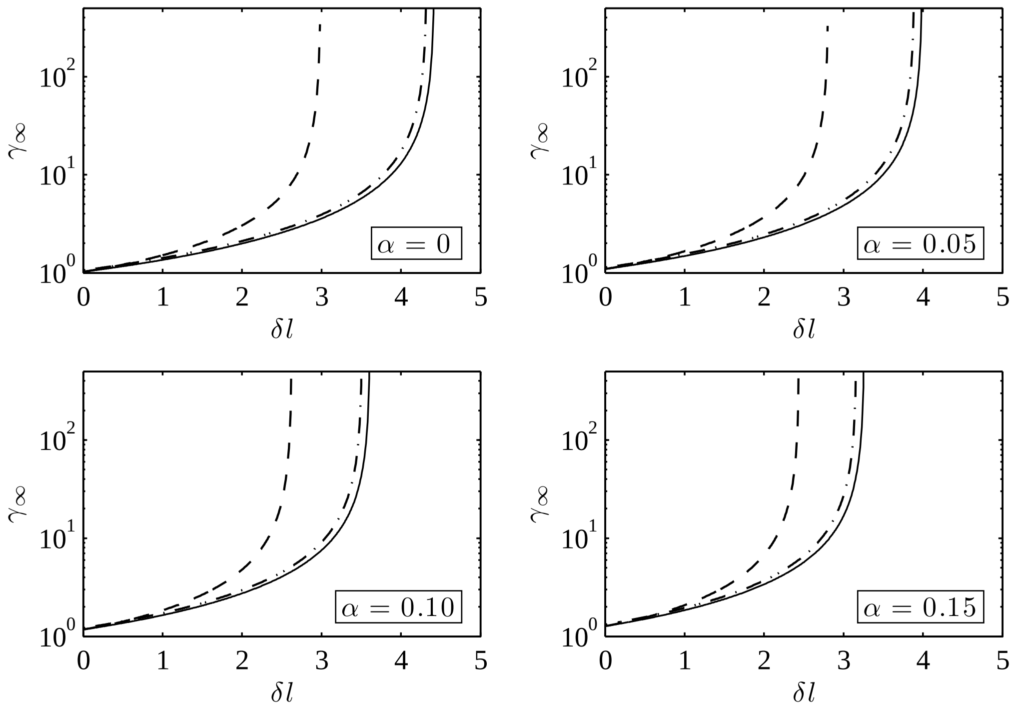

Figure 4.1: Upper bounds of $\mathcal{J}_\infty$ vs. δl for Example 3: Theorem 3/Theorem 4 with $N = 3$ (solid lines), Wu et al. (2009) with $N = 3$ (dashed lines), Qiu and Zhang (2011) (dash-dot lines).

the norm-bounded uncertainty can be described by (4.4) with $D = \alpha I$, E_0, E_1 are identity matrices of appropriate dimension, and E_2 is a null matrix. Loop transformation is applied so that LSTD with ϕ defined above is transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. The computed γ_∞ versus δl for Example 4 are shown in Figure 4.2. It can be seen that Theorem 3 and Theorem 4 always give the upper bound of the worst-case $\mathcal{H}_\infty$ performance less than the results from other criteria. In addition, Theorem 3 and 4 are capable of finding γ_∞ for a wider sector bound δl. These results indicate that the proposed worst-case $\mathcal{H}_\infty$ performance criteria have the advantage over other criteria.

4.5 Discussion

The extended method of Wu et al. (2009) provides γ_∞ greater than that obtained from Theorem 3 and Theorem 4. It can be viewed that Wu *et al.*'s criteria are special cases of our proposed criteria. On the other hand, the delay partitioning proposed by Qiu and Zhang (2011) divides the delay interval into only two segments, so it gives γ_∞ greater than that obtained from our proposed criteria with $N = 3$. Therefore, the proposed criteria are less conservative than the

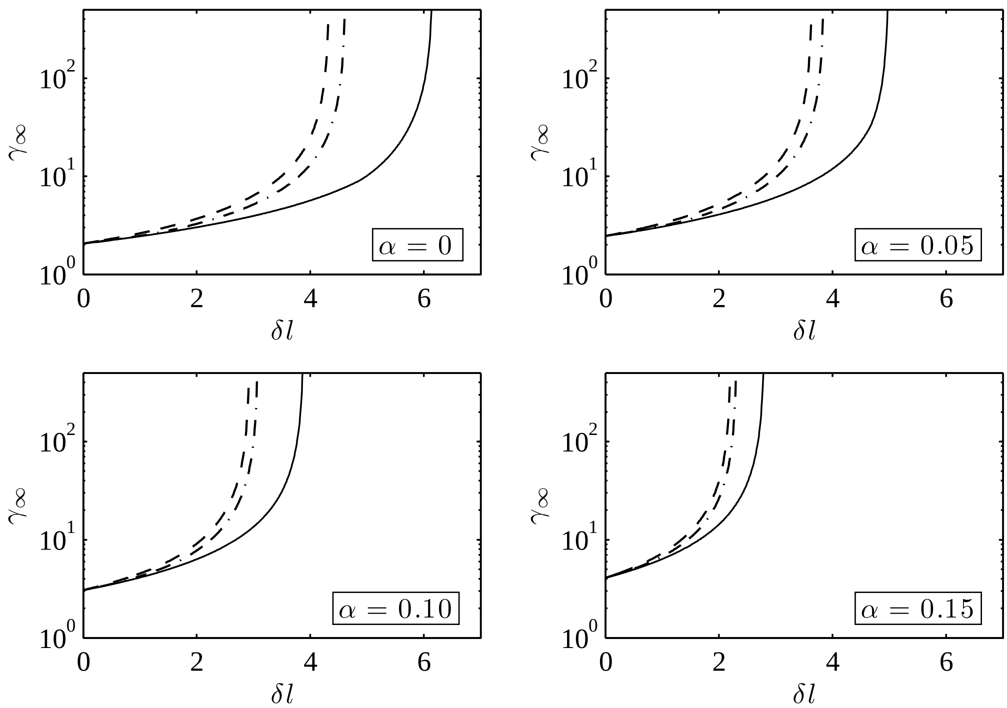

Figure 4.2: Upper bounds of $\mathcal{J}_\infty$ vs. δl for Example 4: Theorem 3/Theorem 4 with $N = 3$ (solid lines), Wu et al. (2009) with $N = 3$ (dashed lines), Qiu and Zhang (2011) (dash-dot lines).

comparative criteria. In addition, from the numerical results, we observe that γ_∞ is increased as the sector-bound is increased.

4.6 Summary

In this chapter, we present the worst-case $\mathcal{H}_\infty$ performance criterion for Lur'e systems with uncertain time-invariant delays. The worst-case performance analysis criterion is formulated by combining the absolute stability analysis and robust performance bounds on the total output energy. The sufficient conditions to ensure the worst-case performance for both nominal and uncertain LSTD are derived along with the proof of the absolute stability criterion in the previous chapter. In addition, the problem of calculating an upper bound of the worst-case $\mathcal{H}_\infty$ performance is cast as a linear objective minimization over LMI constraints. Numerical examples show that the proposed criteria are less conservative than the comparative criteria, and can be served as an effective worst-case performance analysis for LSTD.

CHAPTER V

STATE-FEEDBACK CONTROL DESIGNS

5.1 Introduction

In this chapter, we present design of robust state-feedback stabilization and design of robust state-feedback $\mathcal{H}_\infty$ control for LSTD with time-invariant delays and norm-bounded uncertainty. Both designs use Lyapunov-Krasovskii Theorem with the delay-partitioning Lyapunov-Krasovskii functional and the integral of sector-bounded nonlinearities. The criterion of state-feedback stabilization is established from the absolute stability criterion in §3, and it is cast as a BMI feasibility problem. By modifying problem to be a minimization problem over BMI, we develop the iterative LMI algorithm to solve for the robust state-feedback stabilizations. In addition, the criterion of state-feedback $\mathcal{H}_\infty$ control is developed from the worst-case $\mathcal{H}_\infty$ performance analysis in §4, and it is directly formulated as a linear objective minimization problem over BMI. However, we propose algorithms based on coordinate optimization, namely "V-K" iteration (El Ghaoui and Balakrishnan, 1994), which alternate between two LMI optimization problems, to solve for the robust state-feedback controllers. Two numerical examples are used for illustrating the proposed robust state-feedback stabilization and $\mathcal{H}_\infty$ control.

The organization of this chapter is as follows. In §5.2, we present state-feedback stabilization problem, and give the existence condition to determine a stabilization controller gain. For the $\mathcal{H}_\infty$ control problem, we propose the Theorem for computing a controller gain to guarantee an upper bound of the worst-case performance in §5.3. Lastly, the numerical examples show the comparison of the results between various cases of the proposed state-feedback design in §5.4.

5.2 State-feedback Stabilization

This section introduces the robust stabilization problem with state-feedback control. The existence condition for determining a static state-feedback gain is directly extended from the absolute stability analysis in §3. Although, the design criterion is in terms of BMI, an algorithm based on iterative LMI method is proposed for solving the problem.

Consider LSTD with a time-invariant delay described as follows.

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) \\ &\quad + [B_p + \Delta B_p(t)]p(t) + B_u u(t), \\ q(t) &= C_q x(t), \\ p(t) &= \phi(q(t)), \end{aligned} \tag{5.1}$$

with sector-bounded nonlinearities $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and the initial condition $x(t) = \varphi(t)$, $\forall t \in [-h, 0]$, where $h \in \mathbf{R}_+$ is a time-invariant time delay in the state. The vector $x(t) \in \mathbf{R}^n$ is the state, $q(t)$ and $p(t) \in \mathbf{R}^{n_p}$ are the input and output of ϕ, which is a vector mapping of sector-bounded nonlinearities belongs to the set Φ characterized by memoryless, time-invariant nonlinearities satisfying certain sector conditions. In particular, given an input vector $\sigma := [\sigma_1, \ldots, \sigma_{n_p}]^T$, a lower bound vector $l := [l_1, \ldots, l_{n_p}]^T$ and an upper bound vector $m := [m_1, \ldots, m_{n_p}]^T$, with $l_i < m_i$ for all $i = 1, \ldots, n_p$, the set Φ can be described by

$$\begin{aligned} \Phi(l, m) :=& \{\phi : \mathbf{R}^{n_p} \to \mathbf{R}^{n_p} : \phi(\sigma) = \left[\phi_1(\sigma_1), \ldots, \phi_{n_p}(\sigma_{n_p})\right]^T, \\ & l_i \sigma_i^2 \le \sigma_i \phi_i(\sigma_i) \le m_i \sigma_i^2, \text{ for all } i = 1, \ldots, n_p\}. \end{aligned}$$

In addition, the (A, B_p) pair and (C_q, A) pair are assumed to be controllable and observable, respectively. The uncertainty considered here is assumed to be norm-bounded and has the following form

$$\begin{bmatrix} \Delta A(t) \\ \Delta A_1(t) \\ \Delta B_p(t) \end{bmatrix} = D\,F(t) \begin{bmatrix} E_0 \\ E_1 \\ E_2 \end{bmatrix}, \tag{5.2}$$

where D, E_0, E_1, and E_2 are known constant real matrices of appropriate dimensions, which represent structure and magnitude of the uncertainties, and $F(t)$ is an unknown matrix function with *Lebesgue measurable elements* satisfying $F^T(t)F(t) \le I$. The definition above can be used to efficiently describe norm-bounded uncertainty. For example, the uncertain matrix $\Delta A(t)$ satisfying norm-bounded condition $\|\Delta A(t)\| \le \alpha$ can be represented in the form of (5.2) with $D = \alpha I$ and $E_0 = I$. Next, by assuming the state $x(t)$ is measurable and the pair (A, B_u) is controllable, the robust stabilization problem of LSTD (5.1) is defined as follows.

Problem 5 *Given a maximum allowable time delay, $\bar{h}$, the state-feedback stabilization problem is to search for the control law*

$$u(t) = Kx(t), \tag{5.3}$$

which stabilizes LSTD (5.1) for any time-invariant time delay $h \in (0, \bar{h}]$.

By substituting (5.2) and (5.3) into LSTD (5.1), we obtain the closed-loop LSTD as follows

$$\begin{aligned} \dot{x}(t) &= [A + B_u K + DF(t)E_0]x(t) + [A_1 + DF(t)E_1]x(t-h) \\ &\quad + [B_p + DF(t)E_2]p(t), \\ q(t) &= C_q x(t), \\ p(t) &= \phi(q(t)), \end{aligned} \tag{5.4}$$

with sector-bounded nonlinearities $\phi \in \Phi(\mathbf{0}, \mathbf{1})$. The following theorem gives a sufficient condition for the existence of a state-feedback gain K to guarantee the robust absolute stability of LSTD (5.4).

Theorem 5 *For a given $\bar{h} \in \mathbf{R}_+$, a state-feedback stabilization gain K for the closed-loop LSTD (5.4) with a time-invariant delay, $h \in (0, \bar{h}]$, can be computed by solving the feasibility problem (5.5) over variables K, $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \ldots, N$, $\Lambda \geq 0$, $T \geq 0$, and a scalar $\epsilon > 0$.*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & 0 \\ * & * & \Psi_{33} & \Psi_{34} \\ * & * & * & \Psi_{44} \end{bmatrix} < 0, \tag{5.5}$$

where

$$\Psi_{11} = \begin{bmatrix} \begin{pmatrix} P(A+B_uK) \\ +(A+B_uK)^T P \\ +Q_1 - R_1 + \epsilon E_0^T E_0 \end{pmatrix} & \begin{pmatrix} PA_1 \\ +\epsilon E_0^T E_1 \end{pmatrix} & \begin{pmatrix} PB_p + C_q^T T \\ +(A+B_uK)^T C_q^T \Lambda \\ +\epsilon E_0^T E_2 \end{pmatrix} \\ * & \begin{pmatrix} -Q_N - R_N \\ +\epsilon E_1^T E_1 \end{pmatrix} & \begin{pmatrix} A_1^T C_q^T \Lambda \\ +\epsilon E_1^T E_2 \end{pmatrix} \\ * & * & \begin{pmatrix} B_p^T C_q^T \Lambda + \Lambda C_q B_p \\ -2T + \epsilon E_2^T E_2 \end{pmatrix} \end{bmatrix},$$

$$\Psi_{12} = \begin{bmatrix} R_1 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \end{bmatrix} + \begin{bmatrix} 0_{n\times(N-2)n} & 0 \\ 0_{n\times(N-2)n} & R_N \\ 0_{n\times(N-2)n} & 0 \end{bmatrix}, \quad \Psi_{22} = \begin{bmatrix} \Gamma_1 & R_2 & & \\ * & \Gamma_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & \Gamma_{N-1} \end{bmatrix},$$

$$\Psi_{13} = \begin{bmatrix} \frac{\bar{h}}{N}(A + B_u K)^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} A_1^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} B_p^T \sum_{k=1}^N R_k \end{bmatrix}, \quad \Psi_{14} = \begin{bmatrix} PD \\ 0 \\ \Lambda C_q D \end{bmatrix}, \quad \Psi_{33} = -\sum_{k=1}^N R_k,$$

$$\Psi_{34} = \frac{\bar{h}}{N}\left(\sum_{k=1}^N R_k\right) D, \quad \Psi_{44} = -\epsilon I.$$

with $\Gamma_k = -Q_k + Q_{k+1} - R_k - R_{k+1}$.

Proof: By combining the proof of Theorem 1 and Theorem 2 in §3.3, one can show that (5.5) is a sufficient condition to guarantee the robust absolute stability of LSTD (5.4). ■

It should be noted that the design constraint (5.5) in Theorem 5 is a BMI constraint, because there are product terms involving the decision variables K and $(P, R_1, \ldots, R_N, \Lambda)$ in the constraint. This is a direct consequence of optimizing both the controller parameter and the Lyapunov matrices simultaneously. However, it is observed that if K is fixed, then (5.5) is an LMI in $(P, Q_1, \ldots, Q_N, R_1, \ldots, R_N, \Lambda, T, \epsilon)$. Similarly, if $(P, R_1, \ldots, R_N, \Lambda)$ are fixed, then (5.5) is another LMI in $(K, Q_1, \ldots, Q_N, T, \epsilon)$. This suggests that we may alternate between solving two LMI optimization problems, which is known as *coordinate optimization* (El Ghaoui and Balakrishnan, 1994), to obtain the desired controller. Note that this process is referred to as a "V-K" iteration (El Ghaoui and Balakrishnan, 1994). To proceed the iterative LMI algorithm, it requires that the starting LMI is feasible. Therefore, we reformulate the problem of finding K, which satisfies (5.5) as follows.

$$\begin{aligned} &\text{find} \quad && K, \sigma \\ &\text{such that} \quad && P > 0, Q_1 > 0, \ldots, Q_N > 0, \\ & && R_1 > 0, \ldots, R_N > 0, \Lambda \geq 0, T \geq 0, \epsilon > 0, \\ & && \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & 0 \\ * & * & \Psi_{33} & \Psi_{34} \\ * & * & * & \Psi_{44} \end{bmatrix} < \sigma I. \end{aligned} \tag{5.6}$$

Obviously, the constraint (5.6) is always feasible in K and σ. Moreover, the constraint (5.5) is feasible if and only if $\sigma < 0$. By starting with a positive σ_0, the idea is to find a feasible solution to (5.6) that gives σ smaller than the previous one, *i.e.*,

$$\sigma_i < \sigma_{i-1}, \tag{5.7}$$

Since the constraint (5.7) ensures that the value of σ must be decreased, the iteration continues until σ becomes negative. The algorithm proceeds as follows.

Algorithm 1

1. *Input the parameters of LSTD (5.4), the sector-bounded nonlinearities* (L, M)*, the norm-bounded uncertainty* (D, E_0, E_1, E_2)*, and the desired maximum allowable time delay* $\bar{h}$*.*
2. *Initialize* $K_0 = 0$ *(the open-loop system).*
3. *Let* $i = 0$*.*
4. *Fix* K *at* K_0*. Compute* $(P, Q_1, \ldots, Q_N, R_1, \ldots, R_N, \Lambda, T, \epsilon)$ *and* σ *by solving (5.6). Then, let* $\sigma_0 = \sigma$*.*
5. *Repeat* $\{$
   - *(a) Let* $i = i + 1$*.*
   - *(b) Compute* $(K, Q_1, \ldots, Q_N, T, \epsilon)$ *and* σ *by solving (5.6) with the constraint* $\sigma < \sigma_{i-1}$*, where* $(P, R_1, \ldots, R_N, \Lambda)$ *are fixed at the most recent value. Let* $K_i = K$*.*
   - *(c) Compute* $(P, Q_1, \ldots, Q_N, R_1, \ldots, R_N, \Lambda, T, \epsilon)$ *and* σ *by solving (5.6) with the constraint* $\sigma < \sigma_{i-1}$*, where* K *is fixed at* K_i*. Let* $\sigma_i = \sigma$*.*

   $\}$ *Until* $\sigma_i < 0$*, or the LMI feasibility problem in (b) or (c) is infeasible.*

Note that if a feasibility problem in (b) or (c) is found infeasible, then it is possible that a feasible controller still exists.

5.3 State-feedback $\mathcal{H}_\infty$ Control

In this section, we develop the robust $\mathcal{H}_\infty$ control via state-feedback. Similar to the previous section, the existence condition for calculating the desired controller gain is extended from the worst-case performance analysis in §4. The control design problem, which is a BMI problem, can be solved by using the proposed local LMI optimization procedure.

The LSTD for robust $\mathcal{H}_\infty$ control problem is described as follows.

$$\begin{aligned}
\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) \\
&\qquad + [B_p + \Delta B_p(t)]p(t) + B_w w(t) + B_u u(t), \\
q(t) &= C_q x(t), \\
z(t) &= C_z x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{5.8}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and zero initial condition $x(t) = 0$, $\forall t \in [-h, 0]$. A scalar $h \in \mathbf{R}_+$ is a time-invariant time delay in the state, $x(t) \in \mathbf{R}^n$ is the state variable, $w(t) \in \mathbf{R}^{n_w}$ is the disturbance input which belongs to $\mathcal{L}_2$, $z(t) \in \mathbf{R}^{n_z}$ is the performance output, $q(t) \in \mathbf{R}^{n_p}$, and $p(t) \in \mathbf{R}^{n_p}$ are the input/output of a vector mapping of sector-bounded nonlinearities denoted by ϕ. The vector function ϕ belong to the set Φ characterised by memoryless, time-invariant nonlinearities satisfying certain sector conditions. In addition, the pairs (A, B_p) and (C_q, A) are assumed to be controllable and observable, respectively. The uncertainty considered here is assumed to be norm-bounded and has the following form

$$\begin{bmatrix} \Delta A(t) \\ \Delta A_1(t) \\ \Delta B_p(t) \end{bmatrix} = D\,F(t) \begin{bmatrix} E_0 \\ E_1 \\ E_2 \end{bmatrix}, \tag{5.9}$$

where D, E_0, E_1, and E_2 are known constant real matrices of appropriate dimensions, which represent structure and magnitude of the uncertainties, and $F(t)$ is an unknown matrix function with *Lebesgue measurable elements* satisfying $F^T(t)F(t) \leq I$. Assuming that the state $x(t)$ is measurable and the pair (A, B_u) is controllable, the robust $\mathcal{H}_\infty$ control problem of LSTD (5.8) is defined as follows.

Problem 6 *Given the desired maximum allowable time delay, $\bar{h}$, the state-feedback $\mathcal{H}_\infty$ control problem is to search for the control law*

$$u(t) = Kx(t) \tag{5.10}$$

which minimizes an upper bound γ_∞ of $\mathcal{J}_\infty$ for the LSTD (5.8) for any time-invariant time delay $h \in (0, \bar{h}]$.

After substituting (5.2) and (5.10) into the LSTD (5.8), we obtain the closed-loop LSTD as follows.

$$\begin{aligned}
\dot{x}(t) &= [A + B_u K + DF(t)E_0]x(t) + [A_1 + DF(t)E_1]x(t-h) \\
&\quad + [B_p + DF(t)E_2]p(t) + B_w w(t), \\
q(t) &= C_q x(t), \\
z(t) &= C_z x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{5.11}$$

with sector-bounded nonlinearities $\phi \in \Phi(\mathbf{0}, \mathbf{1})$. Next, we propose the sufficient condition for existence of a static state-feedback gain K which minimizes an upper bound of the worst-case $\mathcal{H}_\infty$ performance of the closed loop LSTD (5.11).

Theorem 6 *Given $\bar{h} \in \mathbf{R}_+$, state-feedback controller K that minimizes γ_∞^2 of the closed-loop LSTD (5.11) with time-invariant delay, $h \in (0, \bar{h}]$, can be computed by minimizing γ_∞^2 subject to the matrix inequality constraint (5.12) over variables K, $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \dots, N$, $\Lambda \geq 0$, $T \geq 0$, a scalar $\epsilon > 0$, and $\gamma_\infty^2 > 0$.*

$$\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & \Psi_{16} \\
* & \Psi_{22} & 0 & \Psi_{24} & 0 & 0 \\
* & * & \Psi_{33} & 0 & 0 & 0 \\
* & * & * & \Psi_{44} & 0 & \Psi_{46} \\
* & * & * & * & \Psi_{55} & 0 \\
* & * & * & * & * & \Psi_{66}
\end{bmatrix} \leq 0, \tag{5.12}$$

where

$$\Psi_{12} = \begin{bmatrix} PB_w \\ 0 \\ \Lambda C_q B_w \end{bmatrix}, \quad \Psi_{13} = \begin{bmatrix} R_1 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \end{bmatrix} + \begin{bmatrix} 0_{n\times(N-2)n} & 0 \\ 0_{n\times(N-2)n} & R_N \\ 0_{n\times(N-2)n} & 0 \end{bmatrix},$$

$$\Psi_{14} = \begin{bmatrix} \frac{\bar{h}}{N}(A + B_u K)^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} A_1^T \sum_{k=1}^N R_k \\ \frac{\bar{h}}{N} B_p^T \sum_{k=1}^N R_k \end{bmatrix}, \quad \Psi_{15} = \begin{bmatrix} C_z^T \\ 0 \\ 0 \end{bmatrix}, \quad \Psi_{16} = \begin{bmatrix} PD \\ 0 \\ \Lambda C_q D \end{bmatrix},$$

$$\Psi_{22} = -\gamma_\infty^2 I, \quad \Psi_{24} = \frac{\bar{h}}{N} B_w^T \sum_{k=1}^{N} R_k, \quad \Psi_{33} = \begin{bmatrix} \Gamma_1 & R_2 & & \\ * & \Gamma_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & \Gamma_{N-1} \end{bmatrix},$$

$$\Psi_{44} = -\sum_{k=1}^{N} R_k, \quad \Psi_{46} = \frac{\bar{h}}{N} \left(\sum_{k=1}^{N} R_k \right) D, \quad \Psi_{55} = -I, \quad \Psi_{66} = -\epsilon I.$$

Ψ_{11} *and* Γ_k *are the same as defined in Theorem 5.*

Proof: By combining the proof of Theorem 3 and Theorem 4 in §4.3, one can show that an upper bound γ_∞ of LSTD (5.11) can be obtained by minimizing γ_∞^2 subject to (5.12). ■

The matrix inequality (5.12) is a BMI condition because of the product terms between decision variables K and $(P, R_1, \ldots, R_N, \Lambda)$. As a result, we cannot find in general the global minimization γ_∞ using convex optimization algorithms. In this work, we apply the *coordinate optimization* (El Ghaoui and Balakrishnan, 1994), *i.e.*, alternating between solving LMI optimization problems in each coordinate, to obtain a suboptimal controller. Consider the following minimization problem

$$\begin{aligned} &\text{minimize} \quad \gamma_\infty^2 \\ &\text{subject to} \quad (5.12), P > 0, Q_1 > 0, \ldots, Q_N > 0, \\ &\qquad\qquad\quad R_1 > 0, \ldots, R_N > 0, \Lambda \geq 0, T \geq 0, \epsilon > 0, \gamma_\infty^2 > 0. \end{aligned} \tag{5.13}$$

By starting with state-feedback stabilization K and a corresponding γ_∞^2, we alternately solve LMI minimization problems by either fixing the analysis variables $(P, R_1, \ldots, R_N, \Lambda)$ or the controller K; the value of γ_∞^2 is monotonically decreasing along the iteration until the improvement in γ_∞^2 less than the desired absolute tolerance. The iterative LMI algorithm for solving (5.13) is as follows.

Algorithm 2

1. *Input the parameters of LSTD (5.11), the sector-bounded nonlinearities* (L, M)*, the norm-bounded uncertainty* (D, E_0, E_1, E_2)*, and the desired* $\bar{h}$*.*

2. *Initialize* K_0 *with state-feedback stabilization for the LSTD with the same* L, M, D, E_0, E_1*,* E_2*, and* $\bar{h}$*.*

3. Let $i = 0$.

4. Fix K at K_0. Compute ($P, Q_1, \ldots, Q_N, R_1, \ldots, R_N, \Lambda, T, \epsilon$) and γ_∞^2 by solving (5.13). Then, let $\gamma_{\infty,0}^2 = \gamma_\infty^2$.

5. Repeat $\{$

(a) Let $i = i + 1$.

(b) Compute $(K, Q_1, \ldots, Q_N, T, \epsilon)$ and γ_∞^2 by solving (5.13), where ($P, R_1, \ldots$,R_N, Λ) are fixed at the most recent value. Let $K_i = K$.

(c) Compute ($P, Q_1, \ldots, Q_N, R_1, \ldots, R_N, \Lambda, T, \epsilon$) and γ_∞^2 by solving (5.13), where K is fixed at K_i. Let $\gamma_{\infty,i}^2 = \gamma_\infty^2$.

$\}$ *Until $\|\gamma_{\infty,i-1}^2 - \gamma_{\infty,i}^2\|$ less than the desired absolute tolerance, or the LMI minimization problem in (b) or (c) is infeasible.*

It is important to note that the initial controller K_0 used in Algorithm 2 should not be restricted only to state-feedback stabilization obtained from Algorithm 1, since the coordinate optimization methods can yield a good design depending on initialization. It is suggested that a uniform grid over a certain region of state-feedback stabilization may be used as initial controllers for Algorithm 2.

5.4 Numerical Examples

To illustrate the proposed design algorithms, the stabilization and $\mathcal{H}_\infty$ control for two numerical examples taken from Han (2005) and Xu and Feng (2007) are synthesized by using Theorem 5 and Theorem 6. Note that the LMI Lab (Gahinet et al., 1995) is used for solving the LMI feasibility problems and the linear objective minimization problems over LMI that arise in the designs.

Example 5: Consider the LSTD of the form (5.8) with the following parameters.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \quad B_w = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} 0.6 & 0.8 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

where the sector-bounded nonlinearity belongs to the set $\Phi(-1-\delta l, 1+\delta l)$, which corresponds to $L = -1 - \delta l$ and $M = 1 + \delta l$. In addition, the uncertain matrices $\Delta A(t)$ and $\Delta A_1(t)$ in (5.8) satisfy norm-bounded condition $\|\Delta A(t)\| \leq \alpha$ and $\|\Delta A_1(t)\| \leq \alpha$ for a nonnegative α. Moreover, $\Delta B_p(t) = 0$. These uncertainties can be represented in the form (5.9) with

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad E_0 = E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where $F^T(t)F(t) \leq I$, $F(t) \in \mathbf{R}^{2\times 2}$. Note that the system data is taken from Han (2005) with slight modifications. To proceed the design, we apply loop transformation to the LSTD with $\phi \in \Phi(-1-\delta l, 1+\delta l)$ so that it is transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(0,1)$. Then, the controller is designed on the equivalent LSTD with $\bar{L} = 0$ and $\bar{M} = 1$.

By Theorem 5 with $N = 2$, state-feedback stabilization K can be designed using Algorithm 1. In practice, small gain of state-feedback is more desirable, thus, we choose the *feasibility radius* (Gahinet et al., 1995) for the decision variables to be 4. This constraint will limit norm of the solution. We design state-feedback stabilization for LSTD for various $\bar{h}$, δl and α. The state-feedback gains and the iteration numbers obtained for Example 5 are shown in Table 5.1. It can be seen that the iteration number ranges from $15 - 45$ and tends to grow as $\bar{h}$ increases. It can be interpreted that the design problem becomes harder to obtain state-feedback stabilization when the maximum allowable time delay is increased. Similar trends are observed when increasing δl and α.

We employ Theorem 6 to synthesize the robust $\mathcal{H}_\infty$ controllers K for the same set of $\bar{h}$, δl and α as in Table 5.1. First, Theorem 6 with $N = 2$ is applied. The initial state-feedback gain $K_0 = [-5, -5]$ is used for the case of $\bar{h} = 2$ and 4, and for the case of $\bar{h} = 6$, K_0 is chosen to be $[-6, -5]$. The desired absolute tolerance for this example is chosen to be 10^{-5}. The calculated state-feedback gain of $\mathcal{H}_\infty$ controllers and the corresponding γ_∞ are summarized in Table 5.2. It is obvious that the guaranteed γ_∞ provided by K for $\bar{h} = 6$ are higher than those for $\bar{h} = 4$, and the values of γ_∞ obtained for $\bar{h} = 4$ are also higher than those of $\bar{h} = 2$. Similarly, the guaranteed γ_∞ in Table 5.2 is increased when the sector bound becomes larger and the size of norm-bounded uncertainty is increased. It can be seen that the required iterations range from $49 - 465$ and do not directly associate with $\bar{h}$, δl and α. In Table 5.2, the refined feedback gains obtained using Theorem 6 with $N = 3$ are also presented. Starting from the feedback gain calculated by using Theorem 6 with $N = 2$, the required iterations range from $1 - 590$ with the improvement in γ_∞ up to 3.5%. Figure 5.1 and 5.2 provide comparison between the

Table 5.1: Stabilization controllers designed using Theorem 5 with $N = 2$ for Example 5 with various $\bar{h}$, δl, and α.

$\bar{h}$	δl	α	Feedback gain	# iter.
2	0	0	$[-0.9306, -0.7722]$	15
		0.05	$[-1.0562, -1.1762]$	19
		0.10	$[-0.9938, -1.2029]$	20
	1	0	$[-1.3181, -0.8803]$	19
		0.05	$[-1.3056, -1.3480]$	21
		0.10	$[-1.2134, -1.3646]$	22
	2	0	$[-0.8380, -1.0082]$	24
		0.05	$[-0.8853, -1.2953]$	26
		0.10	$[-0.9668, -1.4308]$	27
4	0	0	$[-1.5875, -1.5239]$	20
		0.05	$[-1.1128, -1.5762]$	20
		0.10	$[-1.1291, -1.5953]$	19
	1	0	$[-1.3976, -1.6931]$	23
		0.05	$[-0.7497, -1.5487]$	31
		0.10	$[-0.7902, -1.6104]$	31
	2	0	$[-1.2074, -1.7667]$	35
		0.05	$[-1.0524, -1.7255]$	38
		0.10	$[-1.1882, -1.8874]$	40
6	0	0	$[-0.9185, -1.3030]$	19
		0.05	$[-0.7648, -1.4665]$	22
		0.10	$[-0.5322, -1.4130]$	24
	1	0	$[-0.9880, -1.5188]$	32
		0.05	$[-0.8889, -1.7721]$	38
		0.10	$[-0.8850, -1.8191]$	39
	2	0	$[-1.3352, -1.9218]$	39
		0.05	$[-1.2180, -2.0030]$	44
		0.10	$[-1.2897, -2.1671]$	45

Table 5.2: $\mathcal{H}_\infty$ controllers designed using Theorem 6 for Example 5 with various $\bar{h}$, δl, and α.

$\bar{h}$	δl	α	$N=2$			$N=3$		
			Feedback gain	γ_∞	#iter	Feedback gain	γ_∞	#iter
2	0	0	$[-10.09, -8.87]$	1.9691	465	$[-10.10, -8.87]$	1.9638	1
		0.05	$[-17.92, -14.62]$	2.1376	464	$[-17.94, -14.63]$	2.1319	2
		0.10	$[-22.22, -17.40]$	2.3702	345	$[-46.51, -34.43]$	2.3403	145
	1	0	$[-13.41, -11.72]$	2.1779	404	$[-15.10, -13.00]$	2.1738	143
		0.05	$[-21.09, -17.36]$	2.4143	376	$[-21.13, -17.38]$	2.4105	2
		0.10	$[-27.26, -21.47]$	2.7414	386	$[-33.52, -25.98]$	2.7287	63
	2	0	$[-13.80, -12.49]$	2.4382	370	$[-13.82, -12.51]$	2.4382	3
		0.05	$[-23.61, -19.51]$	2.7638	317	$[-23.79, -19.66]$	2.7629	8
		0.10	$[-30.16, -24.00]$	3.2413	314	$[-30.73, -24.42]$	3.2349	14
4	0	0	$[-88.58, -79.00]$	2.2530	174	$[-88.61, -79.03]$	2.2452	1
		0.05	$[-165.81, -144.56]$	2.4721	116	$[-167.29, -145.85]$	2.4657	2
		0.10	$[-243.18, -200.47]$	2.8187	121	$[-268.38, -221.21]$	2.8090	24
	1	0	$[-53.44, -51.25]$	2.4563	142	$[-58.57, -56.17]$	2.4542	40
		0.05	$[-112.19, -103.41]$	2.7669	225	$[-112.72, -103.88]$	2.7655	3
		0.10	$[-128.01, -112.51]$	3.2462	196	$[-129.50, -113.80]$	3.2416	4
	2	0	$[-15.60, -16.27]$	2.7587	196	$[-16.20, -16.92]$	2.7500	99
		0.05	$[-109.13, -101.57]$	3.1138	370	$[-113.18, -105.22]$	3.1117	20
		0.10	$[-114.11, -102.12]$	3.7973	301	$[-125.33, -111.90]$	3.7904	43
6	0	0	$[-173.03, -155.52]$	2.3690	192	$[-179.16, -161.06]$	2.3607	6
		0.05	$[-309.72, -247.78]$	2.7724	214	$[-334.24, -267.96]$	2.7527	62
		0.10	$[-331.74, -251.27]$	3.1675	115	$[-361.11, -273.61]$	3.1511	26
	1	0	$[-77.21, -76.65]$	2.5205	213	$[-77.26, -76.70]$	2.5202	1
		0.05	$[-246.88, -218.48]$	2.9273	280	$[-257.43, -227.93]$	2.9175	14
		0.10	$[-301.33, -233.30]$	3.7656	221	$[-436.34, -339.96]$	3.7090	271
	2	0	$[-17.34, -18.20]$	2.8695	49	$[-17.39, -18.27]$	2.8641	10
		0.05	$[-209.87, -197.92]$	3.2088	282	$[-213.59, -201.45]$	3.2068	9
		0.10	$[-198.89, -157.60]$	4.5725	246	$[-416.19, -333.89]$	4.4096	590

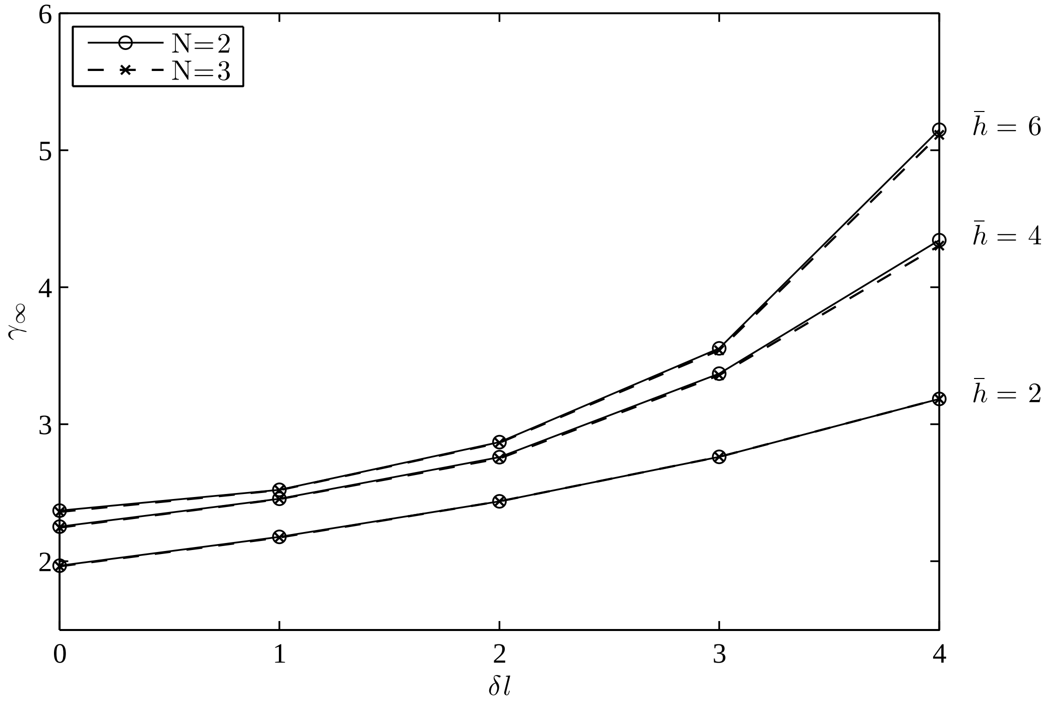

Figure 5.1: Upper bound of $\mathcal{J}_\infty$ vs. δl for Example 5 when $\alpha = 0$.

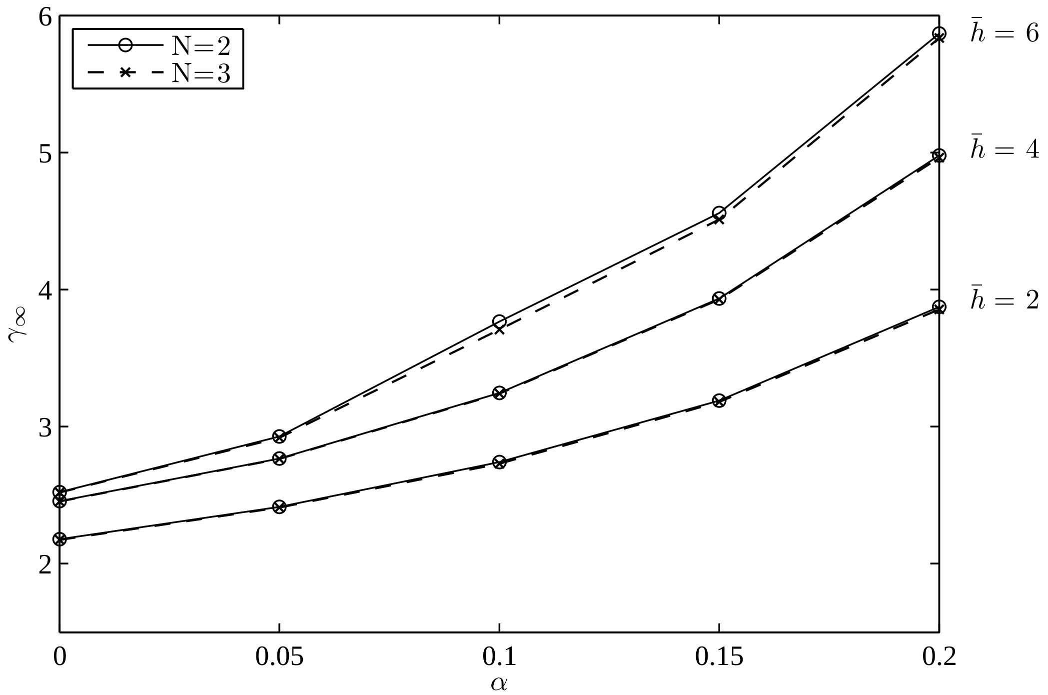

Figure 5.2: Upper bound of $\mathcal{J}_\infty$ vs. α for Example 5 when $\delta l = 1$.

performance plots obtained from Theorem 6 with $N = 2$ and those obtained from Theorem 6 with $N = 3$. It is observed that γ_∞ calculated from the latter case is slightly lower than γ_∞ obtained from the case with $N = 2$. However, the improvement requires a number of calculations which do not associate with $\bar{h}$, δl and α. In Figure 5.1, the plots also show the tendency of γ_∞ from small sector bound to large sector bound. In particular, the guaranteed γ_∞ is increased as δl is larger. Similar trends are observed when the norm of uncertainties is increased as shown in Figure 5.2.

According to the controller gains given in Table 5.2 in case of $N = 2$, simulations are conducted with $\phi(q(t)) = (1 + \delta l)q(t)\sin(q(t)$. The testing disturbance is selected as depicted in Figure 5.3. In Figure 5.4, open-loop $z(t)$ and closed-loop $z(t)$ are shown for $h = 2, 4$ and 6, when $\delta l = 2$ and $\alpha = 0.10$. It can be seen from the figure that the open-loop response is stable for $h = 2$, but unstable for $h = 4$ and 6. The closed-loop responses indicate that the designed controller gains can stabilize the system and reduce the effect from the disturbance with reasonable control effort. Figure 5.5 compares three different cases of sector-bounded nonlinearity, *i.e.*, $\delta l = 0, 1$ and 2 for $h = 2$ and $\alpha = 0.10$. Similarly, Figure 5.6 presents the results from three cases of norm-bounded uncertainty, *i.e.*, $\alpha = 0, 0.05$ and 0.10 for $h = 2$ and $\delta l = 2$. In both experiments, it is clear that the designed controllers can improve the response subject to given disturbance. However, the closed-loop gain does not need to be smaller than the open-loop gain. This is because the designed controllers provide the guarantee on the upper bound γ_∞ for all disturbance inputs in $\mathcal{L}_2$. The comparisons between open-loop gain and closed-loop gain for each case are provided in Table 5.3.

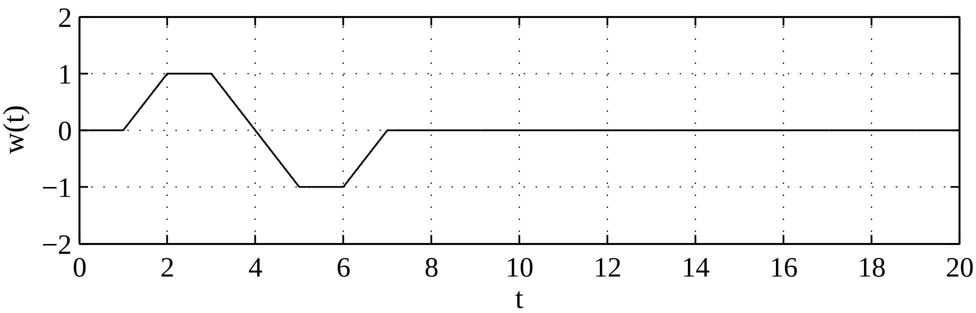

Figure 5.3: Disturbance input used in simulation.

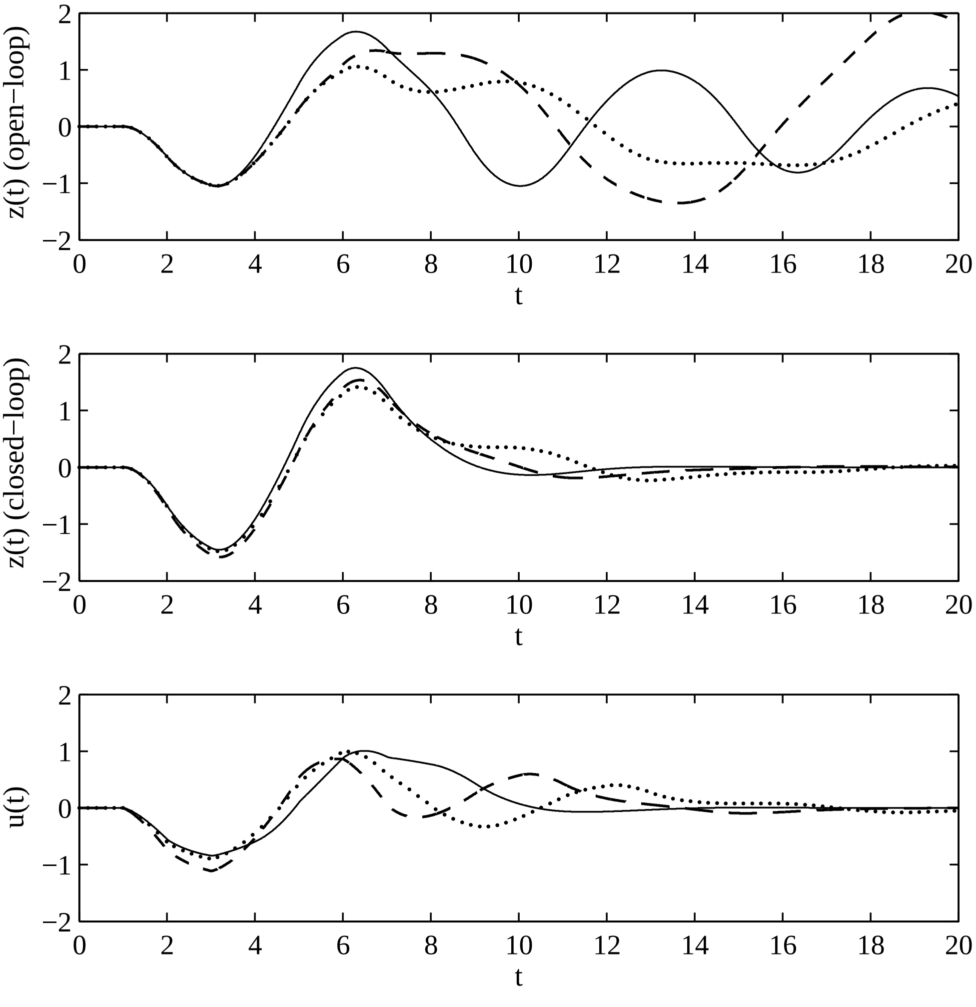

Figure 5.4: Time-response of Example 5: $h = 2$ (solid line), $h = 4$ (dashed line), $h = 6$ (dotted line) when $\delta l = 2$ and $\alpha = 0.10$.

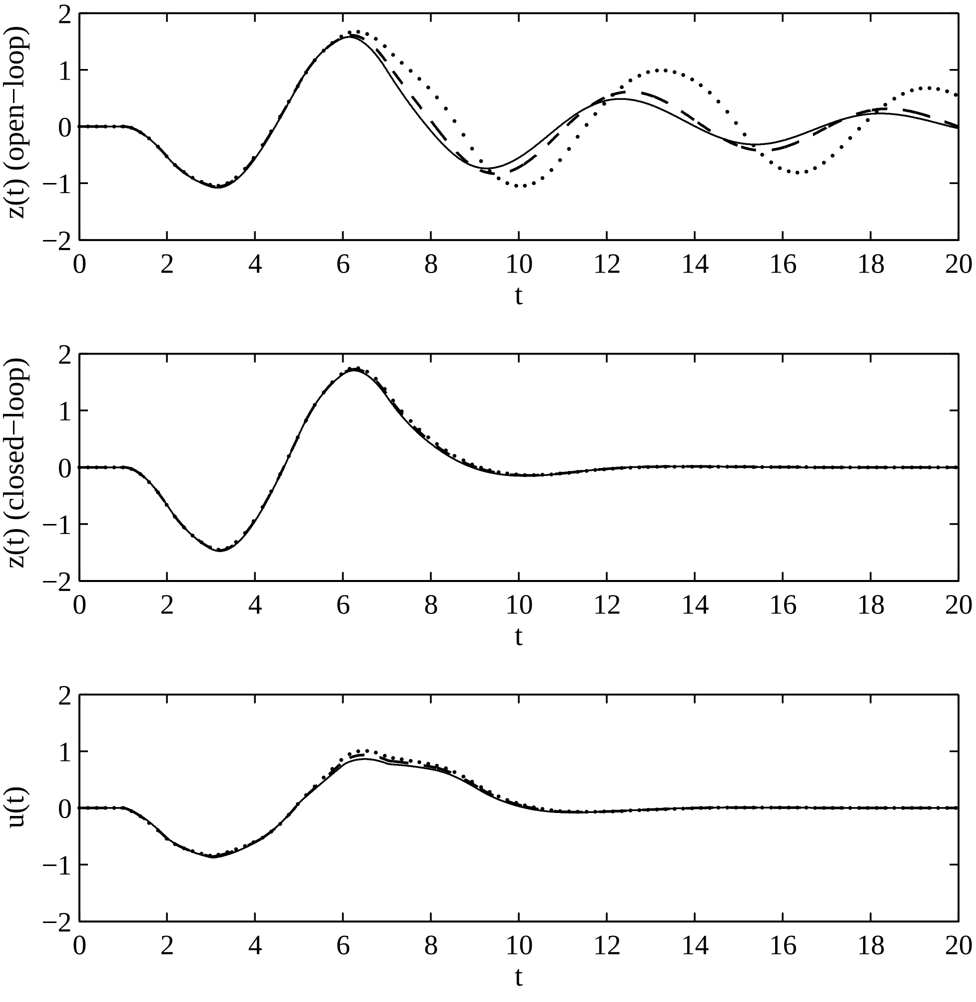

Figure 5.5: Time-response of Example 5: $\delta l = 0$ (solid line), $\delta l = 1$ (dashed line), $\delta l = 2$ (dotted line) when $h = 2$ and $\alpha = 0.10$.

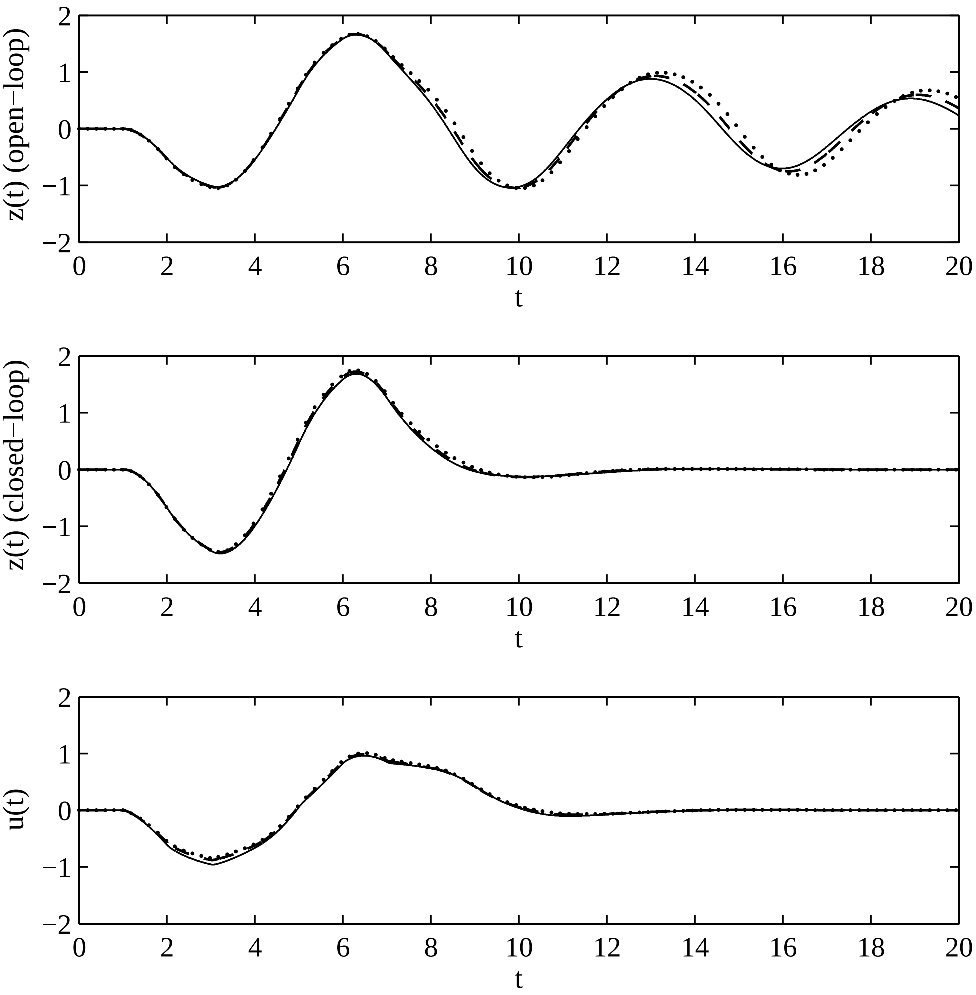

Figure 5.6: Time-response of Example 5: $\alpha = 0$ (solid line), $\alpha = 0.05$ (dashed line), $\alpha = 0.10$ (dotted line) when $h = 2$ and $\delta l = 2$.

Table 5.3: Performance gain for Example 5 with various h, δl and α.

System parameters			Open-loop gain		Closed-loop gain	
h	δl	α	$\|z\|_2/\|w\|_2$	γ_∞	$\|z\|_2/\|w\|_2$	γ_∞
2	0	0.10	1.4667	∞	1.5694	2.3702
2	1	0.10	1.5647	∞	1.5811	2.7414
2	2	0	1.8208	∞	1.5525	2.4382
2	2	0.05	1.8822	∞	1.5703	2.7638
2	2	0.10	1.9644	∞	1.5949	3.2413
4	2	0.10	∞	∞	1.5656	3.7973
6	2	0.10	∞	∞	1.4808	4.5725

Example 6: Consider the LSTD of the form (5.8) with the following parameters.

$$A = \begin{bmatrix} -1.2 & 0 \\ 0.8 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0.6 \\ -0.6 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where the sector-bounded nonlinearities belong to the set $\Phi([0;0],[1+\delta l;3])$, and the corresponding $L = \mathrm{diag}(0,0)$ and $M = \mathrm{diag}(1+\delta l, 3)$. The uncertainty matrices $\Delta A(t)$, $\Delta A_1(t)$ and $\Delta B_p(t)$ in (5.8) are time-varying matrices satisfying norm-bounded constraints $\|\Delta A(t)\| \leq 0.2\alpha$, $\|\Delta A_1(t)\| \leq 0.03\alpha$, and $\|\Delta B_p(t)\| \leq 0.03\alpha$ for a given $\alpha \geq 0$. The uncertainty in this LSTD can be represented with Eq. (5.9) where

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix},$$

and $F^T(t)F(t) \leq I$, $F(t) \in \mathbf{R}^{2\times 2}$. This example is modified from the LSTD given in Xu and Feng (2007). Again, loop transformation is applied so that LSTD with ϕ defined above is transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. Note that the new $\bar{L}$ and $\bar{M}$ are the zero and identity matrices of dimension 2×2, respectively.

First, we apply Theorem 5 with $N = 2$ to design state-feedback stabilization for different values of $\bar{h}$, δl and α. In this example, the feasibility radius for the decision variables is selected to be 11. The calculated state-feedback gains and the iteration numbers are summarized in Table 5.4. The results indicate that the method is successfully applied to the state-feedback

Table 5.4: Stabilization controllers designed using Theorem 5 with $N = 2$ for Example 6 with various $\bar{h}$, δl, and α.

$\bar{h}$	δl	α	Feedback gain	# iter.
2	0	0	$[1.5197, -5.2874]$	12
		0.25	$[1.4208, -8.0037]$	13
		0.50	$[1.3941, -7.6120]$	15
	1	0	$[1.7803, -8.1781]$	13
		0.25	$[1.8093, -7.6666]$	12
		0.50	$[1.7719, -8.4945]$	15
	2	0	$[2.1133, -8.4658]$	15
		0.25	$[2.4117, -8.8797]$	15
		0.50	$[2.1738, -8.7841]$	13
4	0	0	$[1.4935, -4.6359]$	27
		0.25	$[1.9413, -6.8851]$	30
		0.50	$[1.6344, -7.2499]$	31
	1	0	$[1.4536, -5.8564]$	28
		0.25	$[2.3635, -7.8240]$	34
		0.50	$[2.0390, -7.2421]$	41
	2	0	$[2.5455, -8.0545]$	30
		0.25	$[2.7935, -7.9809]$	39
		0.50	$[2.5786, -6.8154]$	38
6	0	0	$[0.8047, -4.8953]$	20
		0.25	$[1.3896, -7.6517]$	20
		0.50	$[1.7029, -6.9601]$	24
	1	0	$[2.0940, -6.6503]$	24
		0.25	$[2.1730, -5.7148]$	26
		0.50	$[2.1893, -7.7078]$	26
	2	0	$[2.7161, -5.9423]$	27
		0.25	$[3.1316, -6.2288]$	27
		0.50	$[3.0628, -5.6396]$	28

Table 5.5: $\mathcal{H}_\infty$ controllers designed using Theorem 6 for Example 6 with various $\bar{h}$, δl, and α.

$\bar{h}$	δl	α	$N=2$			$N=3$		
			Feedback gain	γ_∞	#iter	Feedback gain	γ_∞	#iter
2	0	0	$[92.06, -60.93]$	0.8088	29	$[116.61, -76.71]$	0.8067	30
		0.25	$[85.19, -56.90]$	0.8742	69	$[115.72, -77.15]$	0.8712	72
		0.50	$[100.66, -68.74]$	0.9459	138	$[167.24, -114.09]$	0.9426	69
	1	0	$[96.16, -63.15]$	0.8150	25	$[114.78, -74.80]$	0.8127	25
		0.25	$[87.58, -57.98]$	0.8822	27	$[115.81, -76.55]$	0.8780	94
		0.50	$[98.99, -66.92]$	0.9551	102	$[153.21, -103.55]$	0.9500	86
	2	0	$[98.13, -63.83]$	0.8207	24	$[112.04, -72.26]$	0.8183	24
		0.25	$[91.68, -60.24]$	0.8887	24	$[120.29, -78.96]$	0.8834	114
		0.50	$[102.05, -68.43]$	0.9625	91	$[157.89, -105.99]$	0.9555	119
4	0	0	$[246.33, -149.68]$	0.8303	735	$[246.59, -149.84]$	0.8302	1
		0.25	$[300.29, -185.84]$	0.9019	361	$[369.65, -228.74]$	0.9011	43
		0.50	$[349.31, -219.97]$	0.9868	273	$[415.12, -261.58]$	0.9863	32
	1	0	$[257.16, -155.95]$	0.8319	658	$[257.52, -156.17]$	0.8318	1
		0.25	$[327.29, -202.23]$	0.9030	287	$[422.53, -261.18]$	0.9019	50
		0.50	$[429.69, -270.20]$	0.9873	178	$[502.61, -316.28]$	0.9869	20
	2	0	$[267.54, -162.00]$	0.8333	647	$[305.72, -185.12]$	0.8323	79
		0.25	$[339.10, -209.17]$	0.9043	277	$[444.39, -274.34]$	0.9029	59
		0.50	$[473.16, -297.05]$	0.9881	173	$[630.18, -396.02]$	0.9873	32
6	0	0	$[428.60, -259.17]$	0.8328	116	$[429.04, -259.41]$	0.8327	1
		0.25	$[310.21, -190.14]$	0.9087	359	$[319.52, -196.00]$	0.9083	17
		0.50	$[273.89, -170.93]$	0.9971	329	$[274.02, -171.03]$	0.9970	1
	1	0	$[201.90, -121.62]$	0.8400	115	$[202.05, -121.71]$	0.8399	1
		0.25	$[712.56, -436.20]$	0.9064	399	$[810.97, -496.52]$	0.9061	18
		0.50	$[313.45, -195.27]$	0.9975	149	$[313.50, -195.31]$	0.9973	1
	2	0	$[210.82, -126.68]$	0.8416	535	$[230.08, -138.31]$	0.8402	113
		0.25	$[794.29, -485.94]$	0.9066	385	$[945.55, -578.63]$	0.9063	21
		0.50	$[358.96, -223.28]$	0.9978	134	$[358.96, -223.28]$	0.9975	1

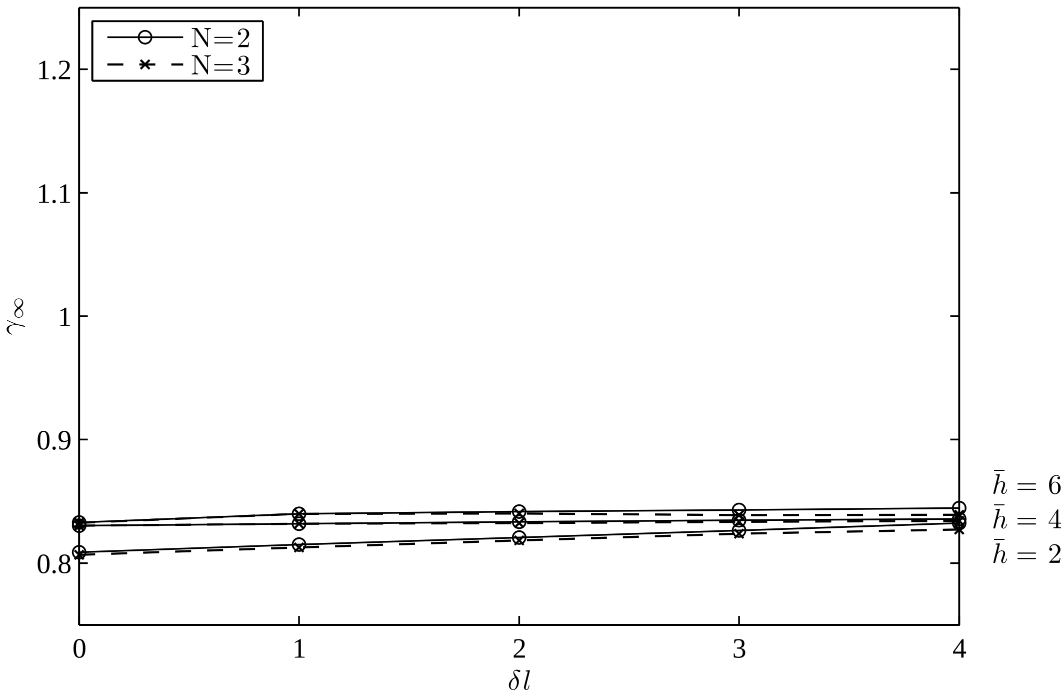

Figure 5.7: Upper bound of $\mathcal{J}_\infty$ vs. δl for Example 6 when $\alpha = 0$.

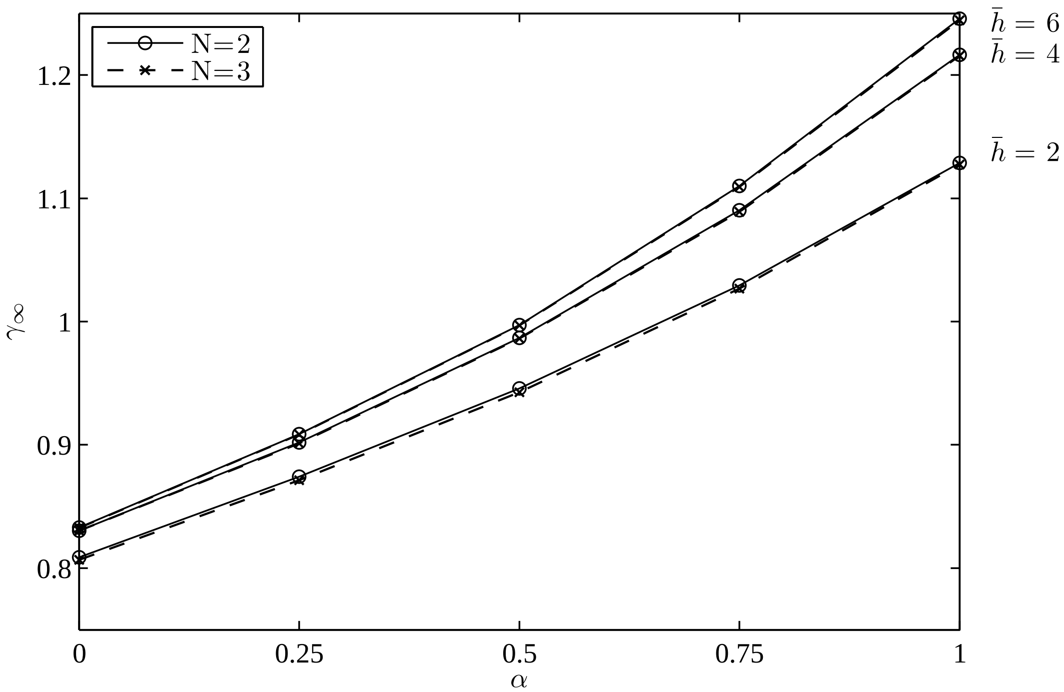

Figure 5.8: Upper bound of $\mathcal{J}_\infty$ vs. α for Example 6 when $\delta l = 0$.

stabilization problem for LSTD with time-invariant delay. It can be seen that the number of iterations ranges from $12 - 41$ and tends to increase associated with $\bar{h}$, δl and α.

Next, Theorem 6 with $N = 2$ is employed to design $\mathcal{H}_\infty$ controllers for different cases. The initial state-feedback gain $K_0 = [75, -50]$ is used for the case of $\bar{h} = 2$ and 4. For the case of $\bar{h} = 6$, the gain K_0 is selected to be $[70, -50]$. The absolute tolerance has been selected to be 10^{-5}. Table 5.5 gives the state-feedback gains of $\mathcal{H}_\infty$ controllers and the corresponding γ_∞ for LSTD with various parameters $\bar{h}$, δl and α as in the stabilization problem. The results shown that the algorithm gives reasonable state-feedback controllers, and the guaranteed worst-case performance is increased when the maximum allowable time delay, the sector bounds, and the norm of uncertainties are increased. We observe that the required iterations range from $24 - 735$ and do not directly associate with $\bar{h}$, δl and α. In Table 5.5, the state-feedback gains and the corresponding γ_∞ designed using Theorem 6 with $N = 3$ are also presented. With the initial feedback gain from the case of $N = 2$, the required iterations for the case of $N = 3$ range from $1 - 119$ with the improvement in γ_∞ up to 0.7%. Figure 5.7 and 5.8 compare the worst-case performance calculated from Theorem 6 with $N = 2$ and Theorem 6 with $N = 3$. It is observed that the design with $N = 3$ provides slight improvements in γ_∞. The plots in Figure 5.7 and 5.8 show that the guaranteed γ_∞ is increased as δl is larger or norm of uncertainties is increased.

In this example, time-response simulations are conducted for the controller gains given in Table 5.5 in case of $N = 2$ with the following nonlinearities,

$$\phi(q(t)) = \begin{bmatrix} (1 + \delta l)\tanh(q_1(t)) \\ 3\tanh(q_2(t)) \end{bmatrix}.$$

The same testing disturbance as used in the previous example is selected. Three experiments are presented. Figure 5.9 shows the open-loop $z(t)$ and closed-loop $z(t)$ for various h. Figure 5.10 compares three different cases of sector-bounded nonlinearity, and Figure 5.11 presents the results from three cases of norm-bounded uncertainty. It is observed that open-loop response is unstable except for the case of $h = 2$, $\delta l = 0$. It is clear that the designed controllers can substantially improve the response using reasonable control effort. The corresponding open-loop and closed-loop gains for each case are given in Table 5.6.

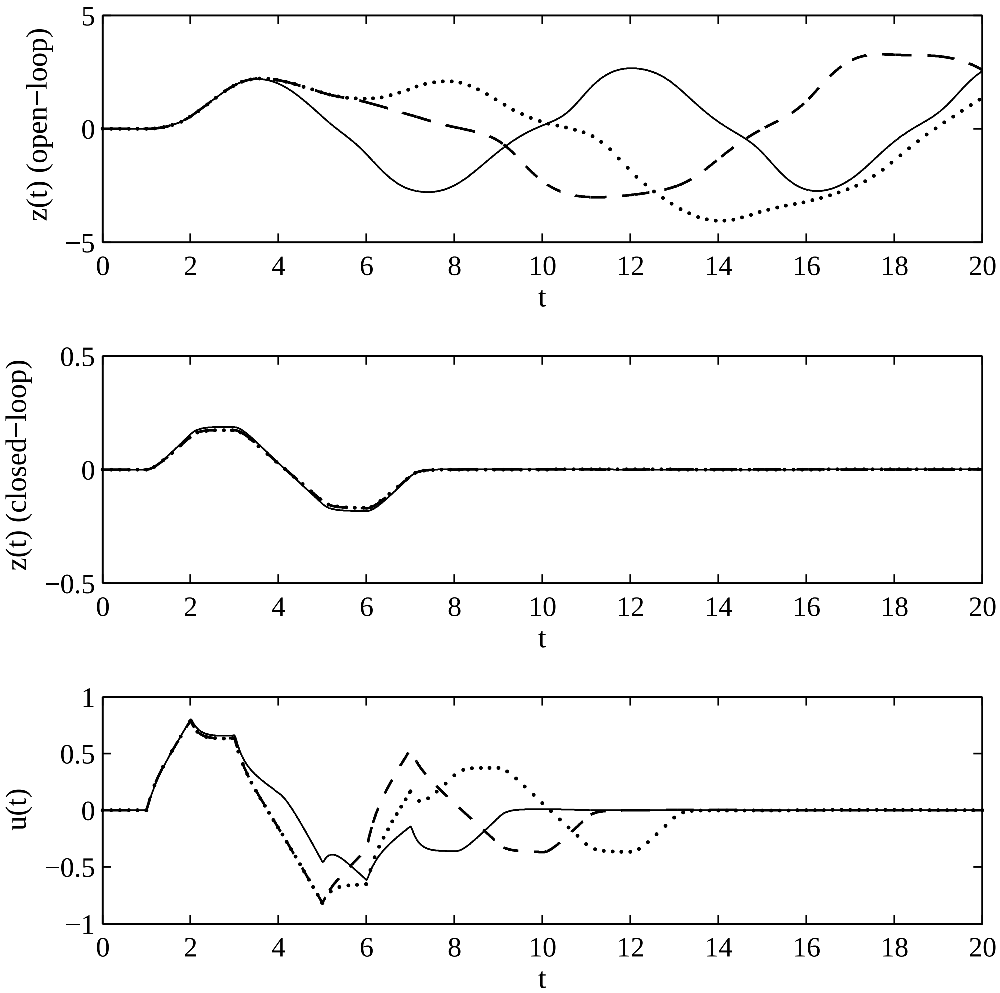

Figure 5.9: Time-response of Example 6: $h = 2$ (solid line), $h = 4$ (dashed line), $h = 6$ (dotted line) when $\delta l = 2$ and $\alpha = 0.50$.

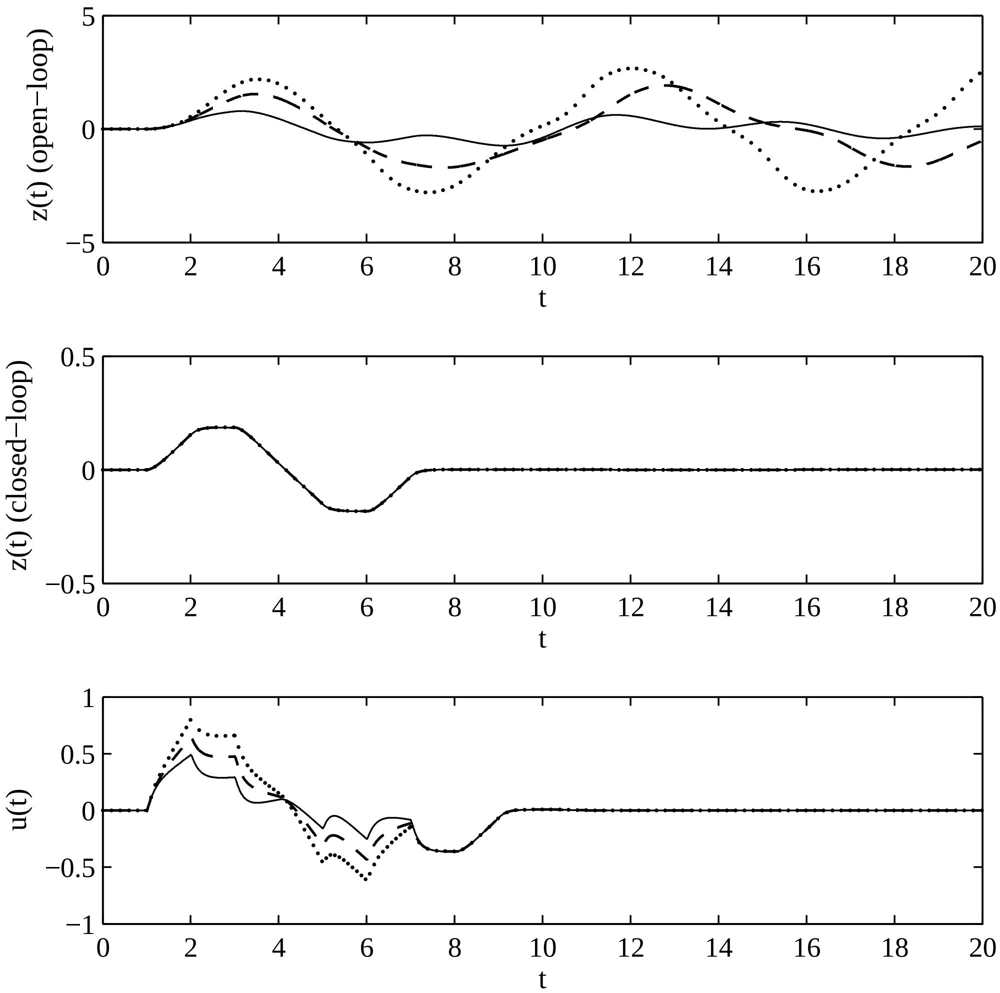

Figure 5.10: Time-response of Example 6: $\delta l = 0$ (solid line), $\delta l = 1$ (dashed line), $\delta l = 2$ (dotted line) when $h = 2$ and $\alpha = 0.50$.

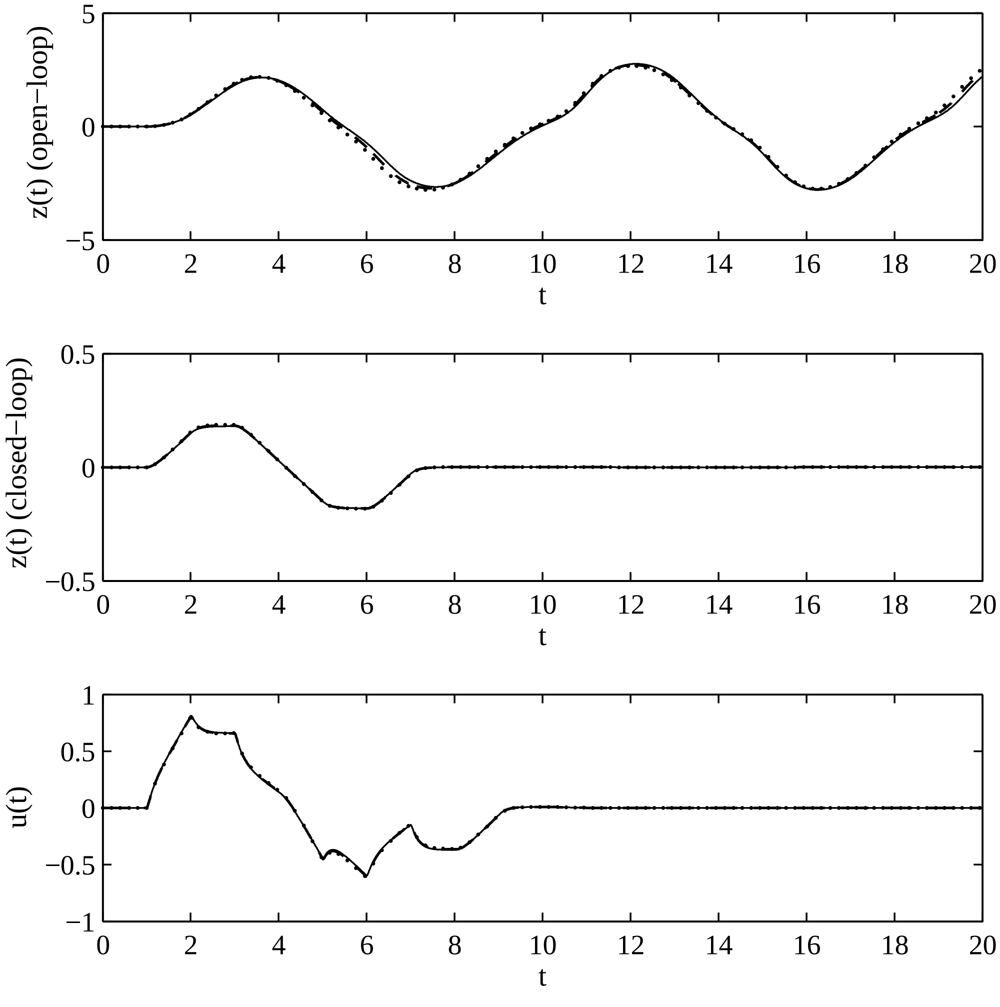

Figure 5.11: Time-response of Example 6: $\alpha = 0$ (solid line), $\alpha = 0.25$ (dashed line), $\alpha = 0.50$ (dotted line) when $h = 2$ and $\delta l = 2$.

Table 5.6: Performance gain for Example 6 with various h, δl and α.

System parameters			Open-loop gain		Closed-loop gain	
h	δl	α	$\|z\|_2/\|w\|_2$	γ_∞	$\|z\|_2/\|w\|_2$	γ_∞
2	0	0.50	1.0058	∞	0.1813	0.9459
2	1	0.50	∞	∞	0.1813	0.9551
2	2	0	∞	∞	0.1774	0.8207
2	2	0.25	∞	∞	0.1791	0.8887
2	2	0.50	∞	∞	0.1815	0.9625
4	2	0.50	∞	∞	0.1685	0.9881
6	2	0.50	∞	∞	0.1674	0.9978

5.5 Discussion

From two numerical examples in the previous section, Algorithm 1 can be used to design state-feedback stabilization for a given maximum allowable time delay. The proposed algorithm is easy and practical to implement. Nevertheless, when the V-K iteration is found infeasible, then it is possible that a feasible controller still exists. For the $\mathcal{H}_\infty$ control problem, Algorithm 2 is not guaranteed to converge globally, but it provides locally optimal solutions El Ghaoui and Balakrishnan (1994). However, for most cases, it yields decent designed controllers depending on the initialization, and can be used as a practical design tool. It is noted that the proposed algorithm requires certain efforts to search for suitable initial controller parameters. We suggest applying initial controller based on state-feedback stabilization for LSTD with the same parameters obtained from Algorithm 1. Furthermore, we find that a uniform grid of controller parameters around the calculated state-feedback stabilization can provide other initial controllers for Algorithm 2. In addition, refining the feedback gain by increasing number of delay partitions for Theorem 6 can slightly improve the guaranteed worst-case performance. The refined worst-case performance is lower or at least equal to that provided by Theorem 6 with less number of delay partitions.

5.6 Summary

This chapter presents robust state-feedback stabilization and state-feedback $\mathcal{H}_\infty$ control for LSTD subject to time-invariant delays. The methods seek design variables to satisfy absolute stability and robust $\mathcal{H}_\infty$ performance condition involving the uniform partitioning Lyapunov-Krasovskii functional with the integral of sector-bounded nonlinearities. The sufficient conditions are formulated in terms of BMI resulting in non-convex optimization problems. In this work, we employ the coordinate optimization to solve such BMI problems and obtain local optimal solutions. The proposed algorithms iterate between two LMI problems, which can be efficiently solved. From two numerical examples, the algorithm works well for state-feedback stabilization design. On the other hand, it requires a relatively large number of iterations to reach a decent robust state-feedback $\mathcal{H}_\infty$ control design. Numerical results illustrate that the proposed algorithms can effectively give robust state-feedback controllers for LSTD.

CHAPTER VI

APPLICATION TO CSTR WITH RECYCLE STREAM

The use of recycling is widespread in industry (Ray, 1981), since it reduces the cost of the reaction by decreasing waste of reagents. The unreacted reagents are returned into the CSTR by feeding through a long pipe. By letting unreacted feed material to make repeated passes, the recycling process greatly increase the overall conversion of a reaction. However, the difficulty arises due to the transportation delay introduced in the recycle line. The goal of this chapter is to present an example of state-feedback $\mathcal{H}_\infty$ controller design proposed in §5. The controller is designed for a continuous stirred tank reactor (CSTR) with recycle stream subject to uncertain reaction rate and flow nonlinearity.

6.1 Dynamic Model of CSTR with Recycle Stream

The process, proposed by Ray (1981), which has been employed as benchmark in the papers (Phoojaruenchanachai et al., 1998; Scali and Ferrari, 1999) with different parameters, consists in two chemical reactors connected in series with a recycle stream from the output of the second to the input of the first as shown in Figure 6.1. The state delay arises due to the transient lags in the recycle stream. Note that c_1, c_2 are the compositions of product streams from the two

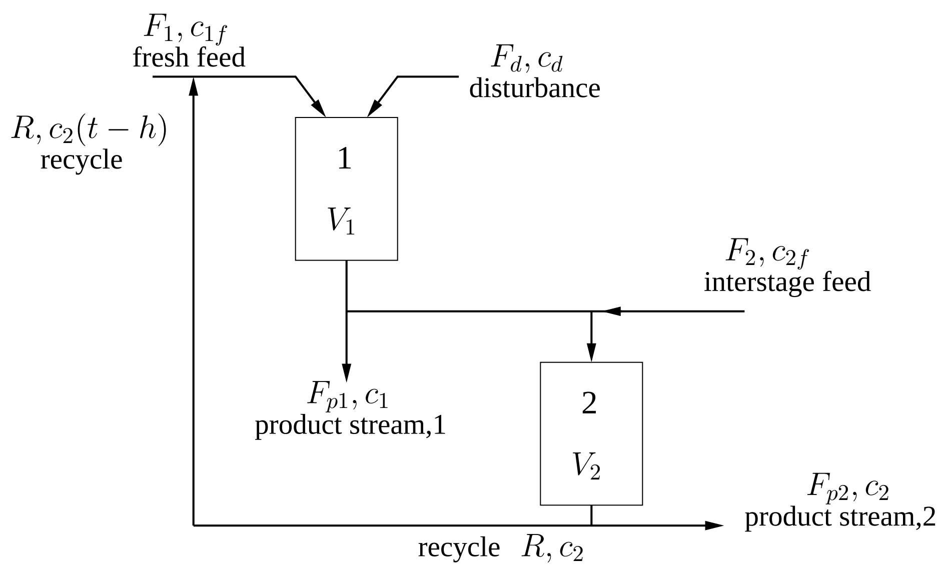

Figure 6.1: Two-stage chemical reactor train with delayed recycle.

reactors, c_{1f}, c_{2f} are the feed compositions to the two reactors, c_d is the composition of an extra feed stream (disturbance), F_{p1}, F_{p2} are the feed rate of product streams from the two reactors, F_1, F_2 are the feed rate to the two reactors, F_d is the extra feed stream (disturbance), and R is the recycle flow rate.

Consider the irreversible reaction $A \to B$ with negligible heat effect which is carried out in the two-stage reactor system. The flow rates to the reactor system are fixed and only the compositions are varied. The control objective is to maintain constant the two output compositions c_1 and c_2, by manipulations the two feed compositions c_{1f} and c_{2f}. The assumptions about the process are as follows.

- The two reactors are perfectly stirred and a first order irreversible reaction, with kinetic coefficients $r_1(t)$ and $r_2(t)$ is carried out.
- Levels, temperatures and flow rates are constant.
- The control objective is to maintain constant the two output compositions c_1 and c_2, by manipulations the two feed compositions c_{1f} and c_{2f}.
- The disturbance entering the process in the composition c_d to the first reactor.
- The output composition c_1 and c_2 can be measured immediately.

The dynamic mathematical model of the reactor is obtained by mass balances of reactants, and the simplified nonlinear dynamic model of the chemical reactor constitutes two differential equations.

$$V_1\dot{c}_1(t) = F_1 c_{1f}(t) + Rc_2(t-h) + F_d c_d(t) - (F_1 + R + F_d)c_1(t) - V_1 r_1(t) c_1(t), \quad (6.1)$$

$$V_2\dot{c}_2(t) = (F_1 + R + F_d - F_{p1})c_1(t) + F_2 c_{2f}(t) - (F_{p2} + R)c_2(t) - V_2 r_2(t) c_2(t), \quad (6.2)$$

where the second product stream F_{p2} is given by $F_{p2} = F_1 + F_d - F_{p1} + F_2$. Supposing the process is initially at steady-state, $c_1(0) = c_{1s}$, $c_2(t) = c_{2s}$, $t \in [-h, 0]$, $c_{1f}(0) = c_{1fs}$, $c_{2f}(0) = c_{2fs}$, $c_d(0) = c_{ds}$, the linearized steady-state model of CSTR system is given by

$$\dot{x}_1(t) = -\left[\frac{F_1 + R + F_d}{V_1} + r_1(t)\right] x_1(t) + \frac{R}{V_1} x_2(t-h) + \frac{F_d}{V_1} w(t) + \frac{F_1}{V_1} u_1(t), \quad (6.3)$$

$$\dot{x}_2(t) = \left[\frac{F_1 + R + F_d - F_{p1}}{V_2}\right] x_1(t) - \left[\frac{(F_{p2} + R)}{V_2} + r_2(t)\right] x_2(t) + \frac{F_2}{V_2} u_2(t), \quad (6.4)$$

where $x_1(t) = c_1(t) - c_{1s}$, $x_2(t) = c_2(t) - c_{2s}$, $u_1(t) = c_{1f}(t) - c_{1fs}$, $u_2(t) = c_{2f}(t) - c_{2fs}$, and $w(t) = c_d(t) - c_{ds}$.

Define the state and input as $x(t) = [x_1(t) \quad x_2(t)]^T$, $u(t) = [u_1(t) \quad u_2(t)]^T$, the state-space model of the CSTR with recycle stream can be written as follows,

$$\begin{aligned}\dot{x}(t) = &\begin{bmatrix} -\frac{F_1+R+F_d}{V_1} - r_1(t) & 0 \\ \frac{F_1+R+F_d-F_{p1}}{V_2} & -\frac{(F_{p2}+R)}{V_2} - r_2(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 & \frac{R}{V_1} \\ 0 & 0 \end{bmatrix} x(t-h) \\ &+ \begin{bmatrix} \frac{F_d}{V_1} \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} \frac{F_1}{V_1} & 0 \\ 0 & \frac{F_2}{V_2} \end{bmatrix} u(t).\end{aligned} \tag{6.5}$$

The $r_1(t)$ and $r_2(t)$ are reaction rate coefficients, which can be written in the form of

$$\begin{aligned} r_1(t) &= r_1 + \delta r_1(t), \\ r_2(t) &= r_2 + \delta r_2(t), \end{aligned}$$

where r_1 and r_2 are constant. Although, the exact value of $\delta r_1(t)$ and $\delta r_2(t)$ are not known, the information on their bounds is available, *i.e.*,

$$\begin{aligned} \|\delta r_1(t)\| &\leq \delta \bar{r}_1, \\ \|\delta r_2(t)\| &\leq \delta \bar{r}_2. \end{aligned}$$

The uncertain parts of $r_1(t)$ and $r_2(t)$ can be formulated as

$$\begin{bmatrix} \delta r_1(t) & 0 \\ 0 & \delta r_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} F(t) \begin{bmatrix} \delta \bar{r}_1 & 0 \\ 0 & \delta \bar{r}_2 \end{bmatrix},$$

where $F(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying the constraint $F^T(t)F(t) \leq I$. Thus, the linearized steady-state model of CSTR system becomes

$$\begin{aligned}\dot{x}(t) = &\begin{bmatrix} -\frac{F_1+R+F_d}{V_1} - r_1 & 0 \\ \frac{F_1+R+F_d-F_{p1}}{V_2} & -\frac{(F_{p2}+R)}{V_2} - r_2 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} F(t) \begin{bmatrix} \delta \bar{r}_1 & 0 \\ 0 & \delta \bar{r}_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & \frac{R}{V_1} \\ 0 & 0 \end{bmatrix} x(t-h) \\ &+ \begin{bmatrix} \frac{F_d}{V_1} \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} \frac{F_1}{V_1} & 0 \\ 0 & \frac{F_2}{V_2} \end{bmatrix} u(t).\end{aligned} \tag{6.6}$$

We assume that the disturbance flow rate, F_d, associating with the state $x_1(t)$ has nonlinear behavior, and can be treated as the sector-bounded nonlinearity

$$\phi(x_1(t)) \in \Phi(F_d - \delta l, F_d + \delta l),$$

where $\delta l > 0$ represents uncertain level in the flow rate F_d. Finally, the model of CSTR system with recycle stream is written as follows

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -\frac{F_1+R}{V_1} - r_1 & 0 \\ \frac{F_1+R-F_{p1}}{V_2} & -\frac{(F_{p2}+R)}{V_2} - r_2 \end{bmatrix} x(t) \\
&\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} F(t) \begin{bmatrix} \delta\bar{r}_1 & 0 \\ 0 & \delta\bar{r}_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & \frac{R}{V_1} \\ 0 & 0 \end{bmatrix} x(t-h) \\
&\quad + \begin{bmatrix} -\frac{1}{V_1} \\ \frac{1}{V_2} \end{bmatrix} p(t) + \begin{bmatrix} \frac{F_d}{V_1} \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} \frac{F_1}{V_1} & 0 \\ 0 & \frac{F_2}{V_2} \end{bmatrix} u(t), \\
q(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{6.7}$$

where $\phi \in \Phi(F_d - \delta l, F_d + \delta l)$.

6.2 Controller Design

It is assumed that the values of the parameters of the presented CSTR with recycle stream are as following: $V_1 = 1$, $V_2 = 1$, $F_1 = 0.4$, $F_2 = 0.5$, $F_{p1} = 0.5$, $F_{p2} = 0.5$, $F_d = 0.1$, $R = 0.25$, $r_1 = 1$, $r_2 = 1$, $\delta\bar{r}_1 = 0.4\alpha$, $\delta\bar{r}_2 = 0.5\alpha$, $h = 1$. Substituting the given parameters into (6.7), we obtained the LSTD

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -1.65 & 0 \\ 0.15 & -1.75 \end{bmatrix} x(t) \\
&\quad + \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} F(t) \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.25 \\ 0 & 0 \end{bmatrix} x(t-h) \\
&\quad + \begin{bmatrix} -1 \\ 1 \end{bmatrix} p(t) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix} u(t), \\
q(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\
p(t) &= \phi(q(t)),
\end{aligned} \tag{6.8}$$

where $\phi \in \Phi(0.1 - \delta l, 0.1 + \delta l)$ represents flow nonlinearity and $F^T(t)F(t) \le I$. In addition, we define the performance output as

$$z(t) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 25 & 0 \\ 0 & 25 \end{bmatrix} u(t).$$

The controller design problem is to search for the state-feedback control law, which minimizes the disturbance $w(t)$ at the performance output $z(t)$ for all $w(t) \in \mathcal{L}_2$, *i.e.*, minimizing the disturbance attenuation level, γ_∞. The desired control law is of the centralized form:

$$u(t) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} x(t), \tag{6.9}$$

or the decentralized form:

$$u(t) = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} x(t). \tag{6.10}$$

We employ Theorem 6 to synthesize the robust state-feedback $\mathcal{H}_\infty$ controller K for the given system (6.8). Note that the desired absolute tolerance for this example is chosen to be 10^{-5}. Choosing K_0 to be the zero matrix with dimension 2×2, the calculated centralized feedback gains for the case of $N = 2$ and $N = 3$ are shown in Table 6.1. From the Table 6.1, we can see that γ_∞ obtained from Theorem 6 with $N = 2$ is very close to that determined using Theorem 6 with $N = 3$. The calculated feedback gains for decentralized case with the corresponding γ_∞ are summarized in Table 6.2. When comparing between centralized and decentralized cases, it is clear that γ_∞ for decentralized case is always greater than that for the centralized case. In addition, each element of K is larger and it requires more iteration for computing the desired controller. For all design cases, the required iterations range from $43 - 167$ and do not directly associate with δl and α. Figure 6.2 and 6.3 confirm that γ_∞ for decentralized case is slightly greater than that of the centralized case. Moreover, the figures also show the tendency of γ_∞ from small δl to large δl and small α to large α. In particular, γ_∞ is increased as δl or α is larger.

Next, time-response of the open-loop and closed-loop reactor system is observed . The test disturbance is selected as depicted in Figure 6.4. It vanishes after $t = 10$, so that we can investigate the settling behaviors of the time-responses. Figure 6.5 shows the state variables

Table 6.1: Centralized $\mathcal{H}_\infty$ controllers designed for CSTR with recycle stream.

δl	α	$N=2$			$N=3$		
		Feedback gain	γ_∞	#iter	Feedback gain	γ_∞	#iter
0.01	0.50	$\begin{bmatrix} -4.24 & -1.87 \\ -0.83 & -12.62 \end{bmatrix}$	4.5186	116	$\begin{bmatrix} -4.23 & -0.72 \\ -0.81 & -11.33 \end{bmatrix}$	4.5186	59
	0.75	$\begin{bmatrix} -4.48 & 0.89 \\ -0.88 & -12.86 \end{bmatrix}$	4.6608	119	$\begin{bmatrix} -4.57 & -3.07 \\ -0.86 & -11.49 \end{bmatrix}$	4.6608	74
	1.00	$\begin{bmatrix} -4.88 & -1.74 \\ -0.94 & -13.28 \end{bmatrix}$	4.8061	143	$\begin{bmatrix} -4.92 & -3.21 \\ -0.91 & -11.58 \end{bmatrix}$	4.8061	96
0.02	0.50	$\begin{bmatrix} -4.27 & -1.99 \\ -0.86 & -12.89 \end{bmatrix}$	4.5310	87	$\begin{bmatrix} -4.28 & -2.37 \\ -0.83 & -11.33 \end{bmatrix}$	4.5310	49
	0.75	$\begin{bmatrix} -4.70 & -7.45 \\ -0.88 & -11.90 \end{bmatrix}$	4.6735	103	$\begin{bmatrix} -4.57 & -1.35 \\ -0.87 & -11.43 \end{bmatrix}$	4.6736	58
	1.00	$\begin{bmatrix} -4.89 & -0.55 \\ -0.96 & -13.20 \end{bmatrix}$	4.8191	115	$\begin{bmatrix} -4.94 & -2.23 \\ -0.93 & -11.58 \end{bmatrix}$	4.8192	67
0.03	0.50	$\begin{bmatrix} -4.34 & -3.80 \\ -0.87 & -12.60 \end{bmatrix}$	4.5435	70	$\begin{bmatrix} -4.30 & -1.96 \\ -0.84 & -11.30 \end{bmatrix}$	4.5435	43
	0.75	$\begin{bmatrix} -4.63 & -3.01 \\ -0.93 & -13.00 \end{bmatrix}$	4.6863	78	$\begin{bmatrix} -4.59 & -1.17 \\ -0.88 & -11.34 \end{bmatrix}$	4.6864	44
	1.00	$\begin{bmatrix} -4.98 & -2.70 \\ -0.99 & -13.19 \end{bmatrix}$	4.8322	78	$\begin{bmatrix} -4.95 & -1.06 \\ -0.93 & -11.22 \end{bmatrix}$	4.8323	57

Table 6.2: Decentralized $\mathcal{H}_\infty$ controllers designed for CSTR with recycle stream.

δl	α	$N = 2$			$N = 3$		
		Feedback gain	γ_∞	#iter	Feedback gain	γ_∞	#iter
0.01	0.50	$\begin{bmatrix} -4.22 & 0 \\ 0 & -55.66 \end{bmatrix}$	4.5209	167	$\begin{bmatrix} -4.23 & 0 \\ 0 & -50.57 \end{bmatrix}$	4.5210	126
	0.75	$\begin{bmatrix} -4.52 & 0 \\ 0 & -56.76 \end{bmatrix}$	4.6634	151	$\begin{bmatrix} -4.52 & 0 \\ 0 & -52.31 \end{bmatrix}$	4.6635	135
	1.00	$\begin{bmatrix} -4.86 & 0 \\ 0 & -58.62 \end{bmatrix}$	4.8091	142	$\begin{bmatrix} -4.86 & 0 \\ 0 & -52.51 \end{bmatrix}$	4.8093	133
0.02	0.50	$\begin{bmatrix} -4.25 & 0 \\ 0 & -56.53 \end{bmatrix}$	4.5336	137	$\begin{bmatrix} -4.25 & 0 \\ 0 & -51.73 \end{bmatrix}$	4.5337	113
	0.75	$\begin{bmatrix} -4.55 & 0 \\ 0 & -58.08 \end{bmatrix}$	4.6764	133	$\begin{bmatrix} -4.55 & 0 \\ 0 & -52.62 \end{bmatrix}$	4.6766	118
	1.00	$\begin{bmatrix} -4.89 & 0 \\ 0 & -59.90 \end{bmatrix}$	4.8224	129	$\begin{bmatrix} -4.89 & 0 \\ 0 & -53.27 \end{bmatrix}$	4.8226	117
0.03	0.50	$\begin{bmatrix} -4.27 & 0 \\ 0 & -58.28 \end{bmatrix}$	4.5463	121	$\begin{bmatrix} -4.27 & 0 \\ 0 & -51.70 \end{bmatrix}$	4.5465	99
	0.75	$\begin{bmatrix} -4.57 & 0 \\ 0 & -59.04 \end{bmatrix}$	4.6896	116	$\begin{bmatrix} -4.57 & 0 \\ 0 & -52.98 \end{bmatrix}$	4.6897	103
	1.00	$\begin{bmatrix} -4.92 & 0 \\ 0 & -60.55 \end{bmatrix}$	4.8359	112	$\begin{bmatrix} -4.92 & 0 \\ 0 & -53.65 \end{bmatrix}$	4.8361	103

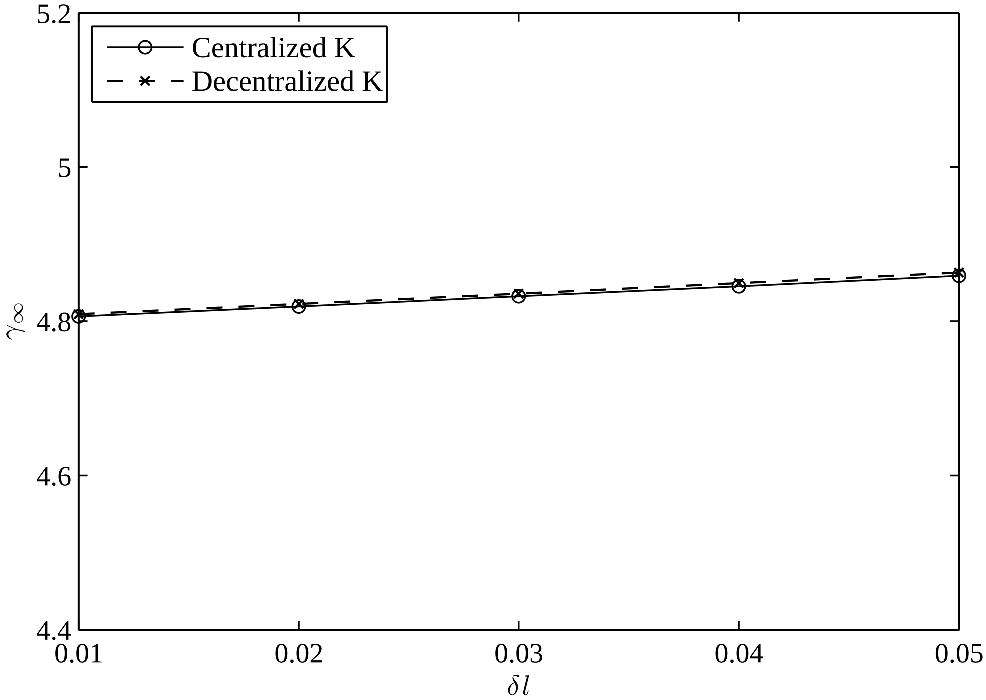

Figure 6.2: Upper bound of $\mathcal{J}_\infty$ vs. δl for CSTR with recycle stream when $\alpha = 1.00$.

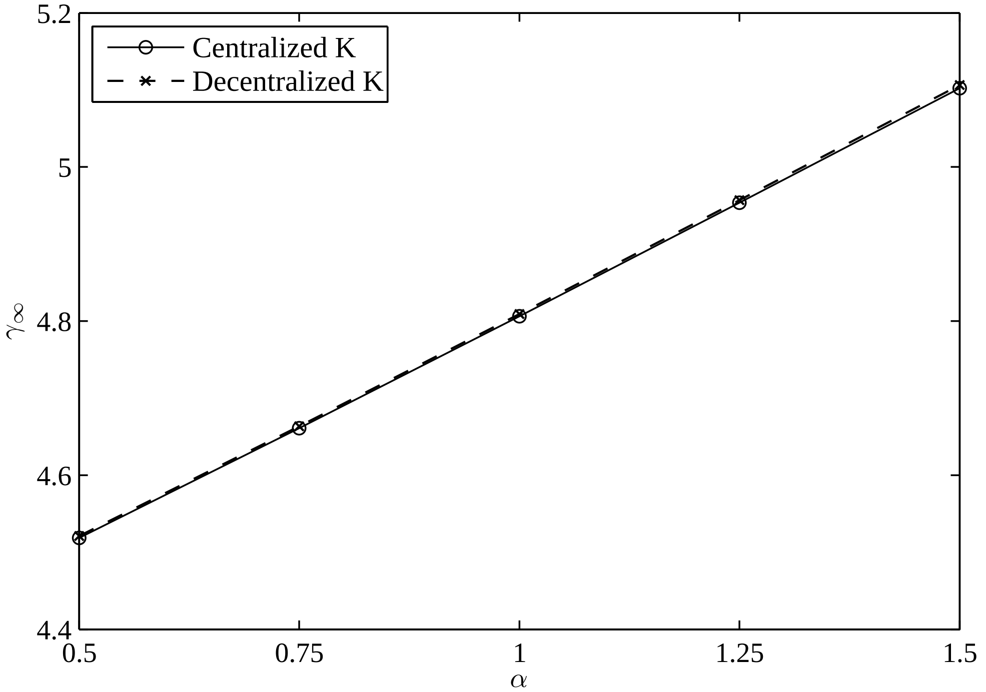

Figure 6.3: Upper bound of $\mathcal{J}_\infty$ vs. α for CSTR with recycle stream when $\delta l = 0.01$.

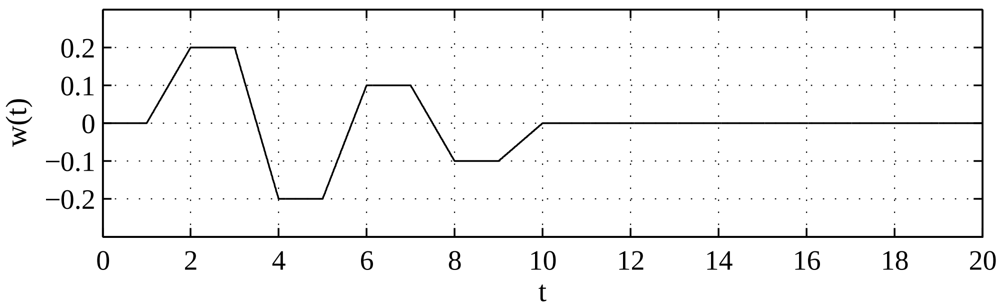

Figure 6.4: Disturbance input.

$x_1(t)$ and $x_2(t)$ of the reactor system subject to the disturbance $w(t)$ for various δl. It is shown that the states $x_1(t)$ and $x_2(t)$ are fluctuated with certain magnitudes due to the presence of disturbance. The small delay of the responses can be observed. It can be seen that the feedback can help reducing the magnitudes of the oscillation in $x_1(t)$ and $x_2(t)$. Moreover, the length of the oscillation period is reduced by a little amount of time. Figure 6.6 presents the corresponding control inputs for the closed-loop response. The control signals are reasonable and they can be practically produced. Other simulations are conducted for the system with various α as shown in Figure 6.7 and 6.8. It is clear that the designed controllers can improve the response subject to the given disturbance. The decreasing in the magnitude indicates that the off-spec product is reduced, thereby cutting down the cost of the reaction.

Table 6.3: Performance gain for CSTR with recycle stream with various δl and α.

System parameters			Open-loop gain		Closed-loop gain	
h	δl	α	$\|z\|_2/\|w\|_2$	γ_∞	$\|z\|_2/\|w\|_2$	γ_∞
1	0.01	0.50	4.5745	6.7568	2.9330	4.5186
1	0.01	0.75	4.6004	7.2798	2.8851	4.6608
1	0.01	1.00	4.6353	7.8987	2.8351	4.8061
1	0.02	1.00	4.6353	7.9369	2.8297	4.8191
1	0.03	1.00	4.6352	7.9760	2.8243	4.8322

Table 6.3 provides the open-loop gain and closed-loop gain for each case of δl and α. The γ_∞ is the calculated upper bound of the worst-case $\mathcal{H}_\infty$ performance. For this design example, the designed feedback control can reduce the gain of $\|z\|_2/\|w\|_2$ by $36 - 39\%$.

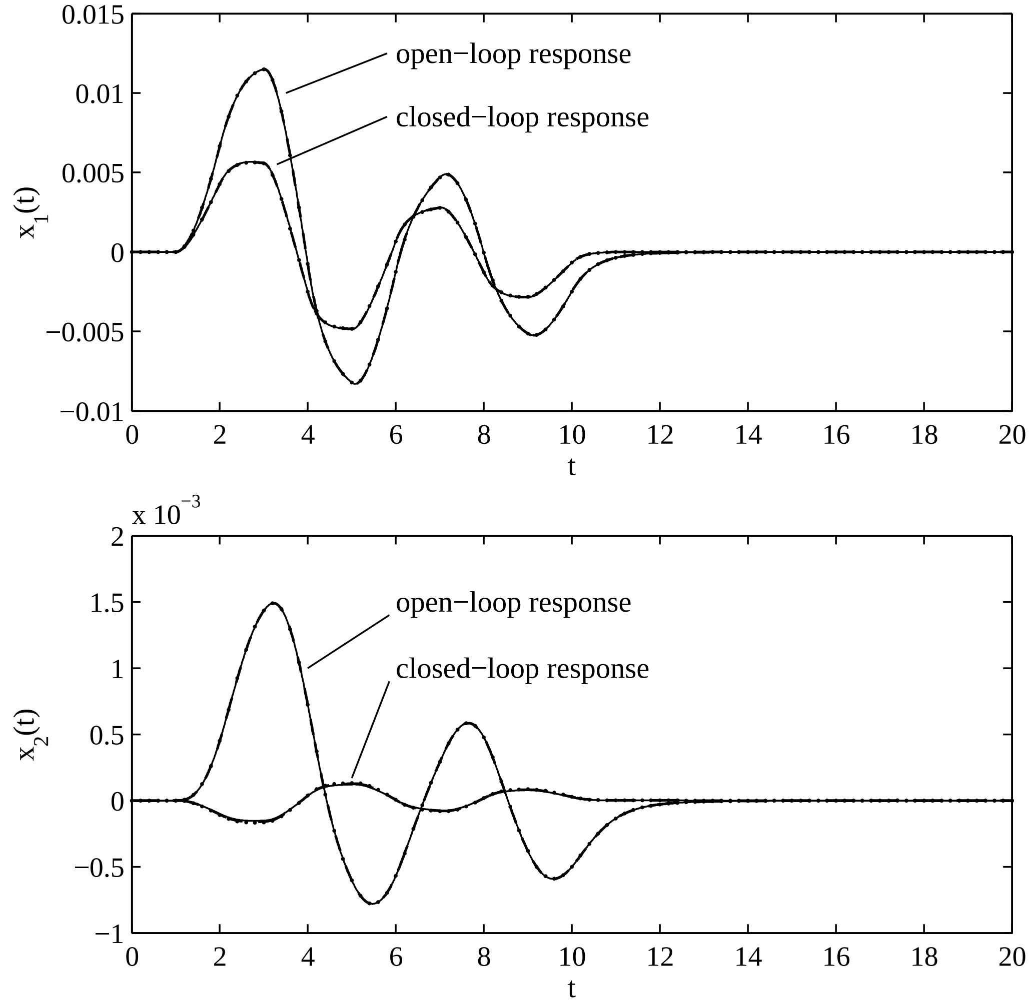

Figure 6.5: Time response of CSTR with recycle stream: $\delta l = 0.01$ (solid lines), $\delta l = 0.02$ (dashed lines), $\delta l = 0.03$ (dotted lines) when $\alpha = 1.00$.

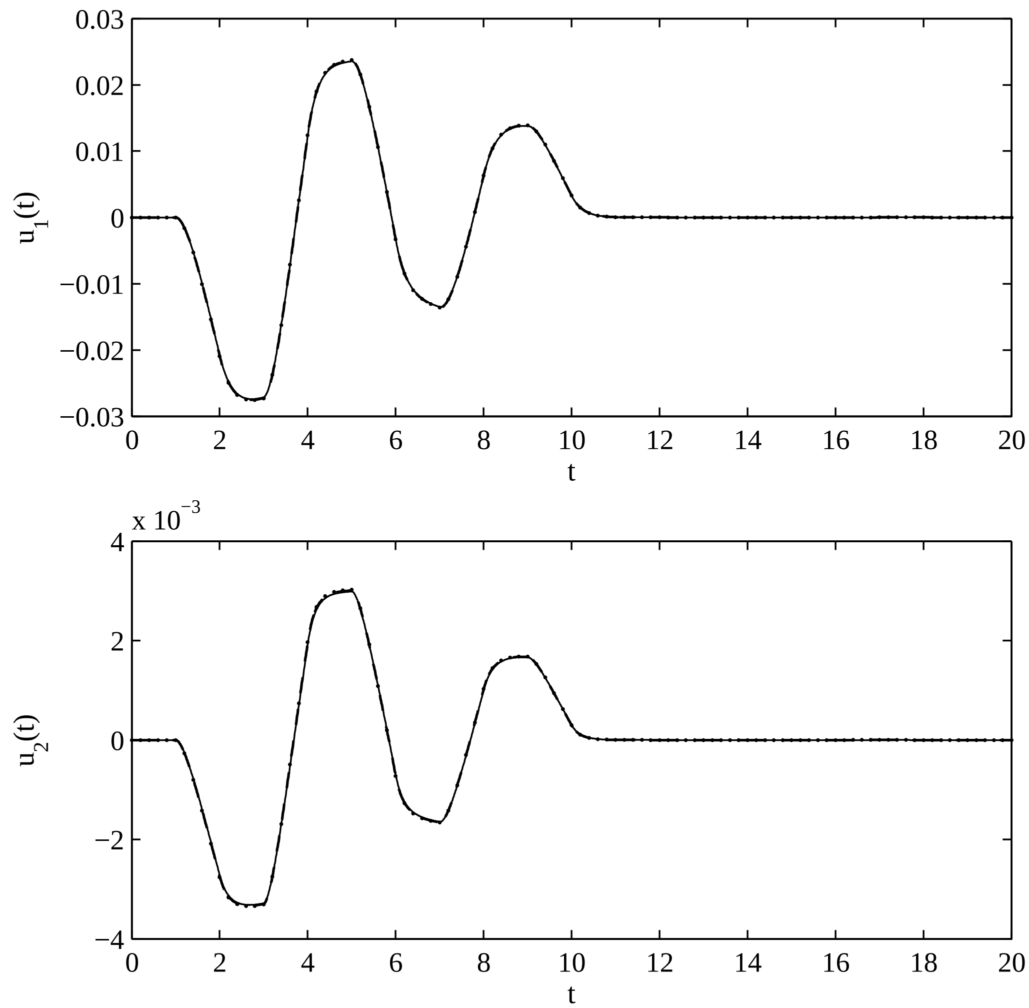

Figure 6.6: Control input: $\delta l = 0.01$ (solid lines), $\delta l = 0.02$ (dashed lines), $\delta l = 0.03$ (dotted lines) when $\alpha = 1.00$.

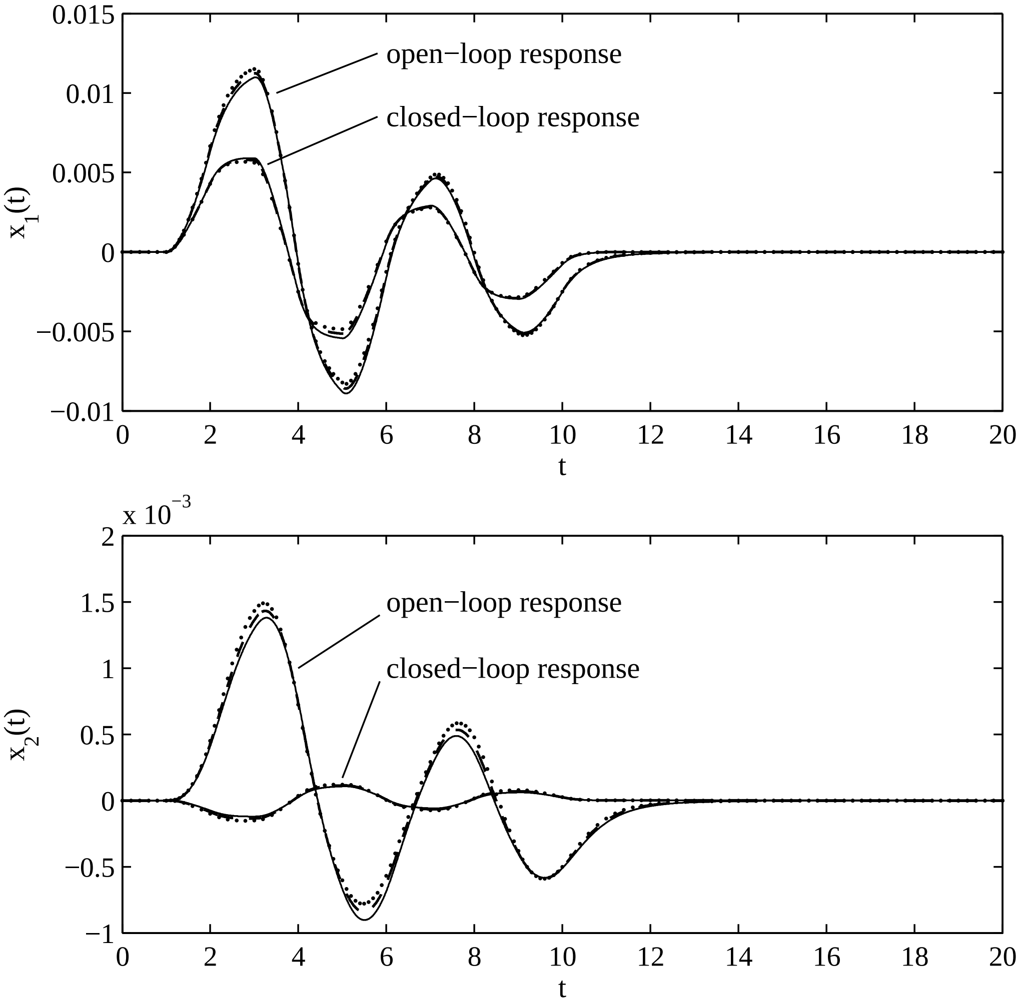

Figure 6.7: Time response of CSTR with recycle stream: $\alpha = 0.5$ (solid lines), $\alpha = 0.75$ (dashed lines), $\alpha = 1.00$ (dotted lines) when $\delta l = 0.01$.

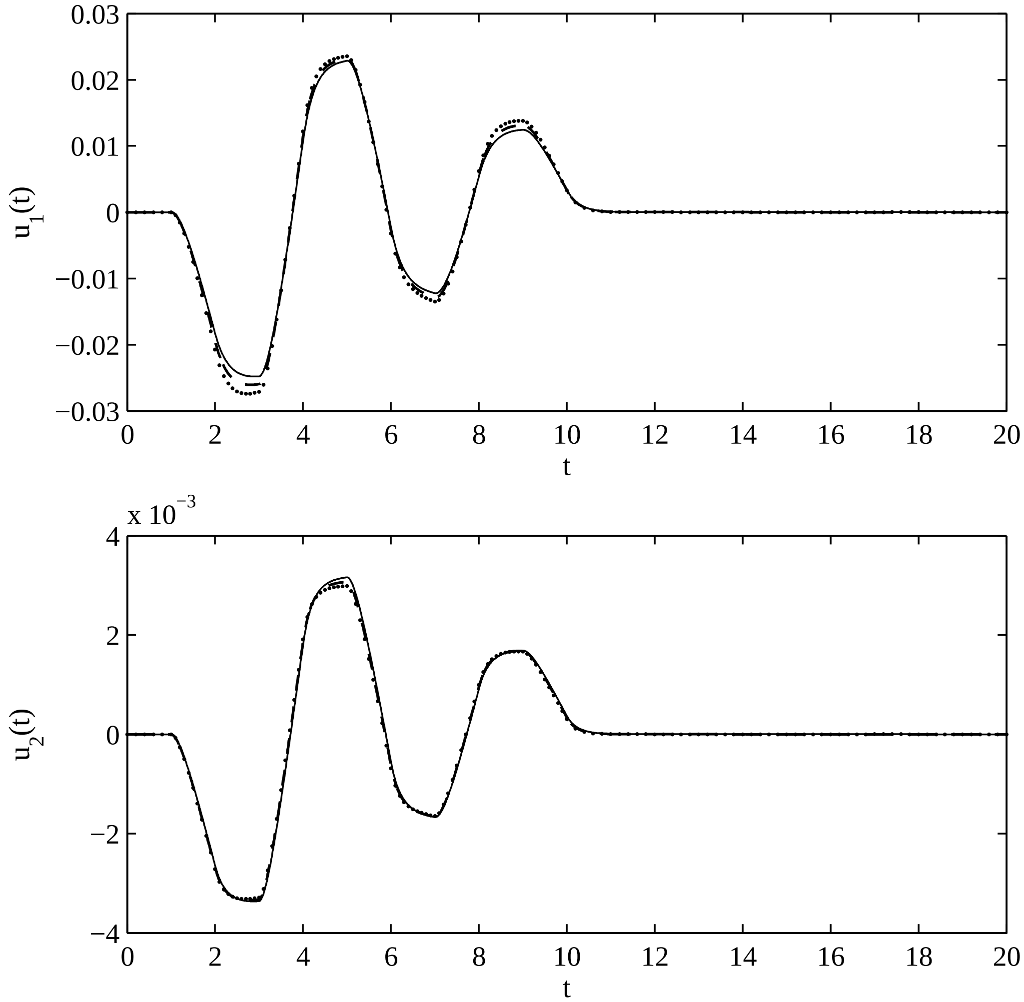

Figure 6.8: Control input: $\alpha = 0.5$ (solid lines), $\alpha = 0.75$ (dashed lines), $\alpha = 1.00$ (dotted lines) when $\delta l = 0.01$.

6.3 Summary

A state-feedback $\mathcal{H}_\infty$ controller design for CSTR with recycle stream is presented. A transportation delay occurs when the unreacted regentes are fed back to the CSTR through a pipe. The parametric uncertainties in reaction rate coefficients are treated as norm-bounded uncertainty, and the uncertain flow rate is captured by using sector-bounded nonlinearity. The simulation results confirm that the proposed design can improve the disturbance attenuation with robustness against uncertain reaction coefficient and nonlinear flow rate.

CHAPTER VII
CONCLUSIONS

7.1 Summary of Results

The dissertation first improves the delay-dependent absolute stability criterion for LSTD with an uncertain time invariant delay. The criterion is derived based on the Lyapunov-Krasovskii functional with the integral of sector-bounded nonlinearities. The length of time delay is uniformly divided into the number of fragments so that the Lyapunov functional terms involving delay are partitioned to be associated with each fragment. The absolute stability criterion is derived from time derivative of the new Lyapunov-Krasovskii functional, then the Jensen inequality and $\mathcal{S}$-procedure are applied to formulate the sufficient condition in terms of LMIs. Furthermore, the criterion for LSTD subject to norm-bounded uncertainty is developed by using the eliminating lemma. Numerical examples in Chapter 3 show that the proposed criteria can enlarge the maximum allowable time delay comparing to other existing criteria especially when the sector bound is comparatively large.

Secondly, the dissertation presents the worst-case $\mathcal{H}_\infty$ performance criterion for Lur'e systems with uncertain time-invariant delays. The criterion is derived by combining the absolute stability analysis in Chapter 3 with robust performance bounds on the total output energy of the system. In addition, the criterion for LSTD subject to norm-bounded uncertainty is developed by using the matrix eliminating lemma. The sufficient condition to ensure the worst-case performance is in terms of LMIs, and the problem of computing an upper bound of the worst-case $\mathcal{H}_\infty$ performance can be cast as an LMI minimization problem. Numerical examples in Chapter 4 show that the proposed criteria are less conservative than the comparative criteria, and can be served as an effective worst-case performance analysis for LSTD.

Thirdly, we develop robust state-feedback stabilization for LSTD subject to time-invariant delays. The feedback stabilization design is established by extending the absolute stability analysis. Particularly, absolute stability of the closed-loop system is guaranteed via the criterion in Chapter 3, when the feedback gain is regarded as the design parameter. The product terms between Lyapunov matrices and feedback gain leads to the BMI constraint, and the controller design problem becomes a BMI feasibility problem. However, a local solution of such BMI problem is determined by alternating between two LMI feasibility problems until the original BMI constraint is satisfied. From two numerical examples in Chapter 5, the algorithm works well, and can effectively give robust state-feedback stabilization for LSTD.

Lastly, the dissertation presents robust state-feedback $\mathcal{H}_\infty$ control for LSTD subject to time-invariant delays. The $\mathcal{H}_\infty$ control is achieved by determining the feedback gain to minimize an upper bound of the worst-case $\mathcal{H}_\infty$ performance. In particular, by applying the worst-case performance analysis in Chapter 4 to the closed-loop systems, regarding the feedback gain as the design parameter. The coupling terms between the variable matrices in LKF and the feedback gain results in the BMI constraint. Since the linear objective minimization over BMI constraints can be locally solved via an LMI-based method, we develop an iterative algorithm to solve such controller design problem for local optimal solutions. Numerical results in Chapter 5 and 6 illustrate that it requires a relatively large number of iterations to reach a decent robust state-feedback $\mathcal{H}_\infty$ control design. Nevertheless, the proposed algorithms can effectively give robust state-feedback controllers for LSTD.

7.2 Conclusions

1. We have presented an improved absolute stability analysis for LSTD with uncertain time-invariant delay. The criteria for both the nominal system and the system with time-varying norm-bounded uncertainty have been developed. It is shown that the partition of the time delay has advantage to reduce the gap of Jensen inequality. Increasing the number of partition, N, the computed $\bar{h}_{\max}$ approaches the actual bound. Furthermore, exploiting the integral of the sector-bounded nonlinearities in the LKF can further increase the maximum allowable time delays. When the sector bound is large, which is harmful to the stability of the system, it is clear to see that the proposed criteria give a larger value of $\bar{h}_{\max}$. Accordingly, the combination of these approaches effectively provides the improved absolute stability analysis. We also propose a bisection method associated with LMI optimization for determining the maximum allowable time delay.

2. We have combined the absolute stability analysis in Chapter 3 with robust performance bounds on the total output energy to establish the worst-case $\mathcal{H}_\infty$ performance analysis. The criterion for LSTD with time-varying norm-bounded uncertainty is also developed in the same way as for the absolute stability analysis. Numerical results confirm that the developed criteria are less conservative than the comparative criteria. In particular, a guaranteed upper bound for the worst-case $\mathcal{H}_\infty$ performance obtained from the developed criteria is always closer to the actual $\mathcal{L}_2$ gain. In some cases, the criteria in Chapter 4 can provide a guarantee on the worst-case performance, while the comparative criteria fail. In

conclusion, the proposed worst-case performance analysis can be served as an effective worst-case $\mathcal{H}_\infty$ performance analysis for LSTD.

3. We have extended the absolute stability analysis in Chapter 3 to the robust state-feedback stabilization. Generally, optimizing both the analysis variables and controller parameter simultaneously leads to the BMI optimization problem. In Chapter 5, the LMI iterative procedure is developed to determine the solution of the control problems. The algorithm comprises of two phases, namely analysis step and synthesis step. By solving different LMI feasibility problems in each step, the solution will be obtained when the original BMI constraint is satisfied. Numerical examples with several cases of time delay, sector-bounded nonlinearities, and uncertainties have been conducted. The experiments show that the iterative algorithm can design state-feedback stabilization for a given maximum allowable time delay. Nevertheless, when the LMI problem in each step is found infeasible, then it is possible that a feasible controller still exists.

4. We have developed a robust state-feedback $\mathcal{H}_\infty$ control by extending the worst-case $\mathcal{H}_\infty$ performance analysis in Chapter 4. The controller design problem is cast as a linear objective minimization over the BMI constraint. To solve such BMI optimization problem, an LMI-based iterative procedure has been proposed. The algorithm is similar to that for the stabilization problem, but each step in the iteration is solving the LMI minimization problem instead of the feasibility problem. It is not guaranteed to converge globally, but it provides locally optimal solutions. However, for most cases, it yields decent designed controllers depending on the initialization, and can be used as a practical design tool. The numerical results in Chapter 5 and the controller design for disturbance rejection problem of CSTR with recycle stream in Chapter 6 confirm that the proposed robust state-feedback $\mathcal{H}_\infty$ control can be served as an effective design tool for LSTD with time-invariant delay.

7.3 Recommendations for Future Work

1. In this dissertation, the delay-partitioning approach with the uniform fragment scheme is utilized. However, the nonuniform fragmentation schemes can reduce the conservatism in certain cases as shown in Briat (2011). In general, there is no common nonuniform partitioning which is optimal for all systems. Some attempts have been made for the case of two partitions by considering the length of the first partition as the variable to be optimized (Gao et al., 2008; Kazemy and Farrokhi, 2013), but there is no systematic

procedure to determine optimal delay partitions. Therefore, further research should be done to investigate the nonuniform fragmentation schemes.

2. The results in this dissertation involve the analysis and synthesis of Lur'e systems with time delays, when the nonlinearity of the Lur'e systems is only sector-restricted. In some applications, information on the slope of the nonlinearity is available. As we known, employing more information on the system leads to a less conservative result. Therefore, the analysis and synthesis of LSTD, when the nonlinearity of the Lur'e systems is sector- and slope-restricted nonlinearity, are valuable to investigate. For example, self and cross properties of sector and slope restrictions (Park, 2002) can be included in the Lyapunov functional in Chapter 3 and 4 to obtained new analysis and designs.

3. In this dissertation, we have presented the LMI-based iterative algorithm for finding the local optimal solution of the controller synthesis problems. Although the algorithm works well for state-feedback stabilization design, it requires a relatively large number of iterations to reach a decent robust state-feedback $\mathcal{H}_\infty$ control design. Hence, it is interesting to compare the solutions computed using the proposed algorithm with those obtained from other local LMI optimization methods such as *dual iteration* (Iwasaki, 1999) which is another variation of coordinate descent method with more suitable choice of coordinates, *path-following method* (Hassibi et al., 1999) which is based on linearization by making use of the first-order perturbation approximation.

4. In this dissertation, we have shown techniques to design robust stabilization and $\mathcal{H}_\infty$ control using state-feedback. When the full state is not available for feedback, the designs in Chapter 5 is not applicable to handle state feedback case. Further research can be carried out to develop a dynamic output feedback control for LSTD based on the absolute stability analysis in Chapter 3 and the worst-case performance analysis in Chapter 4. In particular, the controller design with specific structure of control such as PID control and reduced-order controller should be further studied. In addition, the analysis and synthesis with other performance indices such as $\mathcal{H}_2$ and $\mathcal{L}_\infty$-induced norm can be developed via the proposed technique.

5. It is interesting to employ the proposed control designs with other engineering applications, for example,

   - Delayed Takagi-Sugeno (TS) fuzzy systems (Cao and Frank, 2000, 2001).

- Master-slave synchronization problem of Lur'e systems with time-delay (Yalçin et al., 2001) which can be applied to optical communication.
- Recurrent neural networks (RNNs) with time delays (Cao and Wang, 2003) which is essential in applications to signal and image processing, artificial intelligence, and industrial automation.

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Relevant Publications

1. T. Nampradit and D. Banjerdpongchai, "Performance analysis for Lur'e systems with time delay using linear matrix inequalities," in *Proceedings of International Conference on Electrical Engineering/Electronics Computer Telecommunications and Information Technology (ECTI-CON)*, pp.733–737, 2010.

2. T. Nampradit and D. Banjerdpongchai, "On computing maximum allowable time delay of Lur'e systems with uncertain time-invariant delays," *International Journal of Control, Automation and Systems*, vol. 12, no. 3, pp.497–506, 2014.