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HOLOGRAPHIC MESONS IN EXTERNAL ELECTRIC FIELDS

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หลักสมนัยเอดีเอสซีเอฟทีเป็นวิธีที่ใช้อย่างแพร่หลายสำหรับศึกษาระบบที่ไม่สามารถศึกษาด้วยวิธีการรบกวน หรือที่เรียกว่าระบบนอนเพอร์เทอเบทีพ เราจะใช้หลักสมนัยนี้เพื่อที่จะศึกษาระบบของอนุภาคมีซอนที่มีอุณหภูมิที่วัดค่าได้และอยู่ในสนามไฟฟ้าภายนอก เราสนใจระบบดังกล่าวในสถานการณ์ที่ควาร์กองค์ประกอบของมีซอนมีอันตรกิริยาแบบแรง ระบบในสถานการณ์นี้จัดเป็นระบบนอนเพอร์เทอเบทีพ เพื่อที่จะเข้าใจหลักสมนัยเอดีเอสซี-เอฟที เราจำเป็นต้องศึกษาผลลัพธ์พื้นฐานของทฤษฎีซุปเปอร์สตริงชนิดสองบี จากนั้นเราจะพิจารณาแบบจำลองที่ประกอบด้วยเยื่อสามมิติและเยื่อเจ็ดมิติเรียกว่าแบบจำลองดีสามดีเจ็ด การคำนวณแอคชันของเยื่อเจ็ดมิติสามารถเชื่อมโยงสู่ปริมาณวัดได้ทางฟิสิกส์ รูปร่างของเยื่อเจ็ดมิติสามารถบรรยายการเปลี่ยนสถานะจากมีซอนที่เสถียรไปสู่สถานะมีซอนละลายได้ การเปลี่ยนสถานะของระบบเทียบได้กับการเปลี่ยนสถานะจากฉนวนไปเป็นตัวนำ เมื่อระบบนี้อยู่ภายใต้สนามไฟฟ้าภายนอก เราจะสามารถคำนวณเพื่อหาค่าของสภาพเหนี่ยวนำไฟฟ้าและขนาดของแรงต้านของสื่อทางไฟฟ้าของระบบนี้ได้

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The AdS/CFT correspondence is a popular method which allows us to study non-perturbative systems. We use the AdS/CFT correspondence to study a system of neutral mesons at finite temperature with external electric field. We consider the system in non-perturbative situation that constituent quarks of mesons strongly couple to each other. In order to understand the AdS/CFT correspondence, we will first discuss some basic result of type IIB superstring theory. Then we discuss the AdS/CFT correspondence and D3/D7 model. Calculations of D7-branes action can be matched to physical observables. We explain how shapes of D7-branes can describe the property of phase transition from stable mesons to melting mesons. The phase transition is analogous to metal/insulator phase transition. As external electric field is applied, we can calculate the conductivity and drag force of the system.

Department:.........Physics.........Student's Signature ....

Field of Study:........Physics........Advisor's Signature ....

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Chapter I

INTRODUCTION

Meson is a colorless bound state of a pair quark and anti-quark. The attractive force binding the constituent quarks is the strong nuclear force. We consider a toy model of neutral mesons, its constituent quark and anti-quark have the same color and flavor. Thus the quark and anti-quark have totally opposite electric charge.

We suppose that there is a system of mesons at finite temperature. We consider the system in situation that interaction of the constituent quarks is strong coupling. In this situation, we may say that whole mesons are in strong coupling medium. The system is a non-perturbative system. When the temperature is high enough, the system is in deconfine phase, the phase that gluons decouple from quarks and anti-quarks so they become free. For such system, the quantum chromodynamics or QCD cannot directly make some predictions, because the QCD is a gauge theory base on perturbative technique. There exist non-perturbative approaches e.g. lattice QCD and bag-model which are used to make better predictions for the strong coupling system. In this thesis, we use a new non-perturbative method which is the AdS/CFT correspondence, motivated from the superstring theory.

In this thesis, the system of interest is system of neutral mesons at finite temperature with applied external electric field. Under the electric field, the constituent quarks of mesons try to dissociate due to their opposite electric charge. We hope to see that the dissociation can reduce the mesons binding energy. For fixed temperature, the system becomes deconfine phase easier as result of stronger electric field. The deconfine phase may be referred as mesons melting. The mesons system with external electric field is also a non-perturbative system. In order to study this system, it is necessary to apply the AdS/CFT correspondence. We review this application of AdS/CFT correspondence to describe the phase transition of mesons system by following the Ref. [1].

In Chapter 2, we will discuss the type IIA and type IIB superstring theories.

There exist bosonic and fermionic fluctuations on open and closed superstrings which they are very interesting. Dirichlet boundary conditions of open superstrings give rise to the extended objects called Dirichlet p-brane, or Dp-branes. The massless modes of open superstring, which their endpoints end on Dp-branes, are matched to component fields of a supersymmetric gauge theory. So we can use a configuration of Dp-branes and open superstrings to construct a supersymmetric gauge theory of interested. Notice that, the most context of bosonic string and superstring theories can be found in all superstring text books [2, 3, 4, 5].

We will discuss the AdS/CFT correspondence in Chapter 3. Originally, the AdS/CFT correspondence is suggested by considering two descriptions of a stack of D3-branes: gauge description and gravity description. We will show some information how the two descriptions can be matched together [6, 7]. We modify the original version of the correspondence by adding a stack of D7-branes under probe limit [8]. The configuration now consists of both stacks of D3 and D7 branes so called D3/D7 model. The discussion of the D3/D7 model are also included in Chapter 3.

In Chapter 4, we will perform calculations of D7-branes equation of motion which describes positions of the D7-branes in background space-time. Furthermore, we will illustrate shapes of the D7-branes, they play the crucial role in our consideration. They describe behavior of phase transition of mesons system. We will see that the phase transition is analogous to metal/insulator phase transition. The melting mesons system is considered as a metal. As external electric field is applied, we can find the relation between electric current density and electric field, and we obtain the conductivity of the metal [9]. Finally, the conclusion of this thesis is written in Chapter 5.

Chapter II

Background in String Theory

2.1 Bosonic String

String is a 1-spatial dimensional extended object. As a string evolving in time, it spans a $(1+1)$-dimensional manifold in the space-time. This manifold is called a world-sheet (or a string world-volume). We need two parameters, τ and σ, to label a point on the world-sheet. The parameter τ determines world-sheet time. It is natural to suppose that strings are lines of constant τ and points on the string are labeled by σ.

An action for strings in $(d+1)$-dimensional space-time is proportional to the area of its world-sheet. Note that, we may define $D = d+1$. In order to obtain string action we multiply the area by string tension to make it dimensionless quantity. The string action is Nambu-Goto string action:

$$S_{\mathrm{NG}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-G} \quad . \tag{2.1}$$

Here G is determinant of world-sheet induced metric, G_{ab}, and $\mathrm{d}^2\sigma$ is $\mathrm{d}\tau\mathrm{d}\sigma$. The string tension is written in term of the Regge's Slope α': $T_s = 1/4\pi\alpha'$.

The induced metric allows us to determine length between two separated points on the world-sheet by using world-sheet coordinates, and it must be equal to the length determined by space-time coordinates. Then, we have

$$\mathrm{d}s^2\Big|_{\text{world sheet}} = G_{ab}\mathrm{d}\sigma^a\mathrm{d}\sigma^b = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \quad ,$$

where $a, b = 0, 1$ are world-sheet indices and $\mu, \nu = 0, 1, ..., d$ are space-time indices (or Lorentzian indices). We define $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. We denote that x^μ are components of a position vector in $(d+1)$-dimensional space-time indicating a point on the world-sheet. So x^μ should be functions of world-sheet coordinates: $x^\mu \equiv x^\mu(\tau, \sigma)$. Here $g_{\mu\nu}$ are components of space-time metric. By using the chain rule, components of the induced metric are related to components of the space-time metric by

$$G_{ab} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\nu}{\partial \sigma^b} \quad \text{or} \quad [G_{ab}] = \begin{pmatrix} \dot{x}^\mu \dot{x}_\mu & \dot{x}^\mu x'_\mu \\ x'^\mu \dot{x}_\mu & x'^\mu x'_\mu \end{pmatrix} \quad . \tag{2.2}$$

Note that " $\dot{}$ " and " $'$ " denote the derivative with respect to τ and σ, respectively. Now the Nambu-Goto action has the form:

$$S_{\text{NG}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{(\dot{x}^\mu \dot{x}_\mu)(x'^\nu x'_\nu) - (\dot{x}^\mu x'_\mu)^2} \quad . \tag{2.3}$$

In addition to Lorentz invariance, the action has reparmeterization invariance. For our convenience, we may choose the world-sheet grid such that line of constant τ is perpendicular to line of constant σ. Consequently, the vector $\dot{x}^\mu$ is also perpendicular to the vector x'^μ which gives $\dot{x}^\mu x'_\mu = 0$. We can study string theory by using the Nambu-Goto action with this world-sheet coordinates. However, the Nambu-Goto action has field variable terms being under square root, it is difficult to perform path integral quantization.

On the other hand, we may consider the Polyakov action which is equivalent to the Nambu-Goto action in sense that they give the same equations of motion. The Polyakov action gives two types of equations of motion. The first type describes dynamics of x^μ and the second type is a set of constraint equations which fix component of induced metric and world-sheet metric together. The Polyakov string action is

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{-h}h^{ab}\frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x_\mu}{\partial \sigma^b} \quad , \tag{2.4}$$

where h_{ab} are components of a 2×2 matrix called world-sheet metric tensor. We can perform rising and lowering the world-sheet indices by using this metric.

2.1.1 Properties of String Action

Instead of the Numbu-Goto action, we may derive the equations of motion for bosonic string from the Polyakov action, as in [4]. By varying the Polyakov action with respect to h_{ab}, and using the definition for induce metric Eq. (2.2), we obtain equations of motion of the second type,

$$\frac{G^{ab}}{\sqrt{-G}} = \frac{h^{ab}}{\sqrt{-h}} \quad . \tag{2.5}$$

We may choose the world-sheet metric to be

$$[h_{ab}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

Then substituting this into Eq. (2.5), the constraint equations are

$$\begin{aligned} \dot{x}^\mu x'_\mu &= 0 \quad , \\ \dot{x}^\mu \dot{x}_\mu &= -\sqrt{(\dot{x}^\mu \dot{x}_\mu)(x'^\nu x'_\nu) - (\dot{x}^\mu x'_\mu)^2} \quad , \\ x'^\mu x'_\mu &= \sqrt{(\dot{x}^\mu \dot{x}_\mu)(x'^\nu x'_\nu) - (\dot{x}^\mu x'_\mu)^2} \quad . \end{aligned}$$

Actually, there are only two constraint equations because the equations in the second line and the third line are identical. Then, they can be simplified further and combined into only one equation as

$$\left(\dot{x}^\mu \pm x'^\mu\right)^2 = 0 \quad . \tag{2.6}$$

Then we will calculate the first type of equation of motion. We can think that the action is an action of D massless scalar fields denoted by x^μ. We vary these scalar fields by infinitesimal parameters $\epsilon^\mu(\tau, \sigma)$:

$$x^\mu \to x^\mu + \epsilon^\mu \quad ,$$

and substitute it back into the Polyakov action Eq. (2.4). Then we obtain

$$\delta S = -\frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \epsilon_\mu \partial_a \partial^a x^\mu \quad , \tag{2.7}$$

and non-trivial surface terms which we will discuss later. Let us first consider the equation Eq. (2.7). If we assume the vanishing of the surface terms, we can extract the conserved world-sheet current, $\mathcal{J}^{a\mu}$. The current is defined by

$$\mathcal{J}^{a\mu} \equiv \partial^a x^\mu \quad \text{which satisfy} \quad \partial_a \mathcal{J}^{a\mu} = 0 \quad .$$

Indeed, the conservation condition is string equation of motion which is $(1+1)$-dimensional wave equation:

$$\partial_a \mathcal{J}^{a\mu} = 0 \quad \to \quad -\frac{\partial^2 x^\mu}{\partial \tau^2} + \frac{\partial^2 x^\mu}{\partial \sigma^2} = 0 \quad .$$

We can also define the conjugate momentum densities of the world-sheet coordinates as

$$\mathcal{P}^{a\mu} = g^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_a x^\nu)} = -\frac{1}{2\pi\alpha'} \mathcal{J}^{a\mu} \quad , \tag{2.8}$$

where $\mathcal{L}$ is the Lagrangian of the Polyakov action. The momentum densities also satisfy the conservation condition like $\mathcal{J}^{a\mu}$. The center of mass momentum of moving strings is obtained by integrating $\mathcal{P}^{0\mu}$ overall σ.

Now, let us turn back to consider non-trivial surface term that can be shown explicitly as,

$$\int \mathrm{d}\tau\mathrm{d}\sigma\partial_a\left(\epsilon^\mu\partial^a x_\mu\right) = \int \mathrm{d}\sigma\left(\epsilon^\mu\partial^0 x_\mu\right)\Big|_{\tau_i}^{\tau_f} + \int \mathrm{d}\tau\left(\epsilon^\mu\partial^1 x_\mu\right)\Big|_0^{\sigma_*} \quad .$$

The above equation has been expanded by summation over world-sheet index a and then are performed integral with respect to τ or σ. The ranges of integration are following: from initial world-sheet time τ_i to final world-sheet time τ_f for τ. The τ integration is vanished by setting both $\epsilon^\mu(\tau_i)$ and $\epsilon^\mu(\tau_f)$ to be zero. The range of σ integration for open string differs from closed string. For closed string, the range of integration is just around a closed loop then the result vanishes. For an open string, the σ integration starts from an endpoint $\sigma = 0$ to another endpoint $\sigma = \sigma_*$. So the vanishing of the σ integration is achieved by employing the following boundary conditions

$$\text{boundary conditions}\begin{cases}\epsilon^\mu = 0 & \text{Dirichlet boundary conditions with } \mu \neq 0 \\ \\ \frac{\partial x^\mu}{\partial\sigma} = 0 & \text{Neumann boundary conditions}\end{cases} \tag{2.9}$$

Each direction of endpoints can be separately subjected to either condition. Except the $\mu = 0$ direction, this time direction must be subjected to Neumann boundary condition. For example, for an endpoint, we can have $(p+1)$-directions satisfy the Neumann conditions while the rest $(d-p)$-directions satisfy Dirichlet conditions. Then we may write

$$\underbrace{x^0 \quad x^1 \quad ... \quad x^p}_{\text{Neumann}} \quad \underbrace{x^{p+1} \quad ... \quad x^d}_{\text{Dirichlet}} \quad . \tag{2.10}$$

An endpoint that satisfies Eq. (2.10) can move freely with speed of light in p-spatial direction and also evolving in time. On the other hand, the endpoint is fixed in the $(d-p)$-spatial directions. This looks like the endpoint is freely moving in an extended object which occupies time and p-spatial directions. This object is called Dirichlet p-brane or Dp-brane.

2.1.2 Open String solutions

We will solve string equation of motion with corresponded boundary conditions for open string solution, denoted by x^μ. Let us start from the equation of motion:

$$-\frac{\partial^2 x^\mu}{\partial\tau^2} + \frac{\partial^2 x^\mu}{\partial\sigma^2} = 0 \quad . \tag{2.11}$$

We assume that the solution should be a linear combination of left-moving and right-moving. The general solution is

$$x^\mu(\tau,\sigma) = \frac{1}{2}\left(x^\mu_L(\tau+\sigma) + x^\mu_R(\tau-\sigma)\right) \quad , \tag{2.12}$$

where x^μ_L denotes left-moving part and x^μ_R denotes right-moving part.

We consider the case that $(p+1)$-directions of both string endpoints satisfy the Neumann boundary conditions Eq. (2.9). By performing the derivative with respect to σ in Eq. (2.12) and employing the Neumann conditions at $\sigma = 0$, we obtain

$$\frac{\partial x^\mu}{\partial \sigma}\Big|_{\sigma=0} = 0 \quad \rightarrow \quad \frac{\partial x^\mu_L}{\partial(\tau+\sigma)}\Big|_{\sigma=0} = \frac{\partial x^\mu_R}{\partial(\tau-\sigma)}\Big|_{\sigma=0} \quad . \tag{2.13}$$

Now the index μ is running over $0, 1, ..., p$. We may define $u = \tau+\sigma$ and $v = \tau-\sigma$ where $u = v$ at $\sigma = 0$ but $u = v + 2\pi$ at $\sigma = \pi$.

Thus we can rewrite $\tau \pm \sigma$ to be u and v for our convenience. The relation in Eq. (2.13) implies the same in argument of x'^μ_L and x'^μ_R, then

$$x'^\mu_L(u) = x'^\mu_R(u) \quad \rightarrow \quad x^\mu_L(u) = x^\mu_R(u) - C^\mu \quad .$$

This can be concluded that the right-moving is depending on the left-moving up to a constant vector C^μ. So the solution Eq. (2.12) becomes

$$x^\mu(\tau,\sigma) = \frac{1}{2}C^\mu + \frac{1}{2}\left(x^\mu_L(\tau+\sigma) + x^\mu_R(\tau-\sigma)\right) \quad . \tag{2.14}$$

In case of open string, we can set range of σ as $\sigma \in [0,\pi]$. Now we consider the Neumann conditions at $\sigma = \pi$ by following

$$\frac{\partial x^\mu}{\partial \sigma}\Big|_{\sigma=\pi} = 0 \quad \rightarrow \quad \frac{\partial x^\mu_L}{\partial(\tau+\sigma)}\Big|_{\sigma=\pi} = \frac{\partial x^\mu_R}{\partial(\tau-\sigma)}\Big|_{\sigma=\pi} \quad .$$

Using the definition of u and v, we can see that x'^μ_L are periodic function with 2π period: $x'^\mu_L(v) = x'^\mu_L(v+2\pi)$. Thus, it is natural to expand the Fourier series for the x'^μ_L. Then performing the v integration, we obtain

$$x^\mu_L(v) = C^\mu_2 + \sqrt{2\alpha'}\alpha^\mu_0 v + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\exp\left(-inv\right) \quad .$$

The C^μ_2 is another constant of integration and oscillators, α^μ_n, must satisfy $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ for all n. Notice that we wrote $n \neq 0$ under the $\sum$ for short of the summation starting form $n = -\infty$ to $n = \infty$ but except $n = 0$. We substitute the Fourier expansion of $x^\mu_L(v)$ into Eq. (2.14) and conclude all constants into x^μ_0. So we reach the solution for open string with both endpoints satisfying the Neumann boundary conditions. The solution is following

$$x^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\exp\left(-in\tau\right)\cos\left(n\sigma\right) \quad . \tag{2.15}$$

Calculating the momentum conjugate as defined in Eq. (2.8), then

$$\mathcal{P}^{0\mu} = \frac{\dot{x}^\mu}{2\pi\alpha'} = \frac{1}{2\pi\alpha'}\left(\sqrt{2\alpha'}\alpha_0^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \exp\left(-in\tau\right)\cos\left(n\sigma\right)\right) \quad . \tag{2.16}$$

So the center of mass momentum is

$$p^\mu = \int_0^\pi \mathrm{d}\sigma \frac{\dot{x}^\mu}{2\pi\alpha'} = \frac{\alpha_0^\mu}{\sqrt{2\alpha}}$$

which gives

$$\sqrt{2\alpha'}p^\mu = \alpha_0^\mu \quad . \tag{2.17}$$

Eq. (2.17) gives a relation between momentum and α_0^μ. Thus we may rewrite the solution Eq. (2.15) such that

$$x^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\exp\left(-in\tau\right)\cos\left(n\sigma\right) \quad . \tag{2.18}$$

2.1.3 Dp-branes

An open string solution is a set of D-directions of x^μ. The μ^{th}-direction as Eq. (2.18) is arisen by assumption that both endpoints satisfy Neumann boundary conditions. The $\cos(n\sigma)$ factor implies free motion of string endpoints along that μ^{th}-direction.

Since both endpoints satisfy Dirichlet boundary conditions for some directions, such directions will have $\sin(n\sigma)$ factor instead of $\cos(n\sigma)$. This $\sin(n\sigma)$ factor implies that endpoints are fixed. For a given r^{th}-direction, one endpoint satisfies Neumann condition but the other satisfies Dirichlet condition. In this case, solutions for r^{th}-direction have half integer modes.

We conclude that there are three possible types of directions including in the open string solution: Neumann-Neumann type (NN type), Dirichlet-Dirichlet type (DD type) and Dirichlet-Neumann type (DN type) (see Figure 2.1). The DD type is,

$$x^a(\tau,\sigma) = x_0^a + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^a}{n}\exp\left(-in\tau\right)\sin\left(n\sigma\right) \quad , \tag{2.19}$$

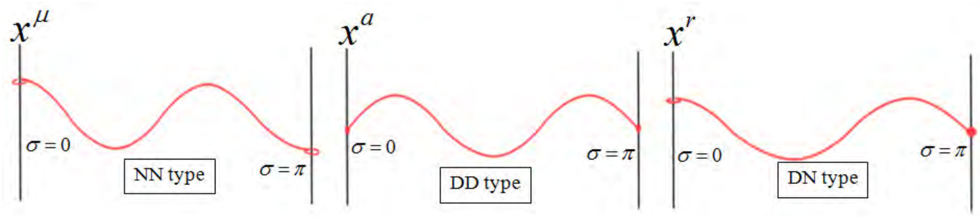

Figure 2.1: The illustration of NN, DD and DN types.

where the index a is running overall DD directions. For DN directions, we solve the equations of motion with imposing the Neumann condition for an endpoint at $\sigma = 0$ and the Dirichlet condition for the other endpoint at $\sigma = \pi$.

$$x^r(\tau,\sigma) = x_0^r + i\sqrt{2\alpha'} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_s^r}{s} \exp\left(-is\tau\right)\cos\left(s\sigma\right) \quad , \tag{2.20}$$

which the index r is running overall DN directions.

Let us consider some examples of open string solution. We suppose that an open string is described by solution including the following directions

$$\underbrace{x^0 \quad x^1 \quad ... \quad x^p}_{\text{NN type}} \quad \underbrace{x^{p+1} \quad ... \quad x^d}_{\text{DD type}} \quad . \tag{2.21}$$

Eq. (2.21) gives a fact that the index μ appearing in equation Eq. (2.18) is running as $\mu = 0, 1, 2, ..., p$. While the rest directions are written by Eq. (2.19), with $a = p + 1, p + 2, ..., d$. In this case, each endpoints ends on Dp-branes. This open string may be referred as p-p string.

Another example, suppose that an open string is described by solution with directions such that

$$\underbrace{x^0 \quad x^1 \quad ... \quad x^p}_{\text{NN type}} \quad \underbrace{x^{p+1} \quad ... \quad x^q}_{\text{DN type}} \quad \underbrace{x^{q+1} \quad ... \quad x^d}_{\text{DD type}} \quad . \tag{2.22}$$

This open string solution has $(p + 1)$ NN-directions and $(d - q)$ DD-directions. Moreover, the solution also includes $(q-p)$ DN-directions which satisfy Eq. (2.20) with index $r = q + 1, q + 2, ..., d$. In this case, this open string has an endpoint ending on Dp-brane while another endpoint ending on Dq-brane. Thus we may refer this open string as p-q string (see Figure 2.2).

2.1.4 Closed String solutions

Because closed string has no endpoint, so the boundary condition is a set of periodic conditions. In case of closed string, we choose a new range of σ as

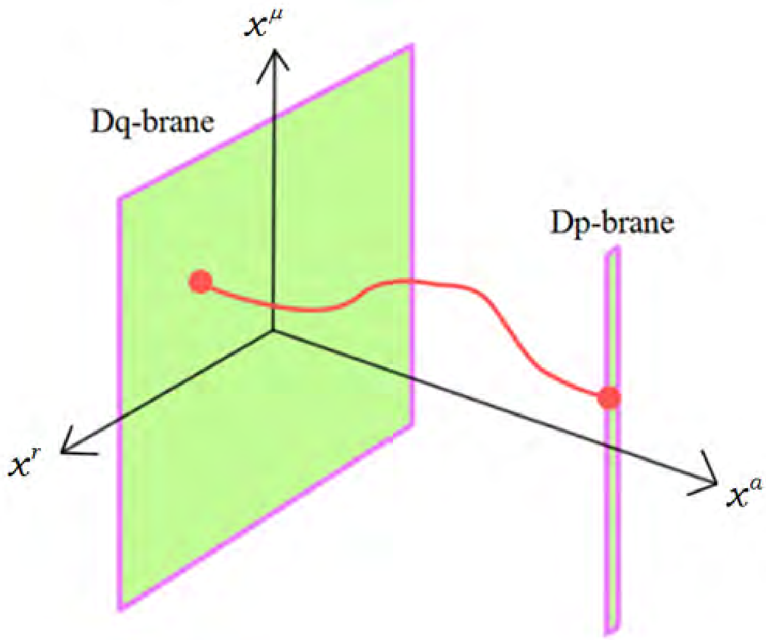

Figure 2.2: An open string stretches between Dp and Dq brane or p-q string. The Dq-brane has more number of directions than Dp-brane, $q > p$.

$\sigma \in [0, 2\pi]$. Thus, the closed string solution contains a set of $x^\mu(\tau,\sigma)$ which satisfies $x^\mu(\tau,\sigma) = x^\mu(\tau,\sigma+2\pi)$ for all $\mu = 0, 1, ..., d$. We will find the explicit form of the $x^\mu(\tau,\sigma)$. We start from a general solution

$$x^\mu(\tau,\sigma) = x^\mu_L(\tau+\sigma) + x^\mu_R(\tau-\sigma) \quad . \tag{2.23}$$

The periodic conditions imply periodicity of σ derivative of x^μ:

$$x'^\mu_L(u) = x'^\mu_L(u+2\pi) \quad \text{and} \quad x'^\mu_R(u) = x'^\mu_R(u+2\pi) \quad . \tag{2.24}$$

We wrote $u = \tau + \sigma$ and $v = \tau - \sigma$, and we saw that the periodic condition does not fix a relation between x^μ_L and x^μ_R. By expanding the $x'^\mu_{L,R}$ using Fourier series then performing integration over σ, we will get the forms of $x^\mu_{L,R}$. Those are

$$x^\mu_L(u) = \frac{1}{2}x^\mu_{0L} + \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0 u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\exp\left(-inu\right) \tag{2.25}$$

$$x^\mu_R(v) = \frac{1}{2}x^\mu_{0R} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}^\mu_0 v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}^\mu_n}{n}\exp\left(-inv\right) \quad . \tag{2.26}$$

In this step, we can check that the periodic condition only fixes α^μ_0 and $\bar{\alpha}^\mu_0$ together by $\alpha^\mu_0 = \bar{\alpha}^\mu_0$. This fixing is called matching condition. In quantum theory of closed string, the matching condition gives equality relation between number of left-moving and right-moving.

We substitute the summation $x^\mu_L + x^\mu_R$, where x^μ_L and x^μ_R are respectively shown by Eq. (2.25) and (2.26), into Eq. (2.23). Then we achieve

$$\begin{aligned} x^\mu(\tau,\sigma) &= \frac{1}{2}\left(x^\mu_{0L} + x^\mu_{0R}\right) + \sqrt{2\alpha'}\alpha^\mu_0\tau \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\exp\left(-in\tau\right)\Big(\bar{\alpha}^\mu_n \exp\left(in\sigma\right) + \alpha^\mu_n \exp\left(-in\sigma\right)\Big). \end{aligned} \tag{2.27}$$

We also calculate a component of the center of mass momentum

$$p^\mu = \int_0^{2\pi}\mathcal{P}^{\tau\mu}\mathrm{d}\sigma \quad ,$$

and then we obtain

$$\sqrt{\frac{\alpha'}{2}}p^\mu = \alpha^\mu_0 \quad . \tag{2.28}$$

We may compare the momentum of closed string to open string (see Eq. 2.17). Then we see that momentum of a closed string is double of momentum for an open string. Closed string is described by solution as $(d+1)$ copies of Eq. (2.28).

This should be concluded that for a given closed string, its solution is written as a summation of left-moving modes and right-moving modes separately, except for α^μ_0 and $\bar{\alpha}^\mu_0$. The α^μ_0 for left and right-moving are fixed with $\bar{\alpha}^\mu_0$ by match condition $\alpha^\mu_n = \bar{\alpha}^\mu_n$. If we put constraints that $\alpha^\mu_n = \bar{\alpha}^\mu_n$ for all $n \neq 0$, the solution in Eq. (2.27) turns to be open string solutions in Eq. (2.18).

2.1.5 Quantum Open String

Now we consider quantization of string theory. We will start from classical string solution. Then we will apply light-cone gauge $x^\mu = (x^+, x^-, x^I)$ with $I = 2, 3, ..., d$ and $x^\pm = x^0 \pm x^1$. We will do the same for any vector v^μ. Note that the definitions and some calculations of light-cone gauge are shown in Appendix A. Under such directions, we can count physical degrees of freedom of quantum string states.

In order to reach the quantum theory, the quantities x^μ, $\mathcal{P}^\mu$ and etc. are promoted to be operators. We use Heisenberg picture where operators are time dependent. For simplicity, we consider NN-type of open string solution. In light-cone gauge, we have

$$x^+ = 2\alpha' p^+\tau \quad , \tag{2.29}$$

$$x^- = x^-_0 + 2\alpha' p^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^-_n}{n}\exp\left(-in\tau\right)\cos\left(n\sigma\right) \quad , \tag{2.30}$$

$$x^I = x^I_0 + 2\alpha' p^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^I_n}{n}\exp\left(-in\tau\right)\cos\left(n\sigma\right) \quad . \tag{2.31}$$

This reflects the fact that x^+ becomes time parameter of string world-sheet. Thus x^+ dose not contribute to physical degrees of freedom. Under constraint equation, Eq. (2.6), x^I determines x^-. Only x^I contributes to physical degrees of freedom. This should be concluded that there are D-directions in string solution but only $D-2$ of them give physical degrees of freedom.

The prove of fixing x^- degree of freedom will be displayed. Rearranging Eq. (2.6) by light-cone dot product, then the result is

$$2\left(\dot{x}^+ \pm x'^+\right)\left(\dot{x}^- \pm x'^-\right) = \left(\dot{x}^I \pm x'^I\right)^2 \quad .$$

Notice that the definition of light-cone dot product is shown in Appendix A. From the above equation, we replace τ and σ derivative of x^+ by $2\alpha' p^+$ and zero, respectively. And we use Eq. (2.30) and (2.31) to rewrite the $(\dot{x} \pm x')$ for both "minus" and I directions in terms of its oscillators. We obtain

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_n \alpha_n^I \alpha_{j-n}^I \quad . \tag{2.32}$$

We define

$$L_n = \frac{1}{2}\sum_n \alpha_n^I \alpha_{j-n}^I \quad ,$$

so

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n \quad . \tag{2.33}$$

We see that Eq. (2.32) fixes the oscillator α_n^- with oscillator α_n^I. This means that all oscillating degrees of freedom of "minus" direction is determined by oscillation in I-directions. The L_n is the Virasoro operator which we may use $p^- = L_0/(2\alpha' p^+)$ as string Hamiltonian (see Appendix A).

We now perform canonical quantization. We establish the equal τ commutation relations between components and its conjugate momentum as follows

$$\left[x^I(\tau,\sigma), \mathcal{P}^{0J}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma') \quad \text{and} \quad \left[x_0^-, p^+\right] = -i \quad . \tag{2.34}$$

The x^+ commutes to all operators. We can reach commutation relation of oscillator by calculating $[\dot{x}^I(\sigma) + x'^I(\sigma'), \dot{x}^J(\sigma) + x'^J(\sigma')]$ with range $\sigma \in [-\pi, \pi]$. The commutation relation of oscillators is

$$[\alpha_n^I, \alpha_{-n}^J] = n\eta^{IJ} \quad \rightarrow \quad [a_n^I, a_n^{J\dagger}] = \eta^{IJ} \quad . \tag{2.35}$$

Here, we have introduced normalization of α_n^I by

$$\alpha_{-n}^I = \sqrt{n}(a_n^I)^\dagger \quad \text{and} \quad \alpha_n^I = \sqrt{n}a_n^I \quad . \tag{2.36}$$

The commutation relation Eq. (2.35) corresponds to commutation relation of annihilation and creation operators.

Virasoro operators, L_n are written in terms of annihilation and creation operators. Since the Hamiltonian is L_0 although we see that L_0 has summation of normal ordering and non-normal ordering terms. We have to introduce a new well defined Hamiltonian. The non-normal ordering terms are substituted by using the commutation relation. The result includes a divergent summation that is $\sum_{n=1}^{\infty} n$. We use the zeta-regularization to deal with the summation. The regularization is following

$$\sum_{n=1}^{\infty} n = \zeta(-1) \to -\frac{1}{12} \quad \text{where} \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad . \tag{2.37}$$

Thus we obtain Hamiltonian as

$$H = \frac{1}{2}\alpha_0^I \alpha_0^I + \sum_{n=0}^{\infty} \alpha_{-n}^I \alpha_n^I - \frac{(D-2)}{24} \quad . \tag{2.38}$$

In the similar way, we have other normal ordering operators such as mass squared operator. Principally the mass squared operator is $M^2 = -p^\mu p_\mu$. The Lorentzian dot product is expanded by using light-cone dot product, then the mass squared operator is written in terms of Hamiltonian. This Hamiltonian is replaced by using Eq. (2.38). Finally we obtain

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I - \frac{(D-2)}{24}\right) \text{ or } M^2 = \frac{1}{\alpha'}\left(N - \frac{(D-2)}{24}\right), \tag{2.39}$$

where number operator is

$$N = \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \quad . \tag{2.40}$$

Now we have the most important physical observable which is mass squared operator. We determine value of mass squared of any string state by acting this mass squared operator to that state.

We now construct string phase space. We suppose that string vacuum is written as $|\lambda = 0; p^+, \vec{p}_T\rangle_{\text{open}}$. Value of center of mass momentum of vacuum state is determined by p^+ and p^I. The λ is number of modes carrying by the state. Moreover, the state is labelled by "open" meaning that this is open string vacuum. This vacuum have to be annihilated by annihilation operators such

$$a_n^I |\lambda = 0; p^+, \vec{p}_T\rangle_{\text{open}} = 0 \quad \text{with} \quad I = 2, 3, ..., 25 \quad .$$

We set $D = 26$ since we need to eliminate an anomaly which brakes the Lorentz symmetry of quantum string theory. The general open string state is

$$|\lambda; p^+, \vec{p}_T\rangle_{\text{open}} = \prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I\dagger})^{\lambda_{n,I}}|0; p^+, \vec{p}_T\rangle \quad \text{with} \quad \lambda = \sum_{n=0}^{\infty}\sum_{I=2}^{25}\lambda_{n,I} \quad .$$

We can check that $a_n^{I\dagger}|0; p^+, \vec{p}_T\rangle_{\text{open}}$ and $|0; p^+, \vec{p}_T\rangle_{\text{open}}$ are massless state and tachyon state. The massless states have $D - 2$ or 24 degrees of freedom which is equal to photon state living in $(25 + 1)$-dimensional space-time.

In conclusion, the massless open string state identifies photon state. Since the field corresponding to photon state is gauge field A^μ. There are possibilities to write gauge field as $A^\mu = a_n^{\mu\dagger}|0; p^+, \vec{p}_T\rangle_{\text{open}}$ with the space-time index $\mu = 0, 1, ..., 25$. There should exist a polarization vector $\xi^\mu = (0, 0, \xi^2, \xi^3, ..., \xi^{25})$ that projects the physical degrees of freedom. So far we calculate only 26 copies of NN type but we will not quantize neither DD type or DN type. Assuming that we have an open string which satisfies boundary condition in Eq. (2.21). So actually the index μ is separated into NN type and DD type. Index running overall NN directions is vector index for Dp-brane. While index running overall DD directions counts the number of scalar representations of the string state.

2.1.6 Quantum Closed String

The context of closed string quantization is the same with open string quantization. Closed string contains independent left-moving and right-moving modes which result in two sets of creation, annihilation and Viresoro operators. At start, we write the classical closed string solution in light-cone directions, x^+, x^- and x^I. Here $x^+ = \alpha' p^+ \tau$ differs from x^+ in open string context. The x^- and x^I replace Lorentz index μ in Eq. (2.27) by "minus" and I, respectively. In quantum theory, all x^+, x^- and x^I become Heisenberg operators. Then we establish commutation relations

$$\left[x^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma') \quad \text{and} \quad \left[x_0^-, p^+\right] = -i \quad . \tag{2.41}$$

We get the commutation of creation and annihilation operators by calculating $\left[\dot{x}^I(\sigma) \pm x'^I(\sigma'), \dot{x}^J(\sigma) \pm x'^J(\sigma')\right]$. The commutation relations are

$$\left[\alpha_m^I, \alpha_n^J\right] = m\eta^{IJ}\delta_{m,-n} \quad \text{and} \quad \left[\bar{\alpha}_m^I, \bar{\alpha}_n^J\right] = m\eta^{IJ}\delta_{m,-n} \quad . \tag{2.42}$$

$$\tag{2.43}$$

We note that

$$\left[\alpha_m^I, \bar{\alpha}_n^J\right] = 0 \quad . \tag{2.44}$$

We may perform the normalization on α operators such that for

$$\begin{aligned} \alpha_n^I &= \sqrt{n} a_n^I \quad \text{and} \quad \alpha_{-n}^I = \sqrt{n} a_n^{I\dagger} \quad \text{for left-moving modes,} \\ \bar{\alpha}_n^I &= \sqrt{n} \bar{a}_n^I \quad \text{and} \quad \bar{\alpha}_{-n}^I = \sqrt{n} \bar{a}_n^{I\dagger} \quad \text{for right-moving modes.} \end{aligned} \tag{2.45}$$

Then we obtain,

$$\left[a_n^I, a_m^J\right] = \eta^{IJ} \delta_{m,n} \quad \text{for left-moving modes} \quad , \tag{2.46}$$

$$\left[\bar{a}_m^I, \bar{a}_n^J\right] = \eta^{IJ} \delta_{m,n} \quad \text{for right-moving modes} \quad . \tag{2.47}$$

For a closed string, left-moving modes are arisen by operating some creation operators $a_n^{I\dagger}$ to closed string vacuum. On the other hand, the right-moving modes on that string arisen by acting some creation operators $\bar{a}_n^{I\dagger}$ to the vacuum.

In context of closed string, there are two Virasoro operators L_n and $\bar{L}_n$, for left and right moving. We will use those Virasoro operators with $n = 0$ as the string Hamiltonian. The left-Virasoro operators L_n are achieved by calculating constraint equation $(\dot{x}^\mu + x'^\mu)^2$. The right-Virasoro operators $\bar{L}_n$ are also obtained by constraint equation $(\dot{x}^\mu - x'^\mu)^2$. Notice that the constraint equation is shown in Eq. (2.6). The calculation for both Virasoro operators is the same in the case of open string. We should notice that the constraint equation fixes oscillators in "minus" direction with I-directions. So the "minus" component does not contribute to physical degrees of freedom. We show both L_n and $\bar{L}_n$ here,

$$L_n = \frac{1}{2} \sum_p \alpha_p^I \alpha_{n-p}^I \quad \text{and} \quad \bar{L}_n = \frac{1}{2} \sum_p \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I \quad . \tag{2.48}$$

The L_n and $\bar{L}_n$ contribute to total Hamiltonian but they contain summation of non-normal ordering terms. In order to obtain normal ordering Hamiltonian we have to write them in normal order form. And we perform zeta-regularization to get value of appearing infinite summations. Under the processes we obtain the Hamiltonian as

$$H = \alpha' p^+ p- = L_0 + \bar{L}_0 - 2 \quad . \tag{2.49}$$

Notice that this equation is true only $D = 26$. This reflects the fact that left-moving modes and right-moving modes independently give rise to total energy (and mass). Here the normal ordering Virasoro operators are

$$L_0 = \frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I \quad \text{and} \quad \bar{L}_0 = \frac{1}{2} \bar{\alpha}_0^I \bar{\alpha}_0^I + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I \quad . \tag{2.50}$$

Notice that the matching condition, which is $\alpha_0^- = \bar{\alpha}_0^-$, fixes the equality of both $n = 0$ Virasoro operators, $L_0 = \bar{L}_0$. The total Hamiltonian in Eq. (2.49) is true only $(25+1)$-dimensional space-time.

By using definition of the Hamiltonian which is written by Eq. (2.49), we can also get mass squared operator as

$$M^2 = \frac{2}{\alpha'}\left(L_0 + \bar{L}_0 - 2\right) - p^I p^I \quad \text{or} \quad M^2 = \frac{2}{\alpha'}\left(N + \bar{N} - 2\right) \quad . \tag{2.51}$$

We have defined the number operators:

$$N = \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \quad \text{and} \quad \bar{N} = \sum_{n=1}^{\infty} n \bar{a}_n^{I\dagger} \bar{a}_n^I \quad .$$

We see that both number operators have to be equal, $N = \bar{N}$ under matching condition. This implies that, for a closed string, left-moving excitation must be equal to right-moving excitation.

We now establish closed string phase space. The closed states are achieved by acting any product of creation operators to closed string vacuum. The closed string vacuum is constructed by

$$|\lambda = 0, \bar{\lambda} = \bar{0}; p^+, \vec{v}_T\rangle_{\text{closed}} \quad \text{which satisfies} \quad a_m^I \bar{a}_n^J |0, \bar{0}; p^+, \vec{v}_T\rangle_{\text{closed}} = 0 \quad .$$

Closed string states have mode numbers λ and $\bar{\lambda}$ counting left-moving modes and right-moving modes, respectively. The matching condition gives constraint relation that $\lambda = \bar{\lambda}$. We show the general closed string state here,

$$|\lambda, \bar{\lambda}; p^+, \vec{v}_T\rangle_{\text{closed}} = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{I\dagger}\right)^{\lambda_{n,I}} \times \prod_{m=1}^{\infty}\prod_{J=2}^{25}\left(\bar{a}_m^{J\dagger}\right)^{\bar{\lambda}_{m,J}} |0, \bar{0}; p^+, \vec{v}_T\rangle_{\text{closed}},$$

where

$$\lambda = \sum_{n=1}^{\infty}\sum_{I=2}^{25} \lambda_{n,I} = \bar{\lambda} = \sum_{n=1}^{\infty}\sum_{I=2}^{25} \bar{\lambda}_{n,I} \quad .$$

We are interested in massless closed string states. The possible massless states are states that carry one left-moving mode and one right-moving mode excitation in any I-directions. So the possible massless states are

$$a_1^I \bar{a}_1^J |\lambda = 0, \bar{\lambda} = \bar{0}; p^+, \vec{v}_T\rangle_{\text{closed}} \quad .$$

We can check that the closed string vacuum is tachyon state. For $D = 26$, the massless states contribute $(D-2)^2$ or 576 degrees of freedom. We may write the massless state for Lorentzian indices as $a_1^\mu \bar{a}_1^\nu |\lambda = 0, \bar{\lambda} = \bar{0}; p^+, \vec{v}_T\rangle_{\text{closed}}$. The states should correspond to elements of a tensor field. We can introduce polarization tensor with elements $\xi_{\mu\nu}$ for $\mu, \nu = 0, 1, ..., 25$. The polarization tensor will project the physical degrees of freedom. Since we use light-cone convention, the elements of the tensor should be zero when one of μ or ν, or both equal to "plus" or "minus". The element which μ or ν equal to I or J are called transverse elements. They can be decomposed into symmetric traceless element $g_{\mu\nu}$, anti-symmetric $B_{\mu\nu}$ and trace ϕ such that $\xi_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu} + \phi\eta_{\mu\nu}$ under Lorentzian indices. The $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi\eta_{\mu\nu}$ will respectively project the degrees of freedom for graviton, Kalb-Ramond fields and dilation fields from massless states.

In conclusion the massless closed string states contain degrees of freedom for three types of fields. Those are background metric tensor, Kalb-Ramond field and a scalar diatonic field. The metric tensor corresponds to graviton field. An open string is a source of Kalb-Ramond field and they play role as electromagnetic ansatz for bosonic string theory. And diatonic field gives rise to string coupling $g_s = \exp\phi$.

2.2 Bosonic String with Charge

2.2.1 Kalb-Ramond Field Interaction

We now include an electromagnetic ansatz into our bosonic string theory. Since strings are 1-dimensional objects, they are possible to interact with a two-form field. We use the Kalb-Ramond field to write the interaction term between string and two-form field. So we turn on an action such that

$$S_{\text{charge}} = -\int \mathrm{d}\tau \mathrm{d}\sigma \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma} B_{\mu\nu} \quad .$$

From the Kalb-Ramond field, it gives rise to the three-form field strength $H_{\mu\nu\rho}$ which

$$H_{\mu\nu\rho} = \partial_\mu B\nu\rho + \partial_\nu B\rho\mu + \partial_\rho B\mu\nu \quad . \tag{2.52}$$

So we can turn on another action meaning that we add a dynamical term into our string action. So the total action is

$$S = S_{\text{Polyakov}} - \int \mathrm{d}\tau \mathrm{d}\sigma \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma} B_{\mu\nu} - \frac{1}{6\kappa^2}\int \mathrm{d}^{26}X H^{\mu\nu\rho}H_{\mu\nu\rho} \quad . \tag{2.53}$$

Here, we use $\mathrm{d}^{26}X = \mathrm{d}X^0\mathrm{d}X^1...\mathrm{d}X^{25}$ and define X^μ to be space-time coordinates which relates to vector x^μ indicating to world-sheet by $x^\mu = X^\mu|_{\text{world-sheet}}$. Notice that S_{Polyakov} is the Polyakov action as shown in Eq. (2.4).

We suppose the action describing an open string with its endpoints are ending on Dp-branes. We may define the anti-symmetric conserved current $j^{\mu\nu}$,

$$j^{\mu\nu}(X) = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}\sigma\delta^{26}(x^\alpha - X^\alpha)\left(\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\sigma} - \frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\mu}{\partial\sigma}\right) \quad .$$

Here we can use the delta function to rewrite the integral overall $\mathrm{d}\tau\mathrm{d}\sigma$ to be integral overall $\mathrm{d}^{26}X$. And the charge action is written in terms of $B_{\mu\nu}j^{\mu\nu}$. Then, by performing variation of $B_{\mu\nu}$, we obtain equation of motion

$$\frac{1}{\kappa^2}\frac{\partial}{\partial x^\rho}H^{\mu\nu\rho} = j^{\mu\nu} \quad . \tag{2.54}$$

We identify some non-zero elements such j^{0i} to be components of electric current density which flows along the considering open string.

We also consider the gauge transformation of action Eq. (2.53). It is easy to check that the gauge transformation such that

$$\delta B_{\mu\nu}(x) = \frac{\partial\Lambda_\nu}{\partial x^\mu} - \frac{\partial\Lambda_\mu}{\partial x^\nu} \quad ,$$

which arises a non-trivial surface term. Here the surface term is

$$\delta S_{\text{charge}} = \int \mathrm{d}\tau \left[\Lambda_m \frac{\partial x^m}{\partial\tau}\right]_{\sigma=0}^{\sigma=\pi}$$

We have already used the Direchlet boundary conditions to fix $\partial x^r \partial\tau = 0$, which r is running over DD-directions. So the appearing index m is running over $m = 0, 1, ..., p$, the NN-directions. To cancel out the surface terms, we may introduce an additional surface term. The S_{charge} with adding an additional surface term is

$$S_{\text{charge}} = -\int \mathrm{d}\tau\mathrm{d}\sigma\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\sigma}B_{\mu\nu} + \int \mathrm{d}\tau\left[A_m\frac{\partial x^m}{\partial\tau}\right]_{\sigma=0}^{\sigma=\pi} \quad . \tag{2.55}$$

Now the S_{charge} is invariant under the following gauge transformation

$$\delta B_{\mu\nu} = \frac{\partial\Lambda_\nu}{\partial x^\mu} - \frac{\partial\Lambda_\mu}{\partial x^\nu} \quad \text{and} \quad \delta A_m = \Lambda_m \quad .$$

We see that the adding term looks like the interaction between point particles and gauge potential A_m. It is natural to introduce a two-form field strength $\mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m$ living on the Dp-branes. Under gauge transformation,

$$\mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m \quad \to \quad \delta\mathcal{F}_{mn} = \partial_m\Lambda_n - \partial_n\Lambda_m = \delta B_{mn} \quad , \tag{2.56}$$

we see that the $\delta\mathcal{F}_{mn}$ completely cancels out all δB_{mn}. This guarantee the gauge invariant of the S_{charge}.

Since two of open string endpoints are confined in Dp-branes, they appear as opposite point charges and give rise the Maxwell theory on that Dp-branes. We have already known that there exist electric density j^{0i}. The current starts at an endpoint which is a positive charge then flows along open string to another endpoint which is a negative charge. This flow leads to the orientation degrees of freedom which we will consider in Chapter 3.

2.2.2 Born-Infeld Electrodynamics

We again consider an open string which its endpoints ending on Dp-branes, moreover the string endpoints play role as point charges. We assume that the Dp-branes behaves like a medium which are affect by those point charges. In this situation, it is difficult to calculate an explicit form of electric field $\vec{E}$, and magnetic field $\vec{\mathcal{B}}$. It is easier to calculate the electric displacement $\vec{D}$ and magnetization field $\vec{H}$. We will apply the action principle to describe such electrodynamics system. Our aim is explicit form of the action. Then we will use the action to find relations between $\vec{E}$ and $\vec{\mathcal{B}}$ with $\vec{D}$ and $\vec{H}$, respectively.

For obtaining some motivations, let us consider an electrodynamics system in (3+1)-space-time. Suppose that there exists a field strength tensor, $G^{\mu\nu}$, which its elements stand for $\vec{D}$ and $\vec{H}$. Note that the Lorentzian indices μ and ν run over $0, 1, 2, 3$. The tensor is

$$[G^{\mu\nu}] = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_x & H_y & -H_x & 0 \end{pmatrix} \quad . \tag{2.57}$$

Furthermore this field strength tensor satisfies equation of motion as

$$\frac{\partial G^{\mu\nu}}{\partial X^\nu} = j^\mu \quad , \tag{2.58}$$

where j^μ is an external source. Now we write an action for this system with an assumption that Lagrangian should depend on field strength tensor $F^{\mu\nu}$. Thus the possible action is

$$S = \int \mathrm{d}^4 X \mathcal{L}(F_{\mu\nu}) + \int \mathrm{d}^4 X A_\mu j^\mu \quad . \tag{2.59}$$

Here A_μ denotes the usual gauge potential which arises the field strength, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. By varying the action with respect to A^μ, we obtain equation of motion. The equation of motion gives relations as

$$D^i = \frac{\partial \mathcal{L}}{\partial E_i} \quad \text{and} \quad H^i = \frac{\partial \mathcal{L}}{\partial \mathcal{B}_i} \quad . \tag{2.60}$$

The Lagrangian must be Lorentz invariant and $U(1)$ gauge invariant. The possible terms in that Lagrangian are $F_{\mu\nu}F^{\mu\nu} \sim (E^2 - \mathcal{B}^2)$ and $F_{\mu\nu}{}^*F^{\mu\nu} \sim \vec{E}\cdot\vec{\mathcal{B}}$. Here the $^*F^{\mu\nu}$ denotes dual field strength which is defined by $^*F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. We choose a form of Lagrangian such that

$$\mathcal{L} \sim -\sqrt{1 - 2\pi\alpha'(E^2 - \mathcal{B}^2) - (2\pi\alpha')^2(\vec{E}\cdot\vec{\mathcal{B}})^2} + 1 \quad . \tag{2.61}$$

We write E and $\mathcal{B}$ terms under a square root because the Dp-brane must contain finite value of norm of electric field, E^2. This means that Dp-brane carries finite energy. Moreover, since E^2 is much smaller than string scale, α', our system is in weak field regime. $2\pi\alpha' = |E|_{\max}$ is the maximum value of electric field on the Dp-branes, electric field with $|E| > |E|_{\max}$ is not allowed on the Dp-branes, see also T-duality written in [5]. In such regime our Lagrangian effectively become usual Maxwell Lagrangian. We may rewrite Lagrangian in Eq. (2.61) as

$$\mathcal{L} \sim -\sqrt{-\det\left(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} + 1 \quad . \tag{2.62}$$

Now we generalize the Lagrangian Eq. (2.61) to be Lagrangian for Dp-brane. First we change the Lorentzian indices to be the indices of Dp-brane, m and n. Then we introduce the Kalb-Ramond field to be under of square root. The result is Lagrangian for Dp-brane, which is invariant under transformation Eq. (2.56). Finally we achieve the Dirac-Born-Infeld Lagrangian (DBI Lagrangian):

$$\mathcal{L} = -T_p\sqrt{-\det\left(\eta_{mn} + B_{mn} + 2\pi\alpha' F_{mn}\right)} \quad . \tag{2.63}$$

Here T_p is tension of Dp-brane. In Chapter 4, we will use this Lagrangian to write an action for stack of D7-branes.

2.3 Superstring Theory

2.3.1 Superstring Action and Equations of Motion

Let us consider superstring theory in $(d+1)$-dimensional space-time. In context of bosonic string, we consider only scalar field x^μ. In order to achieve supersymmetry on a world-sheet, we add a term of two components spinor field Ψ^μ into string action. We suppose that the Ψ^μ is superpartners of x^μ. We note that Ψ^μ may be called vector spinor, since it carries the Lorentzian index, μ. It is spinor in world-sheet but vector in space-time.

To describe the dynamics of Ψ^μ, we add the (1+1)-Dirac action as an additional to the Polyakov action. Thus the superstring action is written as

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\sqrt{-h}\left(h^{ab}\partial_a x^\mu \partial_b x_\mu + \frac{i}{2}\bar{\Psi}^\mu \Gamma^a \partial_a \Psi_\mu\right) \quad . \tag{2.64}$$

Here Γ^a are world-sheet gamma matrices which satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{I}_{2\times 2} \quad , \tag{2.65}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1)$ and $\bar{\Psi}^\mu = (\Psi^\mu)^\dagger \Gamma^0$ denotes the conjugate representation of Ψ^μ. In this thesis, we use representation of the gamma matrix such that

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \tag{2.66}$$

Indeed, the superstring action is not invariant under local supersymmetric transformations $\delta x^\mu = \bar{\xi}(\sigma)\Psi^\mu$ and $\delta \Psi^\mu = -i\Gamma^a \partial_a x^\mu \xi(\sigma)$. Here $\xi(x)$ denotes the infinitesimal Glassmann parameter. It is necessarys to introduce some extra action, S_{ex}, to make string action becomes invariant under those transformations. The S_{ex} is proportional to gravitino field χ. However we may use superconformal invariant of superstring action to fix $\chi = 0$. This fixing results the vanishing of the S_{ex}. Then superstring action is simplified and becomes the action shown in Eq. (2.64). Furthermore, we may set $h_{ab} = \eta_{ab}$ by using Weyl transformation for our convenience.

Superstring action as shown in Eq. (2.64) gives four sets of equations of motion depending on which fields we vary the action with respect to. By varying superstring action with respect to x^μ and Ψ^μ and assuming the vanishing of surface terms, we obtain the first and the second sets of equations of motion that are $(1+1)$-dimensional wave equations and Dirac equations. The first two set of equations of motion are

$$\partial_a \partial^a x^\mu = 0 \quad \text{for bosonic sector,} \tag{2.67}$$

$$i\Gamma^a \partial_a \Psi^\mu = 0 \quad \text{for fermionic sector} \quad . \tag{2.68}$$

The bosonic sector was already discussed in the previous section so here we will pay more attention on fermionic sector. The surface terms associating to the Dirac equation Eq. (2.68) are

$$\delta_\Psi S = -\frac{i}{4\pi\alpha'} \int \mathrm{d}^2\sigma\sqrt{-h}\left(\partial_0(\bar{\Psi}^\mu \Gamma^0 \delta\Psi_\mu) + \partial_1(\bar{\Psi}^\mu \Gamma^1 \delta\Psi_\mu)\right) \quad .$$

The first term becomes zero by setting $\delta\Psi^\mu(\tau_i) = \delta\Psi^\mu(\tau_f) = 0$. And the vanishing of second term depends on choices of boundary condition. The possible boundary conditions are

$$\bar{\Psi}^\mu \Gamma^1 \delta\Psi_\mu \Big|_{0,\pi} = 0 \quad \text{for open string,} \tag{2.69}$$

$$\bar{\Psi}^\mu \Gamma^1 \delta\Psi_\mu \Big|_{\sigma=0} = \bar{\Psi}^\mu \Gamma^1 \delta\Psi_\mu \Big|_{\sigma=2\pi} \quad \text{for closed string} \quad , \tag{2.70}$$

that we will explain them later in quantization section. At the moment, let us consider the two remaining sets of equations of motion. The third set can be obtained by varying the action with respect to h_{ab} (and set $h_{ab} = \eta_{ab}$ at the final result) which yields the vanishing of energy momentum tensor, $T^{ab} = 0$. To obtain the fourth set, we perform supersymmetric transformation of fields in string action, the result is the vanishing of supercurrent, $j^a = (1/2)\Gamma^b\Gamma^a\Psi^\mu\partial_a x_\mu = 0$. Actually the $T^{ab} = 0$ and $j^a = 0$ play the same role as constraint equations in context of superstring theory.

Since Ψ^μ is a two components spinor and a solution of Dirac equation, we may write the spinor by using Weyl basis as

$$\Psi^\mu = \begin{pmatrix} \Psi^\mu_R \\ \Psi^\mu_L \end{pmatrix} \quad \text{and} \quad \bar{\Psi}^\mu = (\Psi^\mu_L \quad -\Psi^\mu_R) \tag{2.71}$$

Here $\Psi^\mu_R \equiv \Psi^\mu_R(\tau - \sigma)$ denotes right-moving and $\Psi^\mu_L \equiv \Psi^\mu_L(\tau + \sigma)$ denotes left-moving. We suppose both $\Psi^\mu_{R,L}$ are real functions. The explicit form of both Ψ^μ_R and Ψ^μ_L can be achieved by considering their boundary conditions.

2.3.2 Open Superstring Quantization

Let us consider the quantum theory for open superstrings. We will find the explicit form of Ψ^μ_R and Ψ^μ_L. We employ the boundary conditions Eq. (2.69) and rewrite it in terms of spinor components. So all possible boundary conditions become

$$\Psi^\mu_R(\tau) = \pm\Psi^\mu_L(\tau) \quad \text{at } \sigma = 0,$$
$$\Psi^\mu_R(\tau - \pi) = \pm\Psi^\mu_L(\tau + \pi) \quad \text{at } \sigma = \pi.$$

Similar to the bosonic case, fermionic boundary conditions can be classified as NN, DD and DN types. This classification is according to a requirement that superconformal symmetry must be preserved at boundary (see Ref. [2]). The appropriate boundary conditions are divided into two sectors which are Ramond sector (or R-sector) and Neveu-Schwarz sectors (or NS-sector). For R-sector, the world-sheet supercurrent is periodic at boundary so it has integer modes. On the other hand, for NS-sector, the world-sheet supercurrent turns to be anti-periodic at boundary thus it acquires half-integer modes.

We start with the NN type. The boundary conditions for each R- and NS-sectors are

$$\Psi^\mu_R(\tau) = \Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = \Psi^\mu_L(\tau+\pi) \quad \text{for NN R-sector,}$$
$$\Psi^\mu_R(\tau) = \Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = -\Psi^\mu_L(\tau+\pi) \quad \text{for NN NS-sector.}$$

We see that conditions at $\sigma = 0$ fix the equality of Ψ^μ_R and Ψ^μ_L. Thus conditions at $\sigma = \pi$ are rewritten as $\Psi^\mu_R(\tau-\pi) = \Psi^\mu_R(\tau+\pi)$ for R-sector and $\Psi^\mu_R(\tau-\pi) = -\Psi^\mu_R(\tau+\pi)$ for NS-sector (and similar for Ψ^μ_L). The conditions at $\sigma = \pi$ enforce the $\Psi^\mu_{R,L}$ to be either periodic (R-sector) or anti-periodic (NS-sector).

In case of NN type, for R-sector, the periodicity for Ψ^μ is the same as for x^μ because world-sheet supercurrent is periodic. In the NS-sector, the Ψ^μ is anti-periodic and flips its sign as σ reaches π. This corresponds to the anti-periodic supercurrent [3].

For the DD type, the boundary conditions are obtained by changing Ψ^μ_L for NN-type to $-\Psi^\mu_L$. Thus the conditions for R- and NS-sectors are

$$\Psi^\mu_R(\tau) = -\Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = -\Psi^\mu_L(\tau+\pi) \quad \text{for DD R-sector,}$$
$$\Psi^\mu_R(\tau) = -\Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = \Psi^\mu_L(\tau+\pi) \quad \text{for DD NS-sector.}$$

The conditions in R-sector and NS-sector enforce the $\Psi^\mu_{R,L}$ to become periodic and anti-periodic functions, respectively, like the NN type. Thus the explicit form of $\Psi^\mu_{R,L}$ for NN and DD types are the same.

The DN type boundary conditions are different from the NN and DD types. In this case, the boundary conditions for R- and NS-sectors are

$$\Psi^\mu_R(\tau) = \Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = -\Psi^\mu_L(\tau+\pi) \quad \text{for DN R-sector,}$$
$$\Psi^\mu_R(\tau) = -\Psi^\mu_L(\tau) \quad \text{and} \quad \Psi^\mu_R(\tau-\pi) = -\Psi^\mu_L(\tau+\pi) \quad \text{for DN NS-sector.}$$

By such boundary conditions, the fermion Ψ^μ are anti-periodic in R-sector and periodic in NS-sector. The modes expansion of Ψ^μ in R-sector acquires half-integer modes which are the same to the x^r as shown in Eq. (2.20). For R-sector, the periodic supercurrent is satisfied by a product between two anti-periodic functions of Ψ^μ and x^r. For NS-sector, modes expansion of Ψ^μ is summation over integer modes. The anti-periodic supercurrent is satisfied by product between periodic function of Ψ^μ and anti-periodic function of x^r.

We now consider the solutions of $\Psi^\mu_{R,L}$ for NN type, which is the same as DD type. These solutions are achieved by performing the Fourier expansion, so they are

$$\Psi^\mu_R(\tau,\sigma) = \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} d^\mu_n \exp\left(-in(\tau-\sigma)\right) \quad \text{for R-sector,} \tag{2.72}$$

$$\Psi^\mu_L(\tau,\sigma) = \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} d^\mu_n \exp\left(-in(\tau+\sigma)\right) \quad \text{for R-sector.} \tag{2.73}$$

As we perform the canonical quantization, all fields x^μ and Ψ^μ become operators. The components $\Psi^\mu_{R,L}$ satisfy the anti-commutation relations

$$\{\Psi^\mu_R(\tau,\sigma), \Psi^\nu_R(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \quad , \tag{2.74}$$

$$\{\Psi^\mu_L(\tau,\sigma), \Psi^\nu_L(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \quad . \tag{2.75}$$

By substituting $\Psi^\mu_{R,L}$ from Eq. (2.72) and (2.73) into above anti-commutation relations, we obtain

$$\{d^\mu_m, d^\nu_n\} = \delta_{m+n,0}\eta^{\mu\nu} \quad . \tag{2.76}$$

This is the anti-commutation relation of annihilation and creation operator which are denoted by d^ν_n and d^ν_{-n}, respectively (for $n > 0$).

For NS-condition, the $\Psi^\mu_{R,L}$ are anti-periodic. They can be written in terms of Fourier expansion such that

$$\Psi^\mu_R(\tau,\sigma) = \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \exp\left(-ir(\tau-\sigma)\right) \quad \text{for NS-sector,} \tag{2.77}$$

$$\Psi^\mu_L(\tau,\sigma) = \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \exp\left(-ir(\tau+\sigma)\right) \quad \text{for NS-sector.} \tag{2.78}$$

They also satisfy the same anti-commutation relations Eq. (2.74) and (2.75) which lead us to the anti-commutation relation

$$\{b^\mu_r, b^\nu_s\} = \delta_{r+s,0}\eta^{\mu\nu} \quad \text{with} \quad r,s\in\mathbb{Z}+\frac{1}{2} \quad . \tag{2.79}$$

Again, this is anti-commutation relation of annihilation and creation operators which are labeled by b^μ_r and b^μ_{-r}, respectively (for $r > 0$). We will use them to construct NS-state.

In case of DN type, $\Psi^\mu_{R,L}$ for R-sector gains half-integer modes due to its anti-periodicity. The Fourier modes expansion is written by

$$\Psi^\mu_R(\tau,\sigma) = \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} d^\mu_r \exp\left(-ir(\tau-\sigma)\right) \quad \text{for R-sector,} \tag{2.80}$$

$$\Psi^\mu_L(\tau,\sigma) = \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} d^\mu_n \exp\left(-ir(\tau+\sigma)\right) \quad \text{for R-sector.} \tag{2.81}$$

And for NS-sector, the modes expansion turns to be summation over integer modes;

$$\Psi^\mu_R(\tau,\sigma) = \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} b^\mu_n \exp\left(-in(\tau-\sigma)\right) \quad \text{for NS-sector,} \tag{2.82}$$

$$\Psi^\mu_L(\tau,\sigma) = \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} b^\mu_n \exp\left(-in(\tau+\sigma)\right) \quad \text{for NS-sector.} \tag{2.83}$$

Oscillators for both R and NS-sector have to satisfy the anti-commutation relations as follow:

$$\{d^\mu_r, d^\nu_s\} = \delta_{r+s,0}\eta^{\mu\nu} \quad \text{with} \quad r,s\in\mathbb{Z}+\frac{1}{2} \quad \text{for R-secctor,} \tag{2.84}$$

$$\{b^\mu_m, b^\nu_n\} = \delta_{m+n,0}\eta^{\mu\nu} \quad \text{with} \quad n,m\in\mathbb{Z} \quad \text{for NS-secctor.} \tag{2.85}$$

Notice that the modes expansion in R-sector for NN and DD types have zero modes d^μ_0 but not for DN type. For DN type, the zero modes b^μ_0 appear in modes expansion in NS-sector instead.

At this time, we consider an open superstring which all of its solutions (x^μ and Ψ^μ) satisfy the NN boundary conditions. We will find the forms of Hamiltonian and mass squared operators. We apply the light-cone gauge to both x^μ and Ψ^μ. The "plus" and "minus" directions are defined by $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ and $\Psi^\pm = (\Psi^0 \pm \Psi^1)/\sqrt{2}$. For open string we choose the "plus" direction so

$$x^+ = 2\alpha' p^+\tau \quad \text{and} \quad \Psi^+ = 0 \quad . \tag{2.86}$$

By using the light-cone gauge, we obtain the constraint equation $j^\mu = 0$ written in terms of Ψ^- and Ψ^I, then we find that Ψ^- can be determined by Ψ^I. Similarly we also express the other constraint equation $T^{ab} = 0$, then we calculate the combination elements as $T^{00} + 2T^{11}$. The substitution of Ψ^μ terms are depending on choices of boundary conditions, either R or NS condition. Then we see that oscillators of x^- is determined by oscillators coming from both x^I and Ψ^I. Since we take the result as $\sqrt{2\alpha'}p^+\alpha^-_0 = L_0$, we obtain the Virasoro operator L_0. The L_0 contains summation of non-normal ordering terms. The non-normal ordering terms are substituted by usual commutation relation of α_n^I oscillators and anti-commutation relation of d_n^I oscillators. Thus we obtain the results that are either

$$L_0 = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I + \sum_{n=1}^{\infty} n d_{-n}^I d_n^I \quad \text{right-R-sector} \tag{2.87}$$

or

$$L_0 = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^I b_r^I + a_{NS} \quad \text{right-NS-sector.} \tag{2.88}$$

Actually the L_0 is for either R or NS right-moving so we denoted "right" at the end of equation. There exist the calculations for L_0 for either R or NS left-moving which give the same result. We can use L_0 in either Eq. (2.87) or (2.88) as the Hamiltonian for respectively R- or NS-sector. Note that, for R-sector, the zero point energy arising from bosonic degrees of freedom is completely canceled by the same arising from fermionic degrees of freedom. While the zero point energy for NS-sector appears in a term denoted by a_{NS} where $a_{\text{NS}} = -(D-2)/16$.

We will find the mass squared operator. Originally, the mass squared operator is defined by $M^2 = -p^\mu p_\mu$. The dot product can be expanded and written in terms of light-cone directions. We substitute $2\alpha' p^+ p^- = L_0$ into the light-cone expression. Thus we obtain that mass squared operator are either

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I + \sum_{n=1}^{\infty} n d_{-n}^I d_n^I\right) \quad \text{for open-R-sector} \tag{2.89}$$

and

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^I b_r^I + a_{NS}\right) \quad \text{for open-NS-sector.} \tag{2.90}$$

Let us consider an open superstring (p-q string) which satisfies the boundary conditions:

$$\begin{matrix} x^2 & x^3 & \dots & x^p & x^{p+1} & \dots & x^q & x^{q+1} & \dots & x^d \end{matrix}$$
$$\underbrace{\Psi^2 \quad \Psi^3 \quad \dots \quad \Psi^p}_{\text{NN type}} \quad \underbrace{\Psi^{p+1} \quad \dots \quad \Psi^q}_{\text{DN type}} \quad \underbrace{\Psi^{q+1} \quad \dots \quad \Psi^d}_{\text{DD type}} \quad .$$

Notation #DN denotes number of DN-directions. When we calculate the mass squared operator, we find that the zero point energy in R-sector is always zero. And for NS-sector, the zero point energy is

$$a_{NS} = -\frac{(D-2)}{16} + \frac{\#\text{DN}}{8} \quad . \tag{2.91}$$

For superstring theory in (9+1)-dimensional space-time, the NS-sector of a p-q string acquires zero point energy $a_{NS} = -1/2 + (\#\mathrm{DN})/8$ [4]. For a specific case where $\#\mathrm{DN} = |p-q| = 4$, a_{NS} becomes zero, R and NS ground states are degenerate. This happens in many string setups, for example D3/D7 model (we will discuss this model in Chapter 3) and D4/D8 in Saki-Sugimoto model. Note that the Dp/Dq is a D-branes configuration which consists of parallel Dp and Dq branes.

2.3.3 Closed Superstring Quantization

Now we turn to consider quantum theory for closed superstring. Since most of the calculations for closed superstring are the same as in the open superstring case. We may quote some results from the open string context.

Let us first consider boundary conditions for closed string, as shown in Eq. (2.70). We rewrite the boundary conditions in terms of right-moving and left-moving components as

$$\Psi^\mu_R(\tau-\sigma) = \pm\Psi^\mu_R(\tau-\sigma-2\pi)$$

and

$$\Psi^\mu_L(\tau+\sigma) = \pm\Psi^\mu_L(\tau+\sigma+2\pi) \quad .$$

The boundary conditions enforce that each component separately satisfies either R-condition or NS-condition. Again, components which satisfy the R-condition are periodic functions with period 2π. On the other hand, components satisfying the NS-condition are periodicity with period 4π.

By performing the Fourier expansions, left-moving modes for both R and NS conditions are the same as in open superstring case. So we can say that in case of closed string, for left-moving component Ψ^μ_L are satisfied either Eq. (2.73) or (2.78). Thus the Ψ^μ_L is in terms of either d^μ_n or b^μ_r up to choices of boundary condition. The right-moving component Ψ^μ_R is in forms of Eq. (2.72) or (2.74). Thus the Ψ^μ_R are written in terms of either $\bar{d}^\mu_n$ or $\bar{b}^\mu_r$ up to choices of boundary condition. Note that the “bar” denotes right-moving modes.

We now perform the canonical quantization for closed superstring. The components $\Psi^\mu_{R,L}$ become operators. Their anti-commutation relations lead to anti-commutation relations between oscillators such that

$$\{d^\mu_m, d^\nu_n\} = \{\bar{d}^\mu_m, \bar{d}^\nu_n\} = \delta_{m+n,0}\eta^{\mu\nu} \tag{2.92}$$

and

$$\{b_r^\mu, b_s^\nu\} = \{\bar{b}_r^\mu, \bar{b}_r^\nu\} = \delta_{r+s,0}\eta^{\mu\nu} \quad . \tag{2.93}$$

We now perform the light-cone direction for x^μ and Ψ^μ. In case of closed string, we choose the "plus" direction such as $x^+ = \alpha' p^+ \tau$ and $\Psi^+ = 0$. Then the element T^{ab} in the constraint equation Eq. (2.69) is expanded by light-cone dot product. The combination $T^{00}+2T^{01}$ gives a Virasoro operator $\bar{L}_0$ for right-moving modes. While the combination $T^{00} - 2T^{01}$ gives a Virasoro operator L_0 for left-moving modes. Then we replace some non-normal ordering terms by commutation relation for α oscillators and anti-commutation relations for fermionic oscillators. After substituting the non-normal ordering terms, the form of L_0 is the same as either Eq. (2.87) or (2.88) up to choices of either R or NS conditions. $\bar{L}_0$ is obtained by reproducing the L_0 and place "bar" over α_n^I, d_n^I and b_r^I. So $\bar{L}_0$ are written in terms of $\bar{\alpha}_n^I$, $\bar{d}_n^I$ and $\bar{b}_r^I$.

We will identify $\bar{L}_0$ as the Hamiltonian for right-moving modes (or H_R) and, on the other hand, L_0 as the Hamiltonian for left-moving modes (or H_L). So the total Hamiltonian is $H = H_R + H_L$. Forms of the H_R and H_L separately depend on boundary condition of the left and right moving modes. There are four distinct closed superstring sectors namely R-R sector, R-NS sector, NS-R sector and NS-NS sector.

The mass squared operator for closed superstring can be written as $M^2 = M_R^2 + M_L^2$. Although the left-moving and the right-moving modes are independent, there are the so-called matching condition $M_R^2 = M_L^2$ that fixes mass contribution from left- and right-moving modes. The explicit form of the mass squared operators for right-moving modes are

$$M_R^2 = \frac{4}{\alpha'}\left(\sum_{n=1} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I + \sum_{n=1} n\bar{d}_{-n}^I \bar{d}_n^I\right) \quad \text{for R-sector} \tag{2.94}$$

or

$$M_R^2 = \frac{4}{\alpha'}\left(\sum_{n=1} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I + \sum_{r=\frac{1}{2}} r\bar{b}_{-r}^I \bar{b}_r^I + a_{NS}\right) \quad \text{for NS-sector} \quad . \tag{2.95}$$

And mass squared operators for left-moving modes can be either

$$M_L^2 = \frac{4}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I + \sum_{n=1}^{\infty} n d_{-n}^I d_n^I\right) \quad \text{for R-sector} \tag{2.96}$$

or

$$M_L^2 = \frac{4}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^I b_r^I + a_{NS}\right) \quad \text{for NS-sector} \tag{2.97}$$

In order to avoid superconformal anomaly, the consistency condition for superstring theory requires $D = d + 1 = 10$ and $a_{NS} = 1/2$. Then the transverse directions are denoted by index $I = 2, 3, ..., 9$. We will discuss the spectrum of both open and closed superstrings in next subsection.

2.3.4 Superstring States and GSO Projection

We will establish string states for open and closed superstring, we follow Ref. [5]. Due to cancelation of superconformal anomaly, we have to set $D = d + 1 = 10$ that means our consideration is in (9+1)-dimensional space-time. We first define an NS-string vacuum such $|0_{\rm NS}\rangle|0\rangle$ for strings in NS-sector. Slots $|0_{\rm NS}\rangle$ and $|0\rangle$ stand for world-sheet fermionic and bosonic vacuum states, respectively. The first slot labeled by $F_{\rm NS} = 0_{\rm NS}$ where $F_{\rm NS}$ counts number of NS-fermionic modes on a string. We can raise or lower the $F_{\rm NS}$ by using NS-creation or annihilation operators, respectively. And the latter slot denoted by $\lambda = 0$, λ counts bosonic modes which rising by bosonic creation operators. Note that the vacuum must be zero by acting an annihilation operator. We show the general NS-state, that is

$$|F_{\rm NS}\rangle|\lambda_{\rm b}\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{9}(\alpha_{-n}^I)^{\lambda_{n,I}}\prod_{r=\frac{1}{2}}^{\infty}\prod_{J=2}^{9}(b_{-r}^J)^{F_{r,J}}|0_{\rm NS}\rangle|0_{\rm b}\rangle \quad \text{open/closed-NS} \,. \tag{2.98}$$

There are more subtle since we consider an R-string vacuum. The associating fermionic creation operators for R-sector are d_{-n}^I for $n > 0$, and zero modes operators, d_0^I. It is easy to show that the d_0^I does not contribute to mass squared value. So there exists the degeneracy for R-state, in the sense that we have different R-state which give the same mass value e.g. d_0^I and $d_0^I d_0^J$ acting on $|\text{R-vacuum}\rangle$ have the same $M^2 = 0$. We define new annihilation and creation operators such $\xi_i = (d_0^{2i} + i d_0^{2i+1})$ and $\xi_i^\dagger = (d_0^{2i} - i d_0^{2i+1})$, respectively, where $i = 1, 2, 3, 4$. Those operators satisfy anit-commutation following $\{\xi_i, \xi_j^\dagger\} = \delta_{ij}$. For a fundamental R-vacuum $|0_R\rangle$, the degenerate states are

$$|0_{\rm Ra}\rangle = \begin{cases} |0_{\rm R}\rangle \\ \\ \xi_1\xi_2|0_{\rm R}\rangle,\;\; \xi_1\xi_3|0_{\rm R}\rangle,\;\; \xi_1\xi_4|0_{\rm R}\rangle,\;\; \xi_2\xi_3|0_{\rm R}\rangle,\;\; \xi_2\xi_4|0_{\rm R}\rangle,\;\; \xi_3\xi_4|0_{\rm R}\rangle \\ \\ \xi_1\xi_2\xi_3\xi_4|0_{\rm R}\rangle \end{cases} \,. \tag{2.99}$$

The degenerate R-vacuum states, which even number of $\xi^\dagger$, are denoted by extra indices a, $a = 1, 2, ..., 8$. On the other hand,

$$|0_{\mathrm{R}\bar{a}}\rangle = \begin{cases} \xi_1|0_\mathrm{R}\rangle, \quad \xi_2|0_\mathrm{R}\rangle, \quad \xi_3|0_\mathrm{R}\rangle, \quad \xi_4|0_\mathrm{R}\rangle \\ \\ \xi_1\xi_2\xi_3|0_\mathrm{R}\rangle, \quad \xi_1\xi_2\xi_4|0_\mathrm{R}\rangle, \quad \xi_1\xi_3\xi_4|0_\mathrm{R}\rangle, \quad \xi_2\xi_3\xi_4|0_\mathrm{R}\rangle \end{cases} . \tag{2.100}$$

The degenerate R-vacuum states with odd number of $\xi^\dagger$ are labeled by index $\bar{a}$, $\bar{a} = \bar{1}, \bar{2}, ..., \bar{8}$. Thus the general R-string states are

$$|F_{\mathrm{R}a}\rangle|\lambda_\mathrm{b}\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{9}(\alpha_{-n}^I)^{\lambda_{n,I}}\prod_{m=1}^{\infty}\prod_{J=2}^{9}(d_{-r}^J)^{F_{n,J}}|0_{\mathrm{R}a}\rangle|0_\mathrm{b}\rangle \quad \text{open/closed-R} \tag{2.101}$$

and

$$|F_{\mathrm{R}\bar{a}}\rangle|\lambda_\mathrm{b}\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{9}(\alpha_{-n}^I)^{\lambda_{n,I}}\prod_{m=1}^{\infty}\prod_{J=2}^{9}(d_{-r}^J)^{F_{n,J}}|0_{\mathrm{R}\bar{a}}\rangle|0_\mathrm{b}\rangle \quad \text{open/closed-R.} \tag{2.102}$$

The first slot of both string vacuum states is $F_\mathrm{R} = 0_\mathrm{R}$, where F_R counts number of fermionic creation operator except the $\xi_i^\dagger$. There exists an operator such $N_\mathrm{FB} = (-1)^{F+1}$ with eigenvalue ± 1. Notice that F in the N_FB operator is valid for both value of F_NS and F_R. Since a state is given value either $+1$ or -1 by N_FB operator, this state is either world-sheet boson or fermion. Thus we may write NS+ and R+ for state which are given $+1$ by the N_FB operator, and vice versa for NS$-$ and R$-$.

We determine mass squared for superstring states. We show some NS-states as shown in the Table 2.1. We see that the ground state for NS$-$ sector is tachyonic

State	sector	mass squared	degree of freedom
$\vert 0_\mathrm{NS}\rangle\vert 0_\mathrm{b}\rangle$	NS$-$	$-1/2\alpha'$	1
$b_{-1/2}^I\vert 0_\mathrm{NS}\rangle\vert 0_\mathrm{b}\rangle$	NS+	0	8
$\alpha_{-1}^I, b_{-1/2}^I b_{-1/2}^J \quad \vert 0_\mathrm{NS}\rangle\vert 0_\mathrm{b}\rangle$	NS$-$	$1/2\alpha'$	$8 + 8 \times 7$

Table 2.1: Mass spectrum of an open superstring with all directions of its solution satisfy the NN boundary condition.

state. Thus this NS$-$ sector will be projected out by performing the Gliozzi, Scherk and Olive projection (or GSO projection). We also see that the ground state of NS+ sector carries a light-cone index and the state also has eight degrees of freedom. Thus, in context of open superstring, we identify the $b_{-1/2}^I|0_\mathrm{NS}\rangle|0_\mathrm{b}\rangle$ as photon field with eight polarizations. And we have already said that the NS+ sector is space-time boson. The GSO projection applying on R-sector will project out either R+ or R−, depending on our choice. After the projection, the surviving R-sector has eight degrees of freedom.

Since we define $d^I_0 \equiv i\sqrt{2}\gamma^I$ where γ^I denotes space-time gamma matrix. The states in sector of R+ and R- are arisen from products of combination of the gamma matrix. Thus we can say that the R-sector is space-time fermion. Moreover, in open string context, the surviving massless R-state after the GSO projection are superpartner of photon states.

For the context of closed superstring, we will first project out the NS− sector, both right-moving and left-moving have to be state in NS+ sector. For R-sector, right-moving and left-moving are separately chosen to be state in either R+ or R− sectors. There is specific name up to choices of R-sector for superstring theories that are type IIA and type IIB superstring theories.

Type IIA superstring theory contains string states in NS+ sector, both R+ and R− sectors. We may choose left-moving modes being either NS+ or R− sector, and right moving modes being either NS+ or R+ sector. Thus we have massless state as shown in Table 2.2. Here the first and second slot in (,) are left-moving

Closed string sector	Closed string state	Degree of freedom
(NS+,NS+)	$b^I_{-1/2}\vert 0_{\mathrm{NS}}\rangle_L \bar{b}^J_{-1/2}\vert 0_{\mathrm{NS}}\rangle_R \vert 0_{\mathrm{b}}\rangle$	64B
(NS+,R+)	$b^I_{-1/2}\vert 0_{\mathrm{NS}}\rangle_L \vert 0_{\mathrm{R\bar{b}}}\rangle_R \vert 0_{\mathrm{b}}\rangle$	64F
(R-,NS+)	$\vert 0_{\mathrm{Ra}}\rangle_L \bar{b}^I_{-1/2}\vert 0_{\mathrm{NS}}\rangle_R \vert 0_{\mathrm{b}}\rangle$	64F
(R-,R+)	$\vert 0_{\mathrm{Ra}}\rangle_L \vert 0_{\mathrm{R\bar{b}}}\rangle_R \vert 0_{\mathrm{b}}\rangle$	64B

Table 2.2: Massless modes of closed superstring excitation in type IIA superstring theory.

and right-moving, respectively. The massless states in (NS+,NS+) sector has 64 bosonic degrees of freedom and carries two light-cone indices. So they match to graviton field $g_{\mu\nu}$ with traceless transverse degrees of freedom, Kalb-Ramond field $B_{\mu\nu}$ and dilaton field ϕ. Both (NS+,R−) and (R−,NS+) sectors have total 128 fermionic degrees of freedom thus they are superpartner for bosonic degrees of freedom. The massless states in (R−,R+) sector also contain 64 bosonic degrees of freedom and have different chirality due to different number of $\xi^\dagger_i$. Actually, the massless states carry one and three light-cone directions so they match to one-form field, A_μ and three-form field, $A_{\mu\nu\rho}$. The one and three form fields play role as electrical gauge potentials which respectively interact with D0- and D2-branes. Those form fields also magnetically interact with D4- and D6-branes. We conclude that type IIA superstring theory contains usual graviton, Kalb-Ramond and dilaton fields. Type IIA theory also contains one and three-form fields. Stable objects living in the type IIA theory are fundamental string and Dp-branes with p =even number.

For type IIB superstring theory, we may choose left-moving modes being either NS+ or R$-$ sector, and right moving modes being either NS+ or R$-$ sector. Thus we show massless state as shown in Table 2.3. Type IIB theory also contains

Closed string sector	Closed string state	Degree of freedom
(NS+,NS+)	$b^I_{-1/2}\|0_{\rm NS}\rangle_L \bar{b}^J_{-1/2}\|0_{\rm NS}\rangle_R\|0_{\rm b}\rangle$	64B
(NS+,R-)	$b^I_{-1/2}\|0_{\rm NS}\rangle_L\|0_{\rm Rb}\rangle_R\|0_{\rm b}\rangle$	64F
(R-,NS+)	$\|0_{\rm Ra}\rangle_L \bar{b}^I_{-1/2}\|0_{\rm NS}\rangle_R\|0_{\rm b}\rangle$	64F
(R-,R-)	$\|0_{\rm Ra}\rangle_L\|0_{\rm Rb}\rangle_R\|0_{\rm b}\rangle$	64B

Table 2.3: Massless modes of closed superstring excitation in type IIB superstring theory.

(NS+,NS+) sector so the theory contains the graviton, Kalb-Ramond and dilaton fields, as in context of type IIA theory. The massless states in $(\mathrm{R}-,\mathrm{R}-)$ sector have the same chirality and actually carry zero, two and four light-cone indices. So they match to zero-form field, A, two-form field, $A_{\mu\nu}$ and four-form field $A_{\mu\nu\rho\sigma}$. The zero-, two- and four-form fields can electrically interact with D(-1)-, D1- and D3-branes, respectively. Those form fields also magnetically interact with D7-, D5- and again D3-branes. We now conclude that the type IIB theory contains graviton, Kalb-Ramond and dilaton fields as in type IIA theory. The differences are that type IIB theory contains zero, two and four form fields which enforce the theory to include Dp-brane with $p = -1$ and odd number.

Chapter III

Introduction to AdS/CFT correspondence

3.1 World-Volume theory

3.1.1 Stack of Dp-Branes

Let us discuss about open string ending on the same Dp-brane (p-p string) in more detail. Quantum fluctuations of the open strings can be divided into two sets acoording to their directions i.e. parallel and perpendicular to the Dp-brane. We are most interested in the massless modes of these fluctuations. The massless modes in parallel directions are components of a vector field A^a with $a = 0, 1, ..., p$ describing $U(1)$ gauge theory in the world-volume of Dp-brane. On the other hand, massless modes along perpendicular directions appear as $9 - p$ massless scalar fields in the world-volume theory. The $9 - p$ scalar fields are denoted by φ^r with $r = p+1, ..., 9$ and transform as scalar under $SO(1, p)$. To be more precise, we introduce x^a and x^r to label the parallel and perpendicular directions to Dp-brane world-volume, respectively. In this notation, $x^r = (x^{p+1}, ..., x^9)$ determine position of Dp-brane. Notice that for supersymmetric theory, there exist fermionic modes with equal degrees of freedom to those bosonic modes, but we will pay attention to bosonic degrees of freedom. We note that fluctuations of p-p string along x^r directions may be considered as fluctuations of Dp-brane itself.

More complications arise when we consider the case that p-p strings attach on two or more Dp-branes. For simplicity, we consider a spacial case when all Dp-branes are completely parallel to each other. There exist additional degrees of freedom for open string states, since endpoints of a p-p string may end at two different Dp-branes. String states describing a p-p string stretching form i^{th} Dp-brane to j^{th} Dp-brane are as $|[i, j]\rangle$. The indices $i, j = 1, 2, ..., N$ are called Chan-Paton indices, N is total number of Dp-branes, see also [5].

Let us discuss a system with two copies Dp-branes which are separated by transverse distance $|x^r| = L$ (see Figure 3.1). The possible of p-p string states are

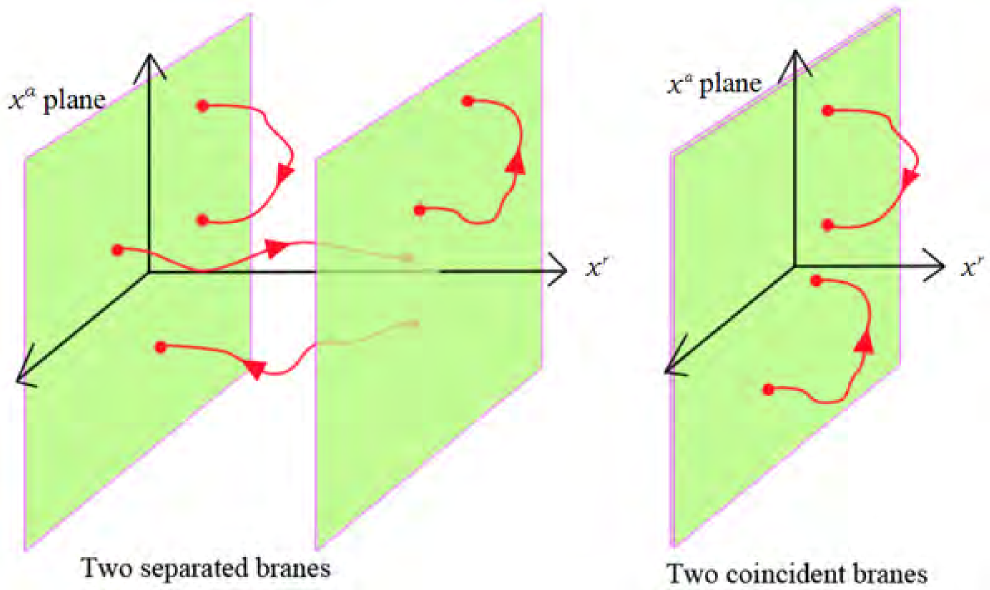

Figure 3.1: When the two Dp-branes are separated (left), there are four sector p-p string. But when the two branes are coincided at the same x^r (right), there are only one sector of p-p string

$|[1,1]\rangle$, $|[1,2]\rangle$ $|[2,1]\rangle$ and $|[2,2]\rangle$. The lowest energy states for open string ending on the same branes $|[1,1]\rangle$ and $|[2,2]\rangle$ are massless. These massless fluctuations are still represented by the massless vector field $(A^a)_{11}$ and $(A^a)_{22}$, and massless scalar fields $(\varphi^r)_{11}$ and $(\varphi^r)_{22}$. The lowest energy states for string ending on different branes $|[1,2]\rangle$ and $|[2,1]\rangle$ are massive. Their masses are given by $M = L/(2\pi\alpha')$, see Ref. [5]. Thus, the fluctuations are represented by components of a massive vector field and scalar fields. They are denoted by $(A^a)_{12}$ and $(A^a)_{21}$ for massive vector fields, and $(\varphi^r)_{12}$ and $(\varphi^r)_{21}$ for massive scalar fields. The world-volume has $U(1) \times U(1)$ gauge symmetry.

The above system becomes more interesting when two Dp-branes are coincided at a same position in the transverse directions, $L = 0$. That conicidence Dp-branes is called a stack of two Dp-branes. The lowest energy states of $|[2,1]\rangle$ and $|[1,2]\rangle$ now become massless, so all the lowest energy states are denoted by massless vector field $(A^a)_{ij}$ and scalar fields $(\varphi^r)_{ij}$. $(A^a)_{ij}$ and $(\varphi^r)_{ij}$ are in the adjoint representations of $U(2)$ gauge group. Consequently, $(A^a)_{ij}$ are identified as gauge fields in $U(2)$ gauge theory on Dp-brane world-volume. Altogether, bosons and fermions form supersymmetry multiplet of supersymmetric Yang-Mills theory (SYM theory) with $U(2)$ gauge group. The number of supersymmetry depends on the dimensions of Dp-branes.

Indeed, the symmetry of world-volume gauge theory is enhanced from $U(1)\times$

$U(1)$ to $U(2)$ when two Dp-branes move closed to each other and form a stack of coincident Dp-branes. If the two Dp-branes move apart from each other, the symmetry is broken from $U(2)$ to $U(1) \times U(1)$ and the vectors $(A^a)_{12}$ and $(A^a)_{21}$ become massive. This is string theory version of the Higgs-mechanism.

We can make the system of two Dp-branes to become more general when we add more Dp-branes and form a stack of N copies Dp-branes. In this system, the vector field $(A^a)_{ij}$ becomes gauge field of a non-abelian $U(N)$ gauge theory. We can conclude that a system consisted of a stack of Dp-branes is possible to construct a non-abelian $U(N)$ gauge theory living in world-volume.

3.1.2 $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a non-abelian $U(N)$ gauge theory with global $U(4)_{\mathrm{R}}$ R-symmetry. The theory is consistent with $(3 + 1)$ Minkowski space-time, where 16 numbers of supercharges are allowed. Those supercharges satisfy the Lie albegra of the Graded Poincaré group, a supersymmetric extension of Poincaré group. Fields in supersymmetric theory form irreducible representations of supersymmetric algebra which is called supermultiplet. Fields from supermultiplet have the same mass but different spins. Massless fields in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory are included in $\mathcal{N} = 4$ vecter supermultiplet. We show a list of physical degrees of freedom in $\mathcal{N} = 4$ vector supermultiplet in Table 3.1. Note that λ is helicity of fields. $\mathcal{N} = 4$ vector supermultiplet includes

Helicity : λ	1	1/2	0	-1/2	-1
Degrees of freedom	1	4	6	4	1

Table 3.1: $\mathcal{N} = 4$ vector supermultiplet.

two bosonic degrees of freedom with $\lambda = \pm 1$, correspond to the 2-polarization of a massless vector field, six bosonic degrees of freedom with $\lambda = 0$ and fermionic degrees of freedom with $\lambda = \pm 1/2$ correspond to 6 real scalar fields and 4 Weyl spinor, respectively.

Next we consider a string configuration. In context of type IIB superstring theory, the $(3+1)$-dimensional non-abelian gauge theory with $U(N_c)$ can be constructed by a stack of N_c D3-branes in Minkowski $(9+1)$-dimensional space-time. We suppose that $(3 + 1)$-directions of D3-brane world-volume coincide with the t, x, y, z directions of the space-time. The remain directions, $x^4, ..., x^9$, are transverse to the world-volume. We show the massless modes fluctuation of 3-3 string in Table 3.2. Massless modes of strings are 2 degrees of freedom for vector field, 6

Space-time coordinates	t	x	y	z	x^4	x^5	x^6	x^7	x^8	x^9
D3-brane world-volume	$\times$	$\times$	$\times$	$\times$						
Bosonic modes	2 DOF of A^a				6 scalar fields $\varphi^4, ..., \varphi^9$					
Fermionic modes	8 DOF $\rightarrow$ 4 Weyl spinors									

Table 3.2: Directions of D3-brane world-volume and degrees of freedom of 3-3 string fluctuations.

scalar fields and 8 fermionic degrees of freedom. Those fermionic degrees of freedom are equal to degrees of freedom of 4 Weyl spinors. All degrees of freedom of string fluctuation can match with fields in $\mathcal{N} = 4$ vector supermultiplet. We can conclude that fluctuations of 3-3 string give rise to field in vector supermultiplet. In other words, the system with a stack of N_c D3-branes has $\mathcal{N} = 4$ SYM theory in D3-brane world-volume.

The gauge field and scalar fields are in adjoint representation of $U(N_c)$ gauge group. This gauge group may be identified with color gauge group, when we apply this gauge theory to explain system of quarks. The gauge group $U(N_c)$ can be usually factorized into $U(N_c) \sim SU(N_c) \times U(1)$, and $U(1)$ is decoupled with fixing position of D3-branes. The vector fields A^a plays a role as gauge field of $SU(N_c)$ gauge group. We can write an effective action which describes those massless open string states. The effective action for $\mathcal{N} = 4$ SYM theory is

$$S_{\mathrm{YM}} = -\frac{1}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \mathrm{Tr} \left(\frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} D_a \varphi^i D_a \varphi_i + \varphi^i \varphi_i \right) + ... \quad . \tag{3.1}$$

Here we show only massless bosonic fluctuation in the action. Terms in "..." are massless fermionic and massive modes. g_{YM} is dimensionless coupling constant of our SYM theory which relates to string coupling g_s by $g_{\mathrm{YM}}^2 = 4\pi g_s$.

3.2 Type IIB Supergravity

The low energy limit of type IIB superstring theory contains only massless states such as graviton $g_{\mu\nu}$, Neveu-Schwarz two-form field $B_{\mu\nu}$, dilaton Φ and Ramond-Ramond $(p+1)$-form fields, where p is odd number. There are also fermionic partners to those states but we can set them to zero at classical level. In other words, we consider only bosonic background fields.

In this subsection, we consider low energy effective action that describes those fields. The action contains Einstein-Hilbert term, dilaton kinetic term, as well as the terms associated with Neveu-Schwarz 2-form fields and Ramond-Ramond $(p+1)$-form fields.

We consider p-brane solution of type IIB supergravity theory. Dp-brane is a special case of p-brane which open string can end on. As massive charged object, a p-brane causes the surrounding space-time to have more curvature. Mass and charge of p-branes are denoted by M and Q. The metric that describes p-brane is called black p-brane solution which is the solution of Einstein equation. The black p-brane solutions can be classified by masses and charges into three cases: $G_{10}M^2 > Q^2$, $G_{10}M^2 = Q^2$ and $G_{10}M^2 < Q^2$. We will consider the p-branes with $G_{10}M^2 > Q^2$, that is non-extremal p-brane solution.

Since we are interested in D3-brane solution, only Ramond-Ramond 4-form field will be turned on. Its kinetic terms can be written in a term of five-form field strength $F_{(5)} = \mathrm{d}C_{(4)}$. The effective action in Einstein frame is

$$S = \frac{1}{16\pi G_{10}} \int \mathrm{d}^D x \sqrt{-g} \left(R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2 \cdot 5!} F_{(5)}^2 \right) \quad , \tag{3.2}$$

G_{10} is 10-dimensional Newton constant which relates to string coupling g_s by $16\pi G_{10} = (2\pi)^2 g_s^2 \alpha'^4$ [7]. $F_{(5)}^2$ can be written in components form as $F_{\alpha_1 \ldots \alpha_5} F^{\alpha_1 \ldots \alpha_5}$.

3.2.1 D3-Brane Solution

Equations of motion for type IIB supergravity can be obtained by varying action in Eq. (3.2) for $n = 5$ with respect to graviton, dilaton and five-form fields. After some rearrangements, the explicit form of equations of motion are

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu \phi \partial_\nu \phi + \frac{1}{2 \cdot 5!} \left(5 F_{\mu\lambda_2 \ldots \lambda_5} F_\nu{}^{\lambda_2 \ldots \lambda_5} - \frac{1}{2} g_{\mu\nu} F_5^2 \right) \quad , \tag{3.3}$$

$$\partial_\rho \partial^\rho \phi = \frac{1}{2 \cdot 5!} F_{(5)}^2 \quad , \tag{3.4}$$

$$\nabla_{\mu_1} \left(F^{\mu_1 \ldots \mu_5} \right) = 0 \tag{3.5}$$

$$\partial_{[\mu_1} F_{\mu_2 \ldots \mu_6]} = 0 \quad . \tag{3.6}$$

Note that Eq. (3.6) comes from the Bianchi identity and five-form field strength is self-dual.

We are interested in the D3-branes solution of the above equations of motion. It is convenient to assume a trial solution

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \quad ,$$

where

$$ds^2 = -e^{2B}dt^2 + e^{2C}\delta_{ij}dx^i dx^j + e^{2F}dr^2 + e^{2G}r^2 d\Omega_5^2 \quad . \tag{3.7}$$

Here index i is running along D3-brane directions so $i = 1, 2, 3$. We imploded some symmetries to our considering trial solution. A symmetry we imploded is spherical symmetry on spatial directions perpendicular to the D3-brane. It is natural to write those perpendicular directions to be five-sphere of variable radius. Thus, in the above metric, the r denotes radial direction of the five-sphere. And the $d\Omega_5^2$ determines infinitesimal length on the five-sphere. Other symmetries are time translational symmetry and rotational symmetry in spatial directions parallel to the D3-brane, x^i.

Since the metric is classified by mass and charge, there should be only two independent factors appearing in the metric. The above metric now have four factors denoted by B, C, F and G, there should exist two constraints which reduce those factors into two independent factors. By using this fact and solving the equations of motion, we can find that $C = -G$ and $B = -F$. And we finally find that

$$e^{2B} = e^{-2F} = \left(1 - \frac{b^4}{r^4}\right) H^{-\frac{1}{2}} \tag{3.8}$$

$$e^{2C} = e^{-2G} = H^{\frac{1}{2}} \quad , \tag{3.9}$$

where

$$H = \left(1 + \frac{R^4}{r^4}\right) \quad .$$

H is harmonic function depended on a constant R, and b is extremal parameter. Mass of the D3-brane is also determined by both Q and b (see Ref. [10]). However we do not show form of the mass in this thesis. The constant R relates to charge and b by

$$R^4 = \sqrt{\frac{Q^2}{16} + b^2} - b \quad . \tag{3.10}$$

By substituting those factors into trial solution Eq. (3.7), we end up with

$$ds^2 = H^{-\frac{1}{2}}\left(-\left(1 - \frac{b^4}{r^4}\right)dt^2 + d\vec{x}^2\right) + H^{\frac{1}{2}}\left(\left(1 - \frac{b^4}{r^4}\right)^{-1} dr^2 + r^2 d\Omega_5^2\right), \tag{3.11}$$

where $d\vec{x}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. This is the non-extremal D3-brane solution. It is easy to check that this metric has two horizons at $r = 0$ and $r = b$, two radial values making $g_{00} = 0$.

3.2.2 $\mathbf{AdS_5 \times S^5}$ Geometry

The extremal limit of metric Eq. (3.11) is obtained by setting $b = 0$. In fact, the D3-brane satisfies the Bonomolnyi Pasad and Somenfield condition (BPS condition), $G_{10}M^2 = Q^2$. Thus the space-time curvature affecting by D3-brane are described by extremal solution. The extremal solution is following

$$\mathrm{d}s^2 = H^{-\frac{1}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + H^{\frac{1}{2}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \quad . \tag{3.12}$$

Then, let us consider the near-horizon limit, $r \to 0$, of metric Eq. (3.12). The harmonic function may be rewritten into $H = (r^4 + R^4)/r^4$, then we suppose that $r \to 0$ is equivalent to $r \ll R$. Under such limit, the harmonic function behaves like $H \to R^4/r^4$. Then we obtain

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{R^2}{r^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \quad . \tag{3.13}$$

The next step, we perform a radial coordinate transformation with $u = \frac{R^2}{r}$. We rewrite the metric Eq. (3.13) in u coordinate as

$$\mathrm{d}s^2 = \frac{R^2}{u^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \mathrm{d}u^2\right) + R^2\mathrm{d}\Omega_5^2 \quad . \tag{3.14}$$

We factorize metric Eq. (3.14) into two factors. Coordinates $t, \vec{x}, u$ belong to the $(4+1)$-dimensional Anti-de-Sitter space (AdS_5) and other coordinates $x^5, ..., x^9$ are compacted into 5-dimensional sphere S^5. Thus, the metric Eq. (3.14) describes the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry, where constant R is now identified by the AdS radius.

3.3 The Correspondence

3.3.1 Two Descriptions

In the previous section, we saw that there are two descriptions which describe a stack of D3-branes. In the first description, 3-3 string fluctuations give rise to the $\mathcal{N} = 4$ non-abelian $SU(N_c)$ SYM theory. In the second description, D3-branes are considered as source of space-time curvature. The two descriptions should be equivalent, since they describe the same system. There exists an equivalent relation between those two descriptions namely gauge/gravity correspondence. The AdS/CFT correspondence is the original version of the gauge/gravity correspondence. In this section, we will discuss the key ideas which lead to statement AdS/CFT correspondence.

3.3.2 Matching Coupling Constants

Recall the action of $\mathcal{N} = 4$ SYM theory Eq. (3.1) with only pure gauge, which is

$$S_{\rm YM} = -\frac{1}{g_{\rm YM}^2} \int {\rm d}^4x {\rm Tr} \left(\frac{1}{4} F^{ab} F_{ab} \right) \quad . \tag{3.15}$$

The field strength tensor is given by $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$. Now we identify $SU(N_c)$ as the color gauge group. $(A^a)_{ij}$ is a $N_c \times N_c$ matrix, and color indices i and j run over $1, 2, ..., N_c$. The gauge fields can self-interact with strength proportional to $1/g_{\rm YM}^2$ [11]. Every closed loop (loop of one line) contributes N_c^2.

We can write loop diagrams of self-interaction SYM theory in double lines notation. There are two types of such diagrams. Diagrams which can be written on the plane are called planar diagrams. On the other hand, diagrams which can not be drawn on the plane without crossing line are called non-planar diagrams. For non-planar diagrams with h crossing, they can be written as planar diagrams on a two-dimensional surface with h number of hole. For example, a non-planar diagram with 1 crossing can be illustrated as a planar diagram in a torus.

A planar diagram with v vertex and l loop (one line loop) has amplitude proportional to $(g_{\rm YM}^2)^{l-v} N_c^v$. We may define the 't Hooft coupling constant $\lambda = g_{\rm YM}^2 N_c$. By writing the amplitude in terms of the 't Hooft coupling, we found that its amplitude is proportional to $\lambda^{l-v} N_c^2$. We see that planar diagram has N_c^2 factor in their amplitude. For a non-planar diagrams with h crossing, we observe that amplitude of such diagram is proportional to N_c^{2-h} multiplied by some positive powers of λ. We may say that the coupling λ controls strength of loop interactions instead of coupling $g_{\rm YM}$, since positive powers of λ appears as amplitude.

When we calculate the total amplitude for collection of loops diagrams, we find that the amplitude can be expressed in $1/N_c$ expansion. Form Ref. [7], the $1/N_c$ expansion is

$$\text{total amplitude} \sim N_c^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N_c^2} f_2(\lambda) + ... \quad . \tag{3.16}$$

The function $f_n(\lambda)$ associates to loop diagrams with n number of crossing. In large N_c limit, $N_c \to \infty$, amplitude of non-planar diagrams are suppressed by negative power of N_c while the amplitude of planar diagrams are dominated. Thus in the large N_c limit, we can neglect all non-planar diagrams.

In context of perturbative theory, closed string interaction can also be written in terms of loops expansion represented by summation of closed string worldsheet diagrams. Diagrams with h closed string loops are a two-dimensional genus

h surface. As shown in Ref. [7], the total closed string amplitude is written by α' expansion such that

$$\text{Amplitude} \sim \sum_{h=0} g_s^{2h-2} F_h(\alpha') = \frac{1}{g_s^2} F_0(\alpha') + F_1(\alpha') + g_s^2 F_2(\alpha') + ... \quad . \tag{3.17}$$

The function $F_h(\alpha')$ are contribution from two-dimensional surface with h hole. In limit that $g_s \to 0$, zero order of Eq. (3.17) is dominate, we can neglect all higher order of loop interactions.

We see that the $1/N_c$ and α' expansion as shown in Eq. (3.16) and (3.17), respectively, have the same structures. By matching f_n to F_h with $n = h \neq 0$, we obtain a relation

$$g_s \sim \frac{1}{N_c} \quad . \tag{3.18}$$

In subsection 3.1.2, we wrote the relation between Yang-Mills and string coupling by $g_{\text{YM}}^2 = 4\pi g_s$. If we substitute g_{Ym}^2 by $g_{\text{Ym}}^2 = \lambda/N_c$, we obtain $4\pi g_s = \lambda/N_c$ which corresponds with Eq. (3.18).

3.3.3 Matching Symmetry Groups

Let us discuss $\mathcal{N} = 4$ SYM theory in more detail. The Lagrangian for $\mathcal{N} = 4$ SYM theory in 4-dimensional space-time given by [12] is invariant under $\mathcal{N} = 4$ Poincaré supersymmetry. Classically it is also scale invariant. In relativistic field theory, scale and Poincaré invariance can combine into a larger symmetry called conformal symmetry. In (3+1)-dimension, it forms the group $SO(2,4) \cong SU(2,2)$. There is an larger superconformal symmetry given by the supergroup $SU(2,2|4)$ which is the combination of $\mathcal{N} = 4$ supersymmetry and conformal invariance.

At quantum level, $\mathcal{N} = 4$ SYM theory has a remarkable property. There is no ultraviolet divergences in perturbative expansion. It is believed that the theory is UV finite. The renormalization group β-function of the theory vanishes in all order of perturbation i.e. scale invariant at quantum level. The superconformal group $SU(2,2|4)$ is a full quantum symmetry of the theory.

One of the necessary requirement for AdS/CFT correspondence is that the global symmetry of the two theories must be identical.

As explain above, the global symmetry for $\mathcal{N} = 4$ SYM theory in its conformal phase is $SU(2,2|4)$. For simplicity, let us consider its maximum bosonic subgroup $SU(2,2) \times SU(4)_{\text{R}} \cong SO(2,4) \times SO(6)_{\text{R}}$. It is the product of conformal group $SO(2,4)$ to the $\mathcal{N}=4$ supersymmetry $SU(4)_{\mathrm{R}}$. The bosonic part matches with isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. The full supergroup $SU(2,2|4)$ arises on the AdS side because 16 of 32 supercharges are preserved by the stack of N_c D3-branes.

3.3.4 AdS/CFT Conjecture

In previous subsection we provide some evidences supporting duality between $\mathcal{N}=4$ SYM theory and type IIB supergravity theory. In this section, we discuss this duality in more detail by following Ref. [6].

Consider a stack of N_c D3-branes. There are two kinds of perturbative string states, closed string states and open string (3-3 string) states. If we consider the system at low energy, only massless modes can survive. The closed string massless states form gravity supermultiplet of $(9+1)$-dimensional type IIB supergravity. Their dynamical evolutions are governed by low energy effective Lagrangian of type IIB superstring theory. On the other hand, open string states give $\mathcal{N}=4$ vector supermultiplet in (3+1)-dimensional world-volume. Their effective Lagrangian is that of $\mathcal{N}=4$ $U(N_c)$ SYM theory. The action for massless string modes have the form

$$S = S_{\text{closed}} + S_{\text{open}} + S_{\text{int}} \quad . \tag{3.19}$$

S_{closed} is the action of type IIB supergravity plus possible higher derivative terms, defined on (9+1)-dimensional bulk. It is not renormalizable and serves as an Wilsonian effective action. S_{open} is defined on the (3+1)-dimensional D3-branes world-volume. It contains $\mathcal{N}=4$ SYM Lagrangian and possible higher derivative terms. S_{int} describes interactions between the brane modes and the bulk fields. The coupling is proportional to the Newton constant which relates to string coupling by $16\pi G_{10} = (2\pi\alpha'^2)^2 g_s^2$.

When we take the low energy limit, $\alpha' \to 0$ with g_s and N_c fixed, we have $G_{10} \to 0$. The interaction action is effectively zero. We can say that both $\mathcal{N}=4$ SYM and type IIB supergravity theories are decouple.

Next we consider the same system by using a different description. D3-branes are massive charged objects which can be geometrically represented by D3-brane solution. We consider low energy effective theory in this background. For the observer at infinity, there are two kinds of excitations. The first kind, there are the massless bulk particles, the second kind are particles near the horizon of D3-brane solution. In observer point of view, the massless modes from near-horizon region lose its energy due to redshift factor. Furthermore, massless modes from near-horizon region are decouple with massless modes at asymptotic region since the strength of interaction is proportional to $1/r^8$. Note that r is the radial coordinate appearing in Eq. (3.13). Thus observer can say that type IIB supergravity theory in flat space is free from interaction with the type IIB superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

We compare those two low energy limit descriptions. Since both descriptions have type IIB supergravity theory in flat space, it is natural to match the $\mathcal{N} = 4$ SYM theory and type IIB superstring theory together. Actually the two theories are living in different dimension of space-time - the $\mathcal{N} = 4$ SYM theory is in (3+1)-dimension while type IIB superstring theory is in (9+1)-dimensional space-time. Matching of two theories can be done through holographic description.

Let us discuss about boundary of the $\mathrm{AdS}_5 \times \mathrm{S}^5$. For our convenient, we may fix $\mathrm{d}s^2$ to be a constant. We also define (infinitesimal) physical distance of (3+1)-space by

$$\mathrm{d}l^2 = \frac{R^2}{u^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) \quad . \tag{3.20}$$

Then the metric as shown in Eq. (3.14) becomes

$$\mathrm{d}s^2 = \mathrm{d}l^2 + \frac{R^2}{u^2}\mathrm{d}u^2 + R^2\mathrm{d}\Omega_5^2 \quad .$$

Now we perform conformal transformation as follows $u \to \epsilon u$. When we take $\epsilon \to 0$ corresponding with going toward boundary, the $(R^2/u^2)\mathrm{d}u^2 + R^2\mathrm{d}\Omega_5^2$ remain constant while the $\mathrm{d}l$ is very large. The $\mathrm{AdS}_5 \times \mathrm{S}^5$ effectively becomes flat (3+1)-dimensional space-time at its boundary. Thus we may say that the $\mathcal{N} = 4$ is living on boundary of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space.

We end up with statement of AdS/CFT correspondence: $\mathcal{N} = 4$ SYM theory in boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ corresponds with type IIB superstring theory $\mathrm{AdS}_5 \times \mathrm{S}^5$ space.

3.3.5 Strong/Weak Conjecture

Statements of the AdS/CFT correspondence have three versions according to limit of two description [8]. The first version is the strongest version of statement. This version claims correspondence between $\mathcal{N} = 4$ SYM theory and type IIB superstring theory $\mathrm{AdS}_5 \times \mathrm{S}^5$ space validing in general. However it is not possible to test this version because no one knows how to quantize the type IIB superstring theory on curved space-time. The second version is obtained in large N_c and 't Hooft limit, $N_c \to \infty$ and λ is fixed. The third version of correspondence is statement which is reached by taking large N_c limit and fixing 't Hooft coupling at large value, $\lambda > 1$. On this case the gauge description becomes strong coupling $\mathcal{N} = 4$ SYM theory due to strange of interaction (proportional to λ) is large. On the other hand, the type IIB superstring theory is now perturbative type IIB supergravity theory, since $\alpha' \to 0$ as $\lambda > 1$.

In conclusion, the AdS/CFT correspondence states that non-perturbative $\mathcal{N} = 4$ SYM theory at boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is dual to perturbative type IIB supergravity theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. This statement realizes only large N_c limit and fixing 't Hooft at large value. By using this statement, we can make prediction for a system in strongly couple gauge theory by performing parturbative calculation in type IIB supergravity theory.

There exists the AdS/CFT dictionary which gives relations between calculation in gravity and gauge sides, see Ref. [9, 13]. The general form of the AdS/CFT dictionary is

$$\langle \mathrm{e}^{\int \phi_0 \mathcal{O} \mathrm{d}^4 x} \rangle = \mathcal{Z}_{\mathrm{SUGRA}}[\phi_{\mathrm{boundary}} = \phi_0] \quad . \tag{3.21}$$

This is relation of partition function of gauge side (left handed side) and gravity side (right handed side). Here ϕ denotes field in $\mathrm{AdS}_5 \times \mathrm{S}^5$ space and $\mathcal{O}$ denotes an observable operator in gauge theory. The vacuum to vacuum expectation value of the observable can be achieved by performing functional derivative at both sides on the relation.

3.4 D3/D7 Model

3.4.1 Adding Probe D7-Branes

We add a stack of N_f copies of D7-branes into the original system so called D3/D7 model. The evolving D7-branes span (7+1)-dimensional world-volume. We choose the (3+1)-directions of D7-brane world-volume completely parallel to all directions of D3-brane world-volume, and the remain directions are all transverse to D3-brane world-volume (see Table 3.3). Directions that transverse to D7-brane world-volume are x^8 and x^9. They determine positions of D7-branes. Stack of D3-branes localizes at $x^8 = x^9 = 0$ and D7-branes may separate from the stack D3-branes in x^8 and x^9 directions by L, $L^2 = (x^8)^2 + (x^9)^2$ (see Figure 3.3).

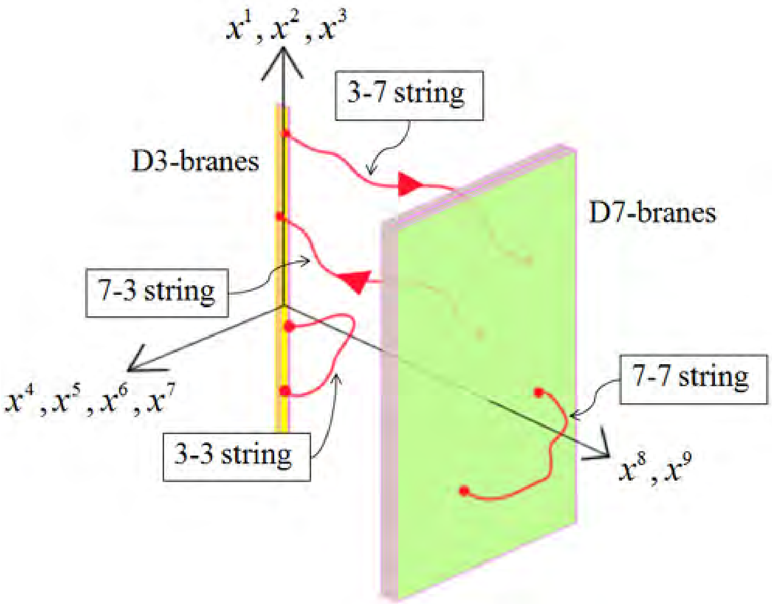

Figure 3.2: D3/D7 configuration. There exist four possible sectors of open superstring.

Space-time coordinates	t	x	y	z	x^4	x^5	x^6	x^7	x^8	x^9
D3-brane world-volume	×	×	×	×						
D7-brane world-volume	×	×	×	×	×	×	×	×		

Table 3.3: The occupation of D3- and D7-brane world-volumes. The D7-branes have freedom to move along the x^8 and x^9 directions.

Configuration that two stacks of D3 and D7 branes coincide in the same position is called Higgs branch, while configuration with separation of two stacks is Coulomb branch. As we add stack of D7-branes, we have more open strings in our system. Now the system contains 3-3, 7-7, 3-7 and 7-3 string. There are two sets of the Chan-Paton indices which we denote the first set by $i, j = 1, 2, ..., N_c$ and the second by $\tilde{i}, \tilde{j} = \tilde{1}, \tilde{2}, ..., \tilde{N}_f$. States of fluctuations on 3-3 string and 7-7 string are included in sectors $|[i, j]\rangle$ and $|[\tilde{i}, \tilde{j}]\rangle$, respectively. Massless states from $|[i, j]\rangle$ sector correspond with massless fields that are shown in Table 2.2. So we will consider the massless field from $|[\tilde{i}, \tilde{j}]\rangle$ sector. Ground states in $|[\tilde{i}, \tilde{j}]\rangle$ sector contribute 6 degrees of freedom for a vector field and 2 degrees of freedom for scalar fields in D7-brane world-volume, and 8 fermionic degrees of freedom as well. The arisen vector field plays a role as gauge field of $U(N_f)$ gauge group which is factorized as $U(N_f) \approx SU(N_f) \times U(1)_{\mathrm{b}}$. We can identify the $U(1)_{\mathrm{b}}$ factor to be group of Maxwell theory in D7-brane world-volume.

In case of 3-7 and 3-7 strings, they are different according to different orientations. For the 3-7 string, its $\sigma = 0$ endpoint is on the stack of D3-branes while $\sigma = \pi$ endpoint is on stack of D7-branes, and vice versa for 7-3 string. Modes fluctuation on the open string are $N_c \times N_f$ matrix which transform as bi-fundamental under two gauge groups, fundamental under $SU(N_c)$ and anti-fundamental under $SU(N_f)$ gauge group.

States of fluctuations on 3-7 and 7-3 strings are included in $|[i, \tilde{i}]\rangle$ and $|[\tilde{i}, i]\rangle$, respectively. The lowerest energy states in these sectors are massive for Coulomb branch and massless for Higgs branch. Fluctuations along t, x, y, z directions contribute to 2 degrees of freedom of massive vector on both D3- and D7-brane world-volumes. Fluctuations along x^4, x^5, x^6, x^7 directions contribute 4 degrees of freedom to scalar and vector fields in D3- and D7-brane world-volumes, respectively. Fluctuations along x^8 and x^9 directions contribute 2 degrees of freedom to scalar fields for both world-volumes.

Because D7-branes are BPS objects, adding a stack of D7-branes breaks half of supersymmetry. Only 8 supercharges are allowed on D3-brane world-volume, so world-volume theory becomes $\mathcal{N} = 2$ SYM theory. We can write fields in the $\mathcal{N} = 4$ vector supermultiplet to be 1 $\mathcal{N} = 2$ vector supermultiplet with its charge-parity-time reversal conjugation (CPT conjugation) and 2 $\mathcal{N} = 2$ chiral supermultiplets. Moreover, 2 additional $\mathcal{N} = 2$ massive chiral supermultiplets are also included in the world-volume theory. Fields content in those massive supermultiplets are ground states in $|[i, \tilde{i}]\rangle$ and $|[\tilde{i}, i]\rangle$ sectors. Mass of fields may be written as M_{sep} where $M_{\mathrm{sep}} = 0$ for Higgs branch but $M_{\mathrm{sep}} \sim L$ Coulomb branch, with separation L (along transverse directions).

In order to construct Lagrangian for the $\mathcal{N} = 2$ theory, we may write component fields in $\mathcal{N} = 2$ supermultiplets as component fields of $\mathcal{N} = 1$ superfields. In fact, the massless 1 $\mathcal{N} = 2$ vector (and its CPT conjugation) and 2 $\mathcal{N} = 2$ chiral supermultiplets are alternatively written as 1 $\mathcal{N} = 1$ vector and 3 $\mathcal{N} = 1$ chiral supermultiplets, with their CPT conjugations. We need at least 1 vector and 3 chiral superfields to write our Lagrangian. The $\mathcal{N} = 1$ vector and three chiral superfields are denoted by W_a and Φ_1, Φ_2, Φ_3, respectively. Similarly, component fields in $\mathcal{N} = 2$ chiral supermultiplets are rewritten as component fields of $\mathcal{N} = 1$ chiral superfields denoted by $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, (see also [14]). In case of Higgs branch, we conclude fields in $\mathcal{N}=2$ theory in Table 3.4. As shown in Table 3.4, we factorize

$\mathcal{N}=2$	Component fields	$SU(2)_\Phi \times SU(2)_\mathcal{R}$	$U(1)_R$	$SU(N_f)$	$U(1)_b$
Φ_1, Φ_2	$\varphi_4, \varphi_5, \varphi_6, \varphi_7$	$(\frac{1}{2}, \frac{1}{2})$	0	singlet	0
	λ_1, λ_2	$(\frac{1}{2}, 0)$	+1	singlet	0
Φ_3, W_a	φ_8, φ_9	$(0,0)$	+2	singlet	0
	λ_3, λ_4	$(0, \frac{1}{2})$	+1	singlet	0
	A_a	$(0,0)$	0	singlet	0
$\mathcal{Q}, \bar{\mathcal{Q}}$	$q, \bar{q}$	$(0, \frac{1}{2})$	0	fundamental	+1
	$\psi, \psi^\dagger$	$(0,0)$	-1, +1	fundamental	+1

Table 3.4: The component fields in the $\mathcal{N}=2$ theory which are written by $\mathcal{N}=1$ superfields.

rotation group of x^4, x^5, x^6, x^7 into $SU(2)_\Phi \times SU(2)_\mathrm{R}$, and rotation group of x^8, x^9 is isomorphic with $U(1)_\mathrm{R}$. Notice that the $U(1)_\mathrm{R}$ is broken in Coulomb branch because the stacks of D3- and D7-branes are separated from each others. Thus, $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ acquire mass M_sep, corresponding to Higgs mechanism for $\mathcal{N}=2$ SYM theory. As shown in reference [8]. Lagrangian of the $\mathcal{N}=2$ SYM theory in terms of those $\mathcal{N}=1$ superfields is

$$\mathcal{L} \sim \int \mathrm{d}^4\theta \left(\Phi^\dagger \mathrm{e}^{2V}\Phi + \mathcal{Q}^\dagger \mathrm{e}^V \mathcal{Q} + \bar{\mathcal{Q}}^\dagger \mathrm{e}^V \bar{\mathcal{Q}}\right) + \int \mathrm{d}^2\theta \left(W_a W^a + W\right) + \mathrm{h.c.}$$

Here W is superpotential, which is

$$W = \epsilon_{ijk}\Phi_i\Phi_j\Phi_k + \bar{\mathcal{Q}}(M_\mathrm{sep} + \Phi_3)\mathcal{Q} \quad .$$

V denotes a vector superfield which governs non-abelian of this $\mathcal{N}=2$ theory.

Stack of D7-branes is treated as probe. The number of D3-branes is much more than the number of D7-branes, $N_c \gg N_f$. In addition, the 7-7 strings decouple with other kinds of string hence the $SU(N_f)$ gauge symmetry becomes global symmetry for D3-brane world-volume. The addition of D7-probe branes does not break conformal symmetry of the $\mathcal{N}=2$ SYM theory effectively, since beta function depending on N_f/N_c goes to zero.

3.4.2 D7-Branes on Background Five-Sphere

In gravity description, the adding D7-branes wrap on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ without back-reaction. The 7-7 strings are added into $\mathrm{AdS}_5 \times \mathrm{S}^5$ space; however, the 7-7 strings decouple to other closed strings. Low energy fluctuation of 7-7 strings in transverse direction on the space can be represented as perturbative fluctuations of wrapping D7-branes. There are two degrees of freedom of D7-branes fluctuations which are fluctuated along x^8 and x^9, and so called scalar fluctuations. Furthermore, the scalar fluctuations describe mass spectrum of scalar mesons:

$$\text{Mass spectrum} = \text{Mass gap} \times \text{Fluctuation modes of D7-branes} \quad . \tag{3.22}$$

As D7-branes wrapping on background, since the first five directions of D7-brane world-volume occupy all directions on the AdS_5, and the remain directions are compacted as S^3. On the $\mathrm{AdS}_5 \times \mathrm{S}^5$, the S^3 part of D7-brane world-volume appears as subspace of background S^5 (see Figure 3.3). The x^8 and x^9 are transformed into two angular coordinates defined by θ and ϕ. Thus we use that angular coordinates to determine positions of D7-branes on the S^5. Perturbative fluctuations of θ and ϕ are approximately the same as fluctuation along x^8 and x^9. We may perform fluctuations along x^8 and x^9 instead of θ and ϕ because calculations using terms of Cartesian coordinates are easier. Here, metric of D7-brane world-volume is

$$\mathrm{d}S^2_{\mathrm{D7}} = \frac{\rho^2 + L^2}{R^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{R^2\rho^2}{(\rho^2+L^2)^2}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_3^2\right) \quad . \tag{3.23}$$

ρ and L are radial directions of D7-brane world-volume and separation of two stacks, respectively. Furthermore $r^2 = \rho^2 + L^2$ and $\rho = r\cos\theta$, where r is radial direction of the $\mathrm{AdS}_5 \times \mathrm{S}^5$. Line element on S^3 is determined by S^3 with radius

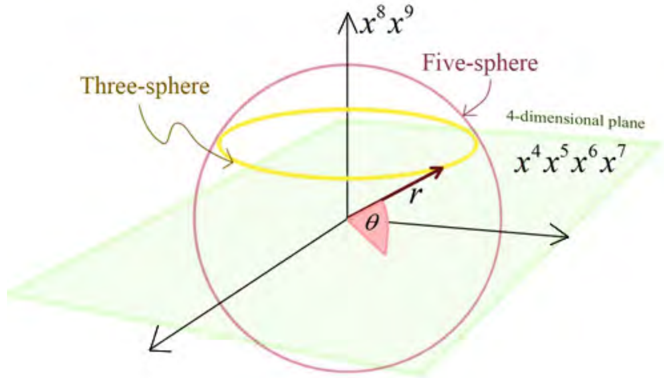

Figure 3.3: This figure shows the five-dimensional sphere illustrated by red sphere. The yellow circle represents the three-dimensional sphere. We can see clearly that $r\cos\theta$ determines size of three-sphere and $r\sin\theta$ determines positions of three-sphere along x^8 and x^9 directions.

$\rho = r\cos\theta$, simultaneously with D7-brane world-volume radial direction. For the Coulomb branch, radial direction ρ does not fill full range of background radius defection but ends at the place where the radius of S^3 shrink to zero, $\rho = 0$ as $\theta = \pi/2$. For the Higgs branch, because $L = 0$ as well as $r = \rho$, radial defection of the D7-brane world-volume fills full range of background radial direction.

In gravity description, dynamics of D7-branes are described by D7-brane action, S_{D7}. Generally, the full action of wrapping D7-branes consists of DBI action, Wess-Zumino terms and contribution form fields fluctuations. DBI action, as shown in Chapter 2, describes S^3 wrapping on S^5 subspace. Wess-Zumino action contains possible interaction between form-fields. Since four-form field $A_{(4)}$, Kalb-Ramond field $B_{(2)}$ and two-form field strength all present in D7-brane world-volume. The interaction terms are written in terms of wedge products of those form-fields. And the last term explains scalar and vector fluctuations. Hence the full D7-branes action is

$$S_{\mathrm{D7}} = S_{\mathrm{DBI}} + S_{\text{Wess-Zumino}} + S_{\text{fluctuation}} \quad ,$$

where

$$S_{\mathrm{DBI}} = -T_{\mathrm{D7}} N_f \int \mathrm{d}^8\sigma \sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \quad . \tag{3.24}$$

Here G_{ab} is the induced metric corresponding to metric Eq. (3.23), B_{ab} and F_{ab} denote components of Kalb-Ramond field $B_{(2)}$ and two-form field strength, respectively.

Chapter IV

D7-branes calculations

4.1 Background Space-time

4.1.1 Background AdS_5-Schwarzchild$\times\mathrm{S}^5$

We focus on system with finite temperature. So space-time background is promoted, from original $\mathrm{AdS}_5\times\mathrm{S}^5$ to be AdS_5-Schwarzchild$\times\mathrm{S}^5$. This background is near-horizon limit of non-extremal D3-brane solution. We use the non-extremal solution in the sense of near-extremal limit, even though D3-branes satisfy the BPS condition $G_{10}M^2 = Q^2$. In this subsection, we show how we achieve the AdS_5-Schwarzchild$\times\mathrm{S}^5$ and how we obtain black hole temperature $\mathcal{T}$.

At the beginning, we recall the non-extremal D3-brane solution as written in Eq. (3.10):

$$\begin{aligned} \mathrm{d}S^2 = &\left(1+\frac{R^4}{u^4}\right)^{-\frac{1}{2}}\left(-\left(1-\frac{b^4}{u^4}\right)\mathrm{d}t^2+\mathrm{d}\vec{x}^2\right) \\ &+\left(1+\frac{R^4}{u^2}\right)^{\frac{1}{2}}\left(\left(1-\frac{b^4}{u^4}\right)^{-1}\mathrm{d}u^2+u^2\mathrm{d}\Omega_5^2\right) \quad . \end{aligned} \tag{4.1}$$

Term $1 + R^4/u^4$ is harmonic function; however, we denote radial direction by u instead of r. R and b are identified with AdS radius and black hole radius, respectively. The $\mathrm{d}\Omega_5^2$ is line-element of unit five-sphere S^5:

$$\mathrm{d}\Omega_5^2 = \mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\Omega_3^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{4.2}$$

$$\text{and}\quad \mathrm{d}\Omega_3^2 = \mathrm{d}\Psi^2 + \cos^2\Psi\mathrm{d}\beta^2 + \sin^2\Psi\mathrm{d}\gamma^2 \quad . \tag{4.3}$$

S^5 contains three-sphere (S^3) subspace where its line-element is denoted by $\mathrm{d}\Omega_3^2$. This choice of $\mathrm{d}\Omega_5^2$ corresponds to D7-brane embedding, as we discussed in subsection 3.4.2. Angular coordinates on S^5 are $\theta, \phi, \Psi, \beta$ and ψ where θ and ϕ determine positions of subspace S^3 on S^5 (see Appendix B). We consider the near-horizon limit of the metric Eq. (4.1). By performing the approximation as illustrated in subsection 3.2.2, we obtain

$$\mathrm{d}s^2 = \frac{u^2}{R^2}\left(-\left(1-\frac{b^4}{u^4}\right)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{R^2}{u^2}\left(\left(1-\frac{b^4}{u^4}\right)^{-1}\mathrm{d}u^2 + u^2\mathrm{d}\Omega_5^2\right). \quad (4.4)$$

Factor $(1 - b^4/u^4)^{-1}$ appearing at $\mathrm{d}u^2$ makes us inconvenient when we analyze shapes of D7-branes on five-sphere. That factor will be cancelled out by introducing coordinate transformation $u \to r$. The result is metric of AdS_5-Schwarzchild$\times\mathrm{S}^5$ written by Poincare$'$ coordinates with radial direction r. In the Poincare$'$ coordinates, terms that contain black hole radius b emerging in factors of $\mathrm{d}t^2$ and $\mathrm{d}\vec{x}^2$ and the $(1 - b^4/u^4)^{-1}$ factor vanishes from radial direction r.

For D7-branes embedding on background AdS_5-Schwarzchild$\times\mathrm{S}^5$, fluctuations of D7-branes can be explained by quantum field theory on the black hole background. We can write the partition function for the field theory, and consider that field theory in language of statistical theory. The partition function can be written by performing Wick rotation $t \to t_{\mathrm{E}} = it$, where t is time direction of metric and t_{E} is Euclidean time. Then the Euclidean time is compact into a circle namely closed Euclidean time loop. Fields in this theory are also periodic functions of t_{E}. The black hole causes the conical deficit in closed Euclidean time loop. In order to remove the conical singularity, it is necessary to introduce a multiplicative factor to t_{E} to redefine period of t_{E}. And that factor determines the black hole temperature (see also Ref. [15]).

We calculate the black hole temperature by a method which is easier than the calculations performed in Ref. [15]. Before introducing a coordinate transformation, the metric Eq. (4.4) is easier to find the temperature. At this point, we read off the time-time component metric tensor g_{00} from metric Eq. (4.4). Then,

$$\begin{aligned} g_{00} &= \frac{u^2}{R^2}\left(1-\frac{b^4}{R^4}\right) \quad , \\ \frac{\mathrm{d}g_{00}}{\mathrm{d}u} &= 4\frac{b^4}{u^3R^2} + \frac{2u}{R^2}\left(1-\frac{b^4}{u^4}\right) \quad , \\ \left.\frac{\mathrm{d}g_{00}}{\mathrm{d}u}\right|_{u=b} &= \frac{4b}{R^2} \quad . \end{aligned}$$

Black hole temperature is defined by

$$\mathcal{T} = \frac{1}{4\pi}\left.\frac{\mathrm{d}g_{00}}{\mathrm{d}u}\right|_{u=b} \quad ,$$

so we obtain

$$\mathcal{T} = \frac{b}{\pi R^2} \quad . \quad (4.5)$$

We see that the temperature is determined by radius of black hole horizon b.

4.1.2 Coordinates Transformation

In this subsection, we derive metric of AdS_5-Schwarzchild$\times \mathrm{S}^5$ written in Poincare$'$ coordinates. Let's introduce coordinate transformation

$$r^2 = \frac{1}{2}\left(u^2 + \sqrt{u^4 - b^4}\right) \quad .$$

We rewrite $1 - b^4/u^4$ and u^2/R^2 factors to be new terms that depends on r. We define $\tilde{f}$ and f by

$$\begin{aligned}\tilde{f} &= 4r^4 + b^4 = 2u^4 + 2u^2\sqrt{u^4 - b^4} \\ f &= 4r^4 - b^4 = 2u^4 + 2u^2\sqrt{u^4 - b^4} - 2b^4 \quad .\end{aligned}$$

Then, we perform the following calculation

$$\begin{aligned}\frac{f}{\tilde{f}} &= \frac{u^4 - u^2\sqrt{u^4 - b^4} - b^4}{u^4 - u^2\sqrt{u^4 - b^4}} = 1 - \frac{b^4}{u^4 - u^2\sqrt{u^4 - b^4}} \quad , \\ \left(\frac{f}{\tilde{f}}\right)^2 &= 1 - \frac{2b^4}{u^4 - u^2\sqrt{u^4 - b^4}} + \frac{b^8}{\left(u^4 - u^2\sqrt{u^4 - b^4}\right)^2} \quad .\end{aligned} \tag{4.6}$$

On the other hand,

$$\frac{\tilde{f}}{r^2} = \frac{4\left(u^4 - u^2\sqrt{u^4 - b^4}\right)}{u^2 - u\sqrt{u^4 - b^4}} = 4u^2 \quad ,$$

thus we obtain

$$\frac{u^2}{R^2} = \frac{\tilde{f}}{4R^2r^2} \quad . \tag{4.7}$$

The component g_{00} is rewritten in terms of $\tilde{f}$, f and r by doing the following process

$$\begin{aligned}\frac{\tilde{f}}{r^2}\left(\frac{f}{\tilde{f}}\right)^2 &= 4u^2\left(1 - \frac{2b^4}{u^4 - u^2\sqrt{u^4 - b^4}} + \frac{b^8}{\left(u^4 - u^2\sqrt{u^4 - b^4}\right)^2}\right) \quad , \\ &= 4u^2\left(1 - \frac{2b^4u^4 - 2b^4u^2\sqrt{u^4 - b^4} - b^8}{\left(u^4 - u^2\sqrt{u^4 - b^4}\right)^2}\right) \quad , \\ &= 4u^2\left(1 - \frac{b^4}{u^4}\frac{2u^4 - 2u^2\sqrt{u^4 - b^4} - b^4}{\left(u^2 - \sqrt{u^4 - b^4}\right)^2}\right) \quad , \\ \frac{\tilde{f}}{r^2}\left(\frac{f}{\tilde{f}}\right)^2 &= 4u^2\left(1 - \frac{b^4}{u^4}\right) \quad .\end{aligned}$$

We obtain

$$\frac{u^2}{R^2}\left(1-\frac{b^4}{R^4}\right)=\frac{\tilde{f}}{4R^2r^2}\left(\frac{f}{\tilde{f}}\right)^2 \quad . \tag{4.8}$$

The left-handed side of Eq.(4.8) is $(-g_{00})$ of metric Eq. (4.4) and the right-handed side is for the Poincare$'$ coordinates. Another calculations are

$$\begin{aligned} r^2 &= \frac{1}{2}\left(u^2+\sqrt{u^4-b^4}\right) \quad , \\ 2r\mathrm{d}r &= u\mathrm{d}u+\frac{u^3\mathrm{d}u}{\sqrt{u^4-b^4}} \quad . \end{aligned}$$

Consequently, we get

$$4r^2\mathrm{d}r^2=\frac{u^2\left(u^2+\sqrt{u^4-b^4}\right)^2}{u^4-b^4}\mathrm{d}u^2.$$

Next, we calculate the following process

$$\frac{1}{r^2}\mathrm{d}r^2=\frac{u^2\left(u^2+\sqrt{u^4-b^4}\right)^2}{4r^4(u^4-b^4)}\mathrm{d}u^2=\frac{u^2\left(u^2+\sqrt{u^4-b^4}\right)^2}{\left(u^2+\sqrt{u^4-b^4}\right)^2(u^4-b^4)}\mathrm{d}u^2,$$

precisely

$$\frac{R^2}{r^2}\mathrm{d}r^2=\frac{R^2}{u^2}\left(1-\frac{b^4}{u^4}\right)^{-1}\mathrm{d}u^2 \quad . \tag{4.9}$$

By substituting the transformations in Eq. (4.8) and (4.9) into metric Eq. (4.4), the AdS$_5$-Schwarzchild$\times$S^5 is written as

$$\mathrm{d}S^2=\frac{1}{4r^2R^2}\left(-\frac{f^2}{\tilde{f}}\mathrm{d}t^2+\tilde{f}\mathrm{d}\vec{x}^2\right)+\frac{R^2}{r^2}\mathrm{d}r^2+R^2\mathrm{d}\Omega_5^2 \quad , \tag{4.10}$$

where $f=4r^4-b^4$ and $\tilde{f}=4r^4+b^4$ as we defined above.

Note that when we set $b=0$ that is $f=\tilde{f}=4r^4$, the metric Eq. (4.10) is simplified to be the metric as shown in (3.12) which describes the AdS$_5\times$S^5. The calculation for D7-branes action with induced metric of AdS$_5$-Schwarzchild$\times$S^5 is more benefit than AdS$_5$-$\times$S^5. After we obtain the action for finite temperature case, it is easy to fix $b=0$ as well as $f=\tilde{f}=4r^4$. If we do, we obtain action for zero temperature case which corresponds to calculation from induced metric of AdS$_5$-$\times$S^5. This simplification is true at action level but not at equation of motion level.

4.2 D7-branes Action

At this time, components of AdS$_5$-Schwarzchild$\times$S^5 induced metric will be computed. Again, D7-branes action is

$$S_{D7} = -T_{D7}N_f \int \mathrm{d}^8\sigma \sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \quad . \tag{4.11}$$

In this thesis, B_{ab} is turned off because we are not interested in string with charge for simplicity. F_{ab} denotes $U(1)_b$ field strength tensor. The induced metric of general space-time metric is defined by

$$G_{ab} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\nu}{\partial \sigma^b} \quad .$$

Here $g_{\mu\nu}$ are components of metric tensor of background space-time which is AdS$_5$-Schwarzchild$\times$S^5. The x^μ is a set of space-time coordinates with index $\mu = 0, 1, 2, ..., 9$. For AdS$_5$-Schwarzchild$\times$S^5, the coordinates are

$$x^\mu = (t, x, y, z, r, R\Psi, R\beta, R\gamma, R\theta, R\phi) \quad ,$$

where coordinates belonging to S^5 are written in angular coordinates. For world-volume coordinates (σ^a), they are fixed by static gauge which makes world-volume coordinates become

$$\sigma^a = (t, x, y, z, r, R\Psi, R\beta, R\gamma) \quad .$$

This coordinates choice corresponds to D7-brane occupation as showed in Table 3.2. For this choice, D7-brane world-volume is AdS$_5$-Schwarzchild$\times$S^3 space. The components of the induecd metric are

$$\begin{aligned}
G_{00} &= -\frac{f^2}{4r^2R^2\tilde{f}} & G_{11} &= \frac{\tilde{f}}{4r^2R^2} \\
G_{22} &= \frac{\tilde{f}}{4r^2R^2} & G_{33} &= \frac{\tilde{f}}{4r^2R^2} \\
G_{44} &= \frac{R^2}{r^2} + R^2\theta'^2 & G_{55} &= \cos^2\theta \\
G_{66} &= \cos^2\theta\cos^2\Psi & G_{77} &= \sin^2\theta\sin^2\Psi
\end{aligned} \quad . \tag{4.12}$$

We have introduced $\theta \equiv \theta(r)$ so the G_{44} has $R^2\theta'^2$ in an addition. The " $'$ " means derivative with respect to r. We introduce t and x components of gauge potential as a function of r. These components are $A_0(r)$ and $A_1 \equiv -Et + B(r)$, where $B(r)$ is a function of r. Other components of the gauge potential are fixed to be zero. The A_1 gives rise to constant electric field E along x-direction. With this gauge potential, the non-zero components of F_{ab} are

$$F_{10} = E \quad , \quad F_{40} = A_0' \quad \text{and} \quad F_{14} = -B' \quad .$$

We may define

$$M_{ab} \equiv G_{ab} + 2\pi\alpha' F_{ab} \quad .$$

The diagonal elements of matrix M_{ab} are

$$M_{ab} = G_{ab} \quad \text{only if} \quad a = b \quad . \tag{4.13}$$

And non-zero off-diagonal elements of the matrix are

$$M_{01} = -M_{10} = -2\pi\alpha' E \quad , \quad M_{04} = -M_{40} = -2\pi\alpha' A_0' \tag{4.14}$$

$$\text{and} \quad M_{14} = -M_{41} = -2\pi\alpha' B' \quad . \tag{4.15}$$

In order to compute matrix determinant for above $[M_{ab}]$, let's consider

$$[M_{ab}] = \begin{pmatrix} M_{00} & M_{01} & 0 & 0 & M_{04} & 0 & 0 & 0 \\ M_{10} & M_{11} & 0 & 0 & M_{14} & 0 & 0 & 0 \\ 0 & 0 & M_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{33} & 0 & 0 & 0 & 0 \\ M_{40} & M_{41} & 0 & 0 & M_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{77} \end{pmatrix} \quad .$$

Its determinant ($M = \det[M_{ab}]$) is

$$\begin{aligned} M &= (M_{00}M_{11} - M_{01}M)M_{22}M_{33}M_{44}M_{55}M_{66}M_{77} \\ &\quad -(M_{00}M_{14} - M_{04}M_{10})M_{41}M_{22}M_{33}M_{55}M_{66}M_{77} \\ &\quad +(M_{01}M_{14} - M_{04}M_{11})M_{22}M_{33}M_{40}M_{55}M_{66}M_{77} \quad , \\ M &= M_{33}M_{55}M_{66}M_{77}\left((M_{00}M_{22} + M_{01}^2)\right. \\ &\quad \left. +M_{00}M_{22}M_{14}^2 + M_{11}M_{22}M_{04}^2\right) \quad . \end{aligned}$$

The anti-symmetric property of off-diagonal elements $M_{ab} = -M_{ba}$ have been used to rearrange the above equations. The matrix determinant can be more simplified by substituting $M_{11} = M_{22} = M_{33}$. Thus we obtain

$$M = M_{55}M_{66}M_{77}\left((M_{00}M_{11}^2 + M_{01}^2M_{11}^2)M_{44} + M_{00}M_{11}^2M_{14}^2 + M_{11}^2M_{04}^2\right) . \tag{4.16}$$

By substituting Eq. (4.13), (4.14) and (4.15) into Eq. (4.16), the determinant is

$$\begin{aligned} -M &= \frac{\cos^6\theta\cos^2\Psi\sin^2\Psi}{256r^{10}R^6}\left(\left(f^2\tilde{f}^2 - 16\,(2\pi\alpha' E)^2\, r^4R^2\tilde{f}^2\right)\left(1 + r^2\theta'^2\right)\right. \\ &\quad \left. +4r^4f^2\tilde{f}\,(2\pi\alpha' B')^2 - 4r^4\tilde{f}^3\,(2\pi\alpha' A_0')^2\right) \quad , \\ \sqrt{-M} &= \frac{\cos^3\theta\cos\Psi\sin\Psi}{16r^5R^3}\left(\left(f^2\tilde{f}^2 - 16r^4R^2\tilde{f}^2\,(2\pi\alpha' E)^2\right)\left(1 + r^2\theta'^2\right)\right. \\ &\quad \left. +4r^4f^2\tilde{f}\,(2\pi\alpha' B')^2 - 4r^4\tilde{f}^3\,(2\pi\alpha' A_0')^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{4.17}$$

D7-branes action is substituted by $\sqrt{-M}$ and world-volume measure

$$\mathrm{d}^8\sigma = R^3 \mathrm{d}t\mathrm{d}^3x\mathrm{d}r\mathrm{d}\Psi\mathrm{d}\beta\mathrm{d}\gamma \quad . \tag{4.18}$$

then the action becomes

$$\begin{aligned} S_{D7} = &-T_{D7}N_f \int \mathrm{d}t\mathrm{d}^3x\mathrm{d}r\mathrm{d}\Psi\mathrm{d}\beta\mathrm{d}\gamma \frac{\cos^3\theta\cos\Psi\sin\Psi}{16r^5}\left(\left(f^2\tilde{f}^2 - 16r^4R^2\tilde{f}^2\left(2\pi\alpha' E\right)^2\right)\right.\\ &\left.\times\left(1+r^2\theta'^2\right) + 4r^4f^2\tilde{f}\left(2\pi\alpha' B'\right)^2 - 4r^4\tilde{f}^3\left(2\pi\alpha' A_0'\right)^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{4.19}$$

In this step, we can integrate out some coordinates such that

$$\int \mathrm{d}^3x \equiv V \quad , \quad \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \mathrm{d}\beta\mathrm{d}\gamma = 2\pi^2 \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin\Psi\cos\Psi\mathrm{d}\Psi = \frac{1}{2} \quad .$$

Hence the action Eq. (4.11) is rewritten as

$$\begin{aligned} S_{D7} = &-2\pi^2T_{\mathrm{D7}}N_fV \int \mathrm{d}t\mathrm{d}r \frac{\cos^3\theta}{16r^5}\left(\left(f^2\tilde{f}^2 - 16r^4R^2\tilde{f}^2\left(2\pi\alpha' E\right)^2\right)\right.\\ &\left.\times\left(1+r^2\theta'^2\right) + 4r^4f^2\tilde{f}\left(2\pi\alpha' B'\right)^2 - 4r^4\tilde{f}^3\left(2\pi\alpha' A_0'\right)^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{4.20}$$

B' and A_0' become fields variable for the above action. Variables in the action may be redefined to new dimensionless quantities;

$$r = b\tilde{r} \quad , \quad B(r) = \frac{b}{2\pi\alpha'}\tilde{B}(\tilde{r}) \quad , \quad E = \frac{b^2}{2\pi\alpha' R^2}\tilde{E} \quad \text{and} \quad A_0(r) = \frac{b}{2\pi\alpha'}\tilde{A}_0(\tilde{r}) \quad .$$

Consequently, we can rescale the black hole radius equal to $1/\sqrt{2}$. Function $\theta(r)$ is redefined and terms f and $\tilde{f}$ are rescaled to be dimensionless terms g and $\tilde{g}$, respectively. Thus, we write

$$\theta(r) \equiv \theta(\tilde{r}) \quad , \quad \frac{f}{b^4} \equiv g = 4\tilde{r}^4 - 1 \quad \text{and} \quad \frac{\tilde{f}}{b^4} \equiv \tilde{g} = 4\tilde{r}^4 + 1 \quad .$$

The derivative of $B(r)$ and $A_0(r)$ with respect to r should be

$$B'(r) = \frac{\partial}{\partial r}B(r) = \frac{1}{2\pi\alpha'}\frac{\partial}{\partial\tilde{r}}\tilde{B}(\tilde{r}) = \frac{1}{2\pi\alpha'}\tilde{B}'(\tilde{r}) \quad , \tag{4.21}$$

$$A_0'(r) = \frac{1}{2\pi\alpha'}\frac{\partial}{\partial\tilde{r}}\tilde{A}_0(\tilde{r}) = \frac{1}{2\pi\alpha'}\tilde{A}_0'(\tilde{r}) \quad . \tag{4.22}$$

The " $'$ " also becomes derivative with respect to $\tilde{r}$. Action in form of new quantities is

$$\begin{aligned} S_{D7} = &-2\pi^2T_{D7}N_fVb^4 \int \mathrm{d}t\mathrm{d}\tilde{r} \frac{\cos^3\theta}{16\tilde{r}^5}\left(\tilde{g}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\right.\\ &\left.\times\left(1+\tilde{r}^2\theta'^2\right) + 4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2 - 4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{4.23}$$

It is natural to define Lagrangian in unit $(-2\pi^2 T_{D7} N_f V b^4)$ by

$$\mathcal{L} \equiv \frac{\cos^3\theta}{16\tilde{r}^5}\left(\tilde{g}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1+\tilde{r}^2\theta'^2\right) + 4\tilde{r}^4 g^2\tilde{g}\tilde{B}'^2 - 4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}} \quad . \tag{4.24}$$

This Lagrangian should give three equations of motion corresponding to particular variables: θ, B and A_0 (three degrees of freedom). We may calculate those equations of motion by using radial Euler-Lagrange equations.

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\frac{\partial\mathcal{L}}{\partial q_i'} = 0 \quad \text{with} \quad q_i = \begin{cases} \theta \\ B \\ A_0 \end{cases} .$$

But, looking at the $\mathcal{L}$ carefully, it contains two cyclic variables which are B and A_0. The cyclic coordinates contribute its constant momentum (radial convention momentum). Consider the radial convention equation of motion,

$$\frac{\partial\mathcal{L}}{\partial\tilde{A}_0'} \equiv \tilde{D} = \text{a constant of motion} \quad , \tag{4.25}$$

$$\text{and} \quad \frac{\partial\mathcal{L}}{\partial\tilde{B}'} \equiv \tilde{T} = \text{another constant of motion} \quad . \tag{4.26}$$

Straightforwardly, we can obtain the constants of motion

$$\tilde{D} = \frac{-\tilde{g}^3\tilde{A}_0'\cos^3\theta}{4\tilde{r}\left(\tilde{g}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1+\tilde{r}^2\theta'^2\right) + 4\tilde{r}^4 g^2\tilde{g}\tilde{B}'^2 - 4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}} \quad , \tag{4.27}$$

$$\tilde{T} = \frac{g^3\tilde{g}\tilde{B}'\cos^3\theta}{4\tilde{r}\left(\tilde{g}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1+\tilde{r}^2\theta'^2\right) + 4\tilde{r}^4 g^2\tilde{g}\tilde{B}'^2 - 4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}} \quad . \tag{4.28}$$

From the above equations, we can write a relation between two momentums as follows

$$-\frac{\tilde{D}}{\tilde{g}^2\tilde{A}_0'} = \frac{\tilde{T}}{g^2\tilde{B}'} \quad \Rightarrow \quad \tilde{A}_0' = -\frac{g^2\tilde{D}}{\tilde{g}^2\tilde{T}}\tilde{B}' \quad . \tag{4.29}$$

Straightforwardly solve for B' and A_0', we get

$$\tilde{A}_0'^2 = \frac{16\tilde{r}^2 g^2\tilde{D}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1+\tilde{r}^2\theta'^2\right)}{64\tilde{r}^6\left(g^2\tilde{g}\tilde{D}^2 - \tilde{g}^3\tilde{T}^2\right) + g^2\tilde{g}^4\cos^6\theta} \quad , \tag{4.30}$$

$$\tilde{B}'^2 = \frac{16\tilde{r}^2\tilde{g}^3\tilde{T}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1+\tilde{r}^2\theta'^2\right)}{64\tilde{r}^6\left(g^4\tilde{D}^2 - g^2\tilde{g}^2\tilde{T}^2\right) + g^4\tilde{g}^3\cos^6\theta} \quad . \tag{4.31}$$

Three degrees of freedom of Lagrangian can be reduced to only one by Routhian procedure:

$$\mathcal{L}\left(\theta;\theta',\tilde{B}',\tilde{A}_0';\tilde{r}\right) \quad \rightarrow \quad \mathcal{R}\left(\theta;\theta';\tilde{T},\tilde{D};\tilde{r}\right) \quad . \tag{4.32}$$

$\mathcal{R}\left(\theta;\theta';\tilde{T},\tilde{D};\tilde{r}\right)$ is called Routhian which is depending on variable $\theta(\tilde{r})$. Instead of $\mathcal{L}$, this is only one equation of motion associated to $\theta(\tilde{r})$. The equation of motion in convention of radial Euler-lagrange equation is

$$\frac{\partial\mathcal{R}}{\partial\theta}-\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\frac{\partial\mathcal{R}}{\partial\theta'}=0 \quad . \tag{4.33}$$

The precise transformation from $\mathcal{L}$ to $\mathcal{R}$ is the Legendre transform for two cyclic variables:

$$\mathcal{R}=\mathcal{L}-\tilde{T}\tilde{B}'-\tilde{D}\tilde{A}_0' \quad . \tag{4.34}$$

Form of $\mathcal{R}$ can be reached by following way

$$\begin{aligned}
\mathcal{R} &= \mathcal{L}+\frac{\cos^3\theta}{4\tilde{r}}\frac{\tilde{g}^3\tilde{A}_0'-g^3\tilde{g}\tilde{B}'}{\left(\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)+4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}}, \\
&= \frac{\cos^3\theta}{16\tilde{r}^5}\frac{\left(\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)+4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)}{\left(\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)+4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}} \\
&\quad +\frac{\cos^3\theta}{16\tilde{r}^5}\frac{4\tilde{r}^4\tilde{g}^3\tilde{A}_0'-4\tilde{r}^4g^3\tilde{g}\tilde{B}'}{\left(\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)+4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}}, \\
\mathcal{R} &= \frac{\cos^3\theta}{16\tilde{r}^5}\frac{\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)}{\left(\tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)+4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2\right)^{\frac{1}{2}}} \quad .
\end{aligned} \tag{4.35}$$

For simplicity, let us consider

$$\begin{aligned}
4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2 &= \frac{64\tilde{r}^6\tilde{g}^4\tilde{T}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)}{64\tilde{r}^6\left(g^2\tilde{D}^2-\tilde{g}^2\tilde{T}^2\right)+g^2\tilde{g}^3\cos^6\theta} \\
&\quad -\frac{64\tilde{r}^6g^2\tilde{g}^2\tilde{D}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)}{64\tilde{r}^6\left(g^2\tilde{D}^2-\tilde{g}^2\tilde{T}^2\right)+g^2\tilde{g}^3\cos^6\theta} \quad .
\end{aligned}$$

Then we write

$$4\tilde{r}^4g^2\tilde{g}\tilde{B}'^2-4\tilde{r}^4\tilde{g}^3\tilde{A}_0'^2=\frac{64\tilde{r}^6\left(\tilde{g}^4\tilde{T}^2-g^2\tilde{g}^2\tilde{D}^2\right)\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)}{64\tilde{r}^6\left(g^2\tilde{D}^2-\tilde{g}^2\tilde{T}^2\right)+g^2\tilde{g}^3\cos^6\theta} \quad .$$

By putting the above calculation into the denominator of $\mathcal{R}$, then the denominator are

$$\begin{aligned}
\mathrm{Deno}^2 &= \tilde{g}^2\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2) \\
&\quad +\frac{64\tilde{r}^6\left(\tilde{g}^4\tilde{T}^2-g^2\tilde{g}^2\tilde{D}^2\right)\left(g^2-16\tilde{r}^4\tilde{E}^2\right)(1+\tilde{r}^2\theta'^2)}{64\tilde{r}^6\left(g^2\tilde{D}^2-\tilde{g}^2\tilde{T}^2\right)+g^2\tilde{g}^3\cos^6\theta} \quad ,
\end{aligned}$$

so it can be written in form

$$\text{Deno}^2 = \left(1 - \frac{64\tilde{r}^6\left(\tilde{g}^2\tilde{T} - g^2\tilde{D}^2\right)}{64\tilde{r}^6\left(\tilde{g}^2\tilde{T} - g^2\tilde{D}^2\right) + g^2\tilde{g}^3\cos^6\theta}\right)$$
$$\times\tilde{g}^2\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1 + \tilde{r}^2\theta'^2\right) \quad .$$

Then, the denominator can be written as

$$\text{Deno}^2 = \frac{g^2\tilde{g}^5\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1 + \tilde{r}^2\theta'^2\right)\cos^6\theta}{64\tilde{r}^6\left(\tilde{g}^2\tilde{T} - g^2\tilde{D}^2\right) + g^2\tilde{g}^3\cos^6\theta} \quad ,$$

$$\text{Deno} = \frac{g\tilde{g}^2\cos^3\theta\sqrt{\tilde{g}\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(1 + \tilde{r}^2\theta'^2\right)}}{\sqrt{64\tilde{r}^6\left(\tilde{g}^2\tilde{T} - g^2\tilde{D}^2\right) + g^2\tilde{g}^3\cos^6\theta}} \quad .$$

Thus, the Routhian Eq. (4.35) with help of the Deno simplifies and takes form as

$$\mathcal{R} = \frac{\sqrt{1 + \tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\sqrt{\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(64\tilde{r}^6\left(g^2\tilde{D}^2 - \tilde{g}^2\tilde{T}^2\right) + g^2\tilde{g}^3\cos^6\theta\right)}. \tag{4.36}$$

Don't forget that this Routhian arisen form Lagrangian in unit $(-2\pi^2 T_{D7}N_f V b^4)$ so the Routhian have to be in unit $(-2\pi^2 T_{D7}N_f V b^4)$, like the Lagrangian. We may introduce a new action by using the Routhian. Thus

$$I_{D7} = -2\pi^2 T_{D7}N_f V b^4\int \mathrm{d}t\mathrm{d}\tilde{r}\frac{\sqrt{1 + \tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}$$
$$\times\sqrt{\left(g^2 - 16\tilde{r}^4\tilde{E}^2\right)\left(64\tilde{r}^6\left(g^2\tilde{D}^2 - \tilde{g}^2\tilde{T}^2\right) + g^2\tilde{g}^3\cos^6\theta\right)} \quad . \tag{4.37}$$

The combination of two terms under square root must be positive, each term must have the same sign and change their sign simultaneously at some radial value, called vanishing locus $\tilde{r}_*$. Thus the two terms have to be zero at $\tilde{r} = \tilde{r}_*$. The first term is

$$g_*^2 - 16\tilde{r}_*^4\tilde{E}^2 = 0 \quad \text{or} \quad g_* = 4\tilde{r}_*^2\tilde{E} \quad .$$

By using $g_* = 4\tilde{r}_*^4 - 1$, we get

$$4\tilde{r}_*^4 - 4\tilde{r}_*^2\tilde{E} - 1 = 0 \quad ,$$

so we obtain

$$\tilde{r}_*^2 = \frac{4\tilde{E} + \sqrt{\tilde{E}^2 + 1}}{2} \quad . \tag{4.38}$$

The second term evaluated at $\tilde{r} = \tilde{r}_*$ is

$$\begin{aligned} 64\tilde{r}_*^6 \left(g_*^2 \tilde{D}^2 - \tilde{g}_*^2 \tilde{T}^2 \right) + g_*^2 \tilde{g}_*^3 \cos^6 \theta_0 = 0 \quad , \\ 64\tilde{r}_*^4 \tilde{E}^2 \tilde{D}^2 - 4\tilde{g}_*^2 \tilde{T}^2 + 4\tilde{g}_*^3 \tilde{E}^2 \cos^6 \theta_0 = 0 \quad . \end{aligned}$$

This is a relation between two constants, $\tilde{T}$ and $\tilde{D}$, and electric field. Here, $\tilde{g}_* = 4\tilde{r}_*^4 + 1$. Then the relation becomes

$$\tilde{T} = \sqrt{\frac{64\tilde{r}_*^4 \tilde{D}^2 + \tilde{g}_*^3 \cos^6 \theta_0}{4\tilde{g}_*^2}} \tilde{E} \quad . \tag{4.39}$$

This relation is identified to Ohm's law since $\tilde{T}$ is dual to electric vacuum expectation value of current for gauge theory.

4.3 Equation of Motion

4.3.1 Exact Equation of Motion

In this section, we will calculate radial Euler-Lagrange equation for equation of motion. Routhian contains only one degree of freedom, so there exists only one equation of motion. The equation of motion associates with variable θ. Let's start with the Routhian,

$$\mathcal{R} = \frac{\sqrt{1 + \tilde{r}^2 \theta'^2}}{16\tilde{r}^5 g \sqrt{\tilde{g}}} \sqrt{\left(g^2 - 16\tilde{r}^4 \tilde{E}^2 \right) \left(-64\tilde{r}^6 \tilde{T}^2 \tilde{g}^2 + g^2 \tilde{g}^3 \cos^6 \theta \right)} \quad . \tag{4.40}$$

The constant of motion $\tilde{D}$ has been set to zero since we are not interested in the system with time-component gauge potential. We rewrite the Routhian as

$$\mathcal{R} = \frac{\sqrt{1 + \tilde{r}^2 \theta'^2}}{16\tilde{r}^5 g \sqrt{\tilde{g}}} \mathcal{A} \quad , \tag{4.41}$$

where the factor $\mathcal{A}$ is defined by

$$\mathcal{A} = \sqrt{\left(g^2 - 16\tilde{r}^4 \tilde{E}^2 \right) \left(-64\tilde{r}^6 \tilde{T}^2 \tilde{g}^2 + g^2 \tilde{g}^3 \cos^6 \theta \right)} \quad . \tag{4.42}$$

This definition of $\mathcal{A}$ makes our calculation more convenient. When $\mathcal{A}$ is operated by derivatives with respect to $\tilde{r}$ or θ, it will repeat itself to denominator of the result. After the Routhian is operated by derivatives, the calculation is easier if we isolate Routhian as a factor. Thus the repetition of $\mathcal{A}$ is necessary. Let's start with

$$\frac{\partial \mathcal{R}}{\partial \theta} = \frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\frac{\partial \mathcal{A}}{\partial \theta} \quad ,$$

$$= \frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\frac{1}{2\mathcal{A}}\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\left(-6g^2\tilde{g}^3\cos^5\theta\sin\theta\right) \quad ,$$

$$= \left(\frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\mathcal{A}\right)\frac{\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\left(-3g^2\tilde{g}^3\cos^5\theta\sin\theta\right)}{\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\left(-64\tilde{r}^6\tilde{T}^2\tilde{g}^2+g^2\tilde{g}^3\cos^6\theta\right)} \quad ,$$

$$\frac{\partial \mathcal{R}}{\partial \theta} = \mathcal{R}\left(\frac{-3g^2\tilde{g}cos^5\theta\sin\theta}{-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta}\right) \quad . \tag{4.43}$$

The derivative of Routhian with respect to θ has itself as a factor. Then, we perform derivative with respect to θ' to Routhian:

$$\frac{\partial \mathcal{R}}{\partial \theta'} = \frac{\mathrm{d}}{\mathrm{d}\theta'}\left(\frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\right)\mathcal{A} \quad ,$$

$$= \frac{\theta'}{16\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}}\mathcal{A} \quad ,$$

$$\frac{\partial \mathcal{R}}{\partial \theta'} = \mathcal{B}\mathcal{A} \quad . \tag{4.44}$$

Factor $\mathcal{B}$ is defined by

$$\mathcal{B} = \frac{\theta'}{16\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}} \quad . \tag{4.45}$$

$\mathcal{B}\mathcal{A}$ as shown in Eq. (4.44) may be called momentum associated to variable θ. By taking a radial derivative to the momentum, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\frac{\partial \mathcal{R}}{\partial \theta'} = \mathcal{A}\mathcal{B}' + \mathcal{A}'\mathcal{B} \quad . \tag{4.46}$$

Calculation for each terms will be shown in detail. It is useful to note that

$$\frac{\mathrm{d}g}{\mathrm{d}\tilde{r}} = \frac{\mathrm{d}\tilde{g}}{\mathrm{d}\tilde{r}} = 16\tilde{r}^3 \quad . \tag{4.47}$$

We calculate the derivative of $\mathcal{B}$ with respect to $\tilde{r}$,

$$\mathcal{A}\mathcal{B}' = \mathcal{A}\left(\frac{\theta''16\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2} - 16\theta'\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}\right)}{\left(\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}\right)^2}\right) \quad ,$$

and then we write

$$\mathcal{A}\mathcal{B}' = \left(\frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\mathcal{A}\right)\left(\frac{\theta''\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)} - \frac{\theta'\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}\right)}{\tilde{r}g\sqrt{\tilde{g}}\left(1+\tilde{r}^2\theta'^2\right)^{3/2}}\right) \quad .$$

We isolate the $\mathcal{R}$ as factor hence the calculation is more simplicity. Then,

$$\begin{aligned}
\mathcal{A}\mathcal{B}' &= \mathcal{R}\left(\frac{\theta''\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)} - \frac{\theta'}{\tilde{r}g\sqrt{\tilde{g}}\,(1+\tilde{r}^2\theta'^2)^{3/2}}\left(\tilde{r}^3 g\sqrt{\tilde{g}}\frac{\tilde{r}\theta'\theta''+\tilde{r}\theta'^2}{\sqrt{1+\tilde{r}^2\theta'^2}}\right.\right.\\
&\quad +\tilde{r}^3 g\sqrt{1+\tilde{r}^2\theta'^2}\frac{8\tilde{r}^3}{\sqrt{\tilde{g}}} + 16\tilde{r}^6\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}\\
&\quad \left.\left. +3\tilde{r}^2 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}\right)\right) \quad ,\\
&= \mathcal{R}\left(\frac{\theta''\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)} - \frac{\theta'\tilde{r}^3\left(\tilde{r}\theta'\theta''+\theta'^2\right)}{(1+\tilde{r}^2\theta'^2)^2} - \frac{8\theta'\tilde{r}^5}{\tilde{g}\,(1+\tilde{r}^2\theta'^2)}\right.\\
&\quad \left. -\frac{16\theta'\tilde{r}^5}{g\,(1+\tilde{r}^2\theta'^2)} - \frac{3\theta'\tilde{r}}{(1+\tilde{r}^2\theta'^2)}\right) \quad ,\\
\mathcal{A}\mathcal{B}' &= \mathcal{R}\left(\frac{\theta''\tilde{r}^2-\theta'^3\tilde{r}^3}{(1+\tilde{r}^2\theta'^2)^2} - \frac{\theta'\tilde{r}}{(1+\tilde{r}^2\theta'^2)}\left(3+\frac{16\tilde{r}^4}{g}+\frac{8\tilde{r}^4}{\tilde{g}}\right)\right) \quad . \qquad (4.48)
\end{aligned}$$

We now calculate the derivative of $\mathcal{A}$ with respect to $\tilde{r}$,

$$\begin{aligned}
\mathcal{A}'\mathcal{B} = &\frac{\theta'}{16\tilde{r}^3 g\sqrt{\tilde{g}}\sqrt{1+\tilde{r}^2\theta'^2}}\frac{1}{2\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\right.\\
&\left.\times\left(-64\tilde{r}^6\tilde{T}^2\tilde{g}^2+g^2\tilde{g}^3\cos^6\theta\right)\right) \quad ,
\end{aligned}$$

and then we write

$$\begin{aligned}
\mathcal{A}'\mathcal{B} = &\left(\frac{\sqrt{1+\tilde{r}^2\theta'^2}}{16\tilde{r}^5 g\sqrt{\tilde{g}}}\mathcal{A}\right)\frac{\theta'\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)}\frac{1}{2\mathcal{A}^2}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\right.\\
&\left.\times\left(-64\tilde{r}^6\tilde{T}^2\tilde{g}^2+g^2\tilde{g}^3\cos^6\theta\right)\right)
\end{aligned}$$

Similarly, we isolate the factor $\mathcal{R}$, then

$$\begin{aligned}
\mathcal{A}'\mathcal{B} = \mathcal{R}&\frac{\theta'\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)}\frac{1}{\mathcal{A}^2}\left(\left(g^2-16\tilde{r}^4\tilde{E}^2\right)\left(-64\times 16\tilde{r}^9\tilde{T}^2\tilde{g}\right.\right.\\
&-64\times 3\tilde{r}^5\tilde{T}^2\tilde{g}^2+24\tilde{r}^3 g^2\tilde{g}^2\cos^6\theta+16\tilde{r}^3 g\tilde{g}^3\cos^6\theta\\
&\left.-3g^2\tilde{g}^3\theta'\cos^5\theta\sin\theta\right)+\left(16\tilde{r}^3 g-32\tilde{r}^3\tilde{E}^2\right)\\
&\left.\times\left(-64\tilde{r}^6\tilde{T}^2\tilde{g}^2+g^2\tilde{g}^3\cos^6\theta\right)\right) \quad . \qquad (4.49)
\end{aligned}$$

The $(-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta)$ will be isolated as a factor. Finally,

$$\begin{aligned}
\mathcal{A}'\mathcal{B} = \mathcal{R}&\frac{\theta'\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)}\frac{1}{\left(-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta\right)}\left(\frac{-64\tilde{r}^5\tilde{T}^2\left(16\tilde{r}^4+3\tilde{g}\right)}{\tilde{g}}\right.\\
&+8\tilde{r}^3\left(3g^2+2g\tilde{g}\right)\cos^6\theta-3g^2\tilde{g}\theta'\cos^5\theta\sin\theta\\
&\left.+16\tilde{r}^3\left(g-2\tilde{E}^2\right)\left(\frac{-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta}{g^2-16\tilde{r}^4\tilde{E}^2}\right)\right) \quad . \qquad (4.50)
\end{aligned}$$

Now, we have all ingredients for equation of motion. Substituting them into

$$\frac{\partial \mathcal{R}}{\partial \theta} = \mathcal{A}\mathcal{B}' + \mathcal{A}'\mathcal{B} \quad .$$

Factor $\mathcal{R}$ in each ingredients is cancelled out. The equation of motion should be written in the from

$$\begin{aligned}
-3g^2\tilde{g}cos^5\theta \sin\theta = & \frac{\theta'\tilde{r}^2}{(1+\tilde{r}^2\theta'^2)} \left(\frac{-64\tilde{r}^5\tilde{T}^2\left(16\tilde{r}^4+3\tilde{g}\right)}{\tilde{g}} \right. \\
& +8\tilde{r}^3\left(3g^2+2g\tilde{g}\right)\cos^6\theta - 3g^2\tilde{g}\theta'\cos^5\theta\sin\theta \\
& \left. +16\tilde{r}^3\left(g-2\tilde{E}^2\right)\left(\frac{-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta}{g^2-16\tilde{r}^4\tilde{E}^2}\right)\right) \\
& +\left(-64\tilde{r}^6\tilde{T}^2+g^2\tilde{g}\cos^6\theta\right)\left(\frac{\theta''\tilde{r}^2-\theta'^3\tilde{r}^3}{(1+\tilde{r}^2\theta'^2)^2} \right. \\
& \left.\left. -\frac{\theta'\tilde{r}}{(1+\tilde{r}^2\theta'^2)}\left(3+\frac{16\tilde{r}^4}{g}+\frac{8\tilde{r}^4}{\tilde{g}}\right)\right)\right) \quad .
\end{aligned} \tag{4.51}$$

This is a non-linear second order differential equation. In order to solve this, it needs at least two conditions. Furthermore the equation has two singularities at black hole radius $\tilde{r} = 1/\sqrt{2}$ and vanishing locus $\tilde{r} = \tilde{r}_*$. The singularity at vanishing locus is removable by a condition which is later used as an initial condition for the equation of motion.

4.3.2 Asymptotic Equation of Motion

We can analyze the asymptotic behavior of the exact equation of motion Eq. (4.51). It is convenient to start at the Euler-Lagrange equation which consists of Eq. (4.43), (4.48) and (4.51). The asymptotic equation of motion is obtained by taking approximation such $\tilde{r}$ is much greater than $\tilde{r}_*, \tilde{T}$ and $\tilde{E}$, and $\theta \to 0$. Consequently, we perform the approximation with the use of $g \approx \tilde{g} \approx 4\tilde{r}^4$ and $\mathcal{O}(\tilde{r}^a) + \mathcal{O}(\tilde{r}^b) \approx \mathcal{O}(\tilde{r}^a)$ for $a - b > 2$. Euler-Lagrange consists of three terms: $\partial\mathcal{R}/\partial\theta$, $\mathcal{A}\mathcal{B}'$ and $\mathcal{A}'\mathcal{B}$. We perform the approximation term by term. For the first part, we start from Eq. (4.43) then we obtain

$$\frac{\partial \mathcal{R}}{\partial \theta} \approx -3\theta\mathcal{R} \quad . \tag{4.52}$$

Starting form the Eq. (4.48), the approximation of the second term is as follow

$$\mathcal{A}\mathcal{B}' \approx \mathcal{R}\left(\frac{\tilde{r}^2\theta''-9\tilde{r}\theta'}{(1+\tilde{r}\theta'^2)} - \frac{\tilde{r}^4\theta'^2\theta''+\theta r^3\theta'^3}{(1+\tilde{r}\theta'^2)^2}\right) \quad ,$$

we write

$$\mathcal{AB}' \approx \mathcal{R}\frac{(\tilde{r}^2\theta'' - 10\theta r^3\theta'^3 - 9\theta r\theta')}{(1+\tilde{r}\theta'^2)^2} \quad . \tag{4.53}$$

And from the Eq. (4.51), the approximation of the third term is given by

$$\mathcal{A}'\mathcal{B} \approx \mathcal{R}\frac{\tilde{r}^2\theta'}{1+\tilde{r}^2\theta'^2}\left(\frac{14}{\tilde{r}} + 3\theta\theta'\right) \quad ,$$

so we write

$$\mathcal{A}'\mathcal{B} \approx \mathcal{R}\frac{(14\tilde{r}\theta' + 14\tilde{r}^3\theta'^3 - 3\theta r^2\theta\theta'^2 - 3\theta r^4\theta\theta'^4)}{(1+\tilde{r}^2\theta'^2)^2} \quad . \tag{4.54}$$

It is not necessary to perform the asymptotic approximation for factor $\mathcal{R}$ because this factor is cancelled in Euler-Lagrange equation. We substitute the Eq. (4.52), (4.53) and (4.54) into Euler-Lagrange equation:

$$\frac{\partial\mathcal{R}}{\partial\theta} = \mathcal{A}'\mathcal{B} + \mathcal{AB}' \quad .$$

Then, we obtain the asymptotic equation of motion such that

$$\theta''\tilde{r}^2 + 5\theta'\tilde{r} + 3\theta + 3\theta\theta'^2\tilde{r}^2 + 4\theta'^3\tilde{r}^3 = 0 \quad . \tag{4.55}$$

The equation of motion can be more simplified. Let us introduce the asymptotic solution as

$$\theta(\tilde{r}) \sim \frac{\alpha_n}{\tilde{r}^n} \quad , \quad \theta'(\tilde{r}) \sim -\frac{n\alpha_n}{\tilde{r}^{n+1}} \quad \text{and} \quad \theta''(\tilde{r}) \sim \frac{n(m+1)\alpha_n}{\tilde{r}^{n+2}} \quad .$$

By substituting them into Eq. (4.55), we see that

$$\frac{n(n+1)\alpha}{\tilde{r}^n} - \frac{5n\alpha}{\tilde{r}^n} + \frac{3\alpha}{\tilde{r}^n} + \frac{3n^2\alpha}{\tilde{r}^{3n}} + \frac{4n^3(n+1)^3\alpha^3}{\tilde{r}^{3n}} = 0 \quad .$$

We can ignore the terms which depend on $\tilde{r}^{-3}$. Thus the equation of motion becomes

$$\theta''\tilde{r}^2 + 5\theta'\tilde{r} + 3\theta = 0 \quad . \tag{4.56}$$

It is easy to solve for asymptotic solution, which is

$$\theta(\tilde{r}) = \frac{\tilde{m}}{\tilde{r}} + \frac{\tilde{c}}{\tilde{r}^3} \quad . \tag{4.57}$$

$\tilde{m}$ and $\tilde{c}$ are constants [1, 9, 16], (actually α_1 and α_3 respectively). The $\tilde{m}$ and $\tilde{c}$ are fixed to be quark mass and chiral condensate respectively.

4.4 Initial Conditions

The solutions $\theta(\tilde{r})$ is solved by numerical method. There exists many possible solutions depending on choices of initial conditions. However, they can be grouped into two types. One type is collection of solutions which are passing through vanishing locus and falling into black hole horizon. This type is called black hole embedding. Initial conditions of the black hole embedding are given by equation of motion itself. And the other type is called Minkowski embedding. Collection of solutions for this type does not pass through vanishing locus. Initial conditions for the Minkowski embedding are analyzed by cancellation of a singularity in D7-brane world-volume.

4.4.1 Initial Conditions of Black Hole Embedding

All solutions of this type are initiated at the vanishing locus $\tilde{r}_*$. Thus we use exact equation of motion Eq. (4.51) evaluated at $\tilde{r}_*$ as initial conditions. Let us consider the equation of motion at vanishing locus. We define Δ as,

$$\Delta \equiv \frac{-64\tilde{r}^6\tilde{T}^2 + g^2\tilde{g}\cos^6\theta}{g^2 - 16\tilde{r}^4\tilde{E}^2} \quad .$$

This term behaves like $0/0$ as $\tilde{r} \to \tilde{r}_*$, making the equation of motion become singularity at vanishing locus. In order to remove this singular behavior, it is necessary to find limit of Δ as $\tilde{r} \to \tilde{r}_*$. Thus

$$\Delta_* = \lim_{\tilde{r}\to\tilde{r}_*} \frac{-64\tilde{r}^6\tilde{T}^2 + g^2\tilde{g}\cos^6\theta}{g^2 - 16\tilde{r}^4\tilde{E}^2} \quad . \tag{4.58}$$

From Eq. (4.40), each term under square root must have the same sign and change their sign at the vanishing locus. So we have

$$\tilde{T}^2 = \frac{g_*^2\tilde{g}_*\cos^6\theta_0}{64\tilde{r}_*^6} \quad \text{and} \quad \tilde{E}^2 = \frac{g_*^2}{16\tilde{r}_*^6} \quad .$$

We substitute these $\tilde{T}^2$ and $\tilde{E}^2$ into Eq. (4.58). Then, Δ_* is resolved by using L'Hôpital's rule as follow

$$\begin{aligned}
\Delta_* &= \lim_{\tilde{r}\to\tilde{r}_*} \frac{\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\frac{\tilde{r}^6}{\tilde{r}_*^6} g_*^2\tilde{g}_*\cos^6\theta_0 + g^2\tilde{g}\cos^6\theta\right)}{\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(g^2 - \frac{\tilde{r}^4}{\tilde{r}_*^4}g_*^2\right)} \quad , \\
&= \lim_{\tilde{r}\to\tilde{r}_*} \frac{-6\frac{\tilde{r}^5}{\tilde{r}_*^6}\cos^6\theta_0 + 32\tilde{r}^3 g\tilde{g}\cos^6\theta + 16\tilde{r}^3 g^2\cos^6\theta - 6g^2\tilde{g}\theta'\cos^5\theta\sin\theta}{32\tilde{r}^3 g - 4g_*^2\frac{\tilde{r}^4}{\tilde{r}_*^3}} ,
\end{aligned}$$

$$\Delta_* = \frac{(-3g_*\tilde{g}_* + 16\tilde{r}_*^4\tilde{g}_* + 8\tilde{r}_*^4 g_*)\cos^6\theta_0 - 3\tilde{r}_* g_*\tilde{g}_*\theta_0'\cos^5\theta_0\sin\theta_0}{16\tilde{r}_*^4 - 2g_*} \quad . \tag{4.59}$$

Notice that the $\theta_0 \equiv \theta(\tilde{r} = \tilde{r}_*)$ and $\theta'_0 \equiv \theta'|_{\tilde{r}=\tilde{r}_*}$. As a result, the Δ_* turns to be finite. So singularity at vanishing locus is removed by $\Delta(\tilde{r} = \tilde{r}_*) \equiv \Delta_*$. Other terms in equation of motion are simply substituted by $\tilde{r} = \tilde{r}_*$. It is easy to compute

$$\begin{aligned} 16\tilde{r}^3 \left(g - 2\tilde{E}^2\right) \Delta\Big|_{\tilde{r}=\tilde{r}_*} &= \frac{g_*}{\tilde{r}_*} \cos^6\theta_0 \left(-3g_*\tilde{g}_* + 16\tilde{r}_*^4\tilde{g}_* + 8\tilde{r}_*^4 g_*\right) \\ &\quad -3g_*^2\tilde{g}_*\theta'_0 \cos^5\theta_0 \sin\theta_0 \quad . \end{aligned} \tag{4.60}$$

Another calculation is

$$\begin{aligned} &\left(\frac{-64\tilde{r}^5\tilde{T}^2\left(16\tilde{r}^4 + 3\tilde{g}\right)}{\tilde{g}} + 8\tilde{r}^3\left(3g^2 + 2g\tilde{g}\right)\cos^6\theta\right)\Bigg|_{\tilde{r}=\tilde{r}_*} \\ &\qquad = \frac{g_*}{\tilde{r}_*}\cos^6\theta_0\left(8\tilde{r}_*^4 g_* - 3g_*\tilde{g}_* + 16\tilde{r}_*^4\tilde{g}_*\right) \quad . \end{aligned} \tag{4.61}$$

With Eq. (4.60) and (4.61), the equation of motion Eq. (4.51) at vanishing locus are

$$\begin{aligned} -3g_*^2\tilde{g}_* cos^5\theta_0 \sin\theta_0 &= \frac{2\theta'_0\tilde{r}_*}{(1 + \tilde{r}_*^2\theta_0'^2)}\Big(g_*\cos^6\theta_0\left(8\tilde{r}_*{}^4 g_* - 3g_*\tilde{g}_*\right. \\ &\quad \left. +16\tilde{r}_*^4\tilde{g}_*\right) - 3g_*^2\tilde{g}_*\theta'_0\cos^5\theta_0\sin\theta_0\Big) \quad , \\ -3g_*\tilde{g}_*\tan\theta_0 &= 2\tilde{r}_*\theta'_0\left(8\tilde{r}_*^4 g_* - 3g_*\tilde{g}_* + 16\tilde{r}_*^4\tilde{g}_*\right) - 3\tilde{r}_*^2 g_*\tilde{g}_*\theta_0'^2\tan\theta_0 \quad . \end{aligned}$$

Then this may be rearranged to

$$\begin{aligned} 3\tilde{r}_*^2 g_*\tilde{g}_*\theta_0'^2\tan\theta_0 - 2\tilde{r}_*\theta'_0\left(8\tilde{r}_*^4 g_* - 3g_*\tilde{g}_* + 16\tilde{r}_*^4\tilde{g}_*\right) - 3g_*\tilde{g}_*\tan\theta_0 &= 0 \\ \text{then, } 3\tilde{r}_*^2\left(16\tilde{r}_*^8 - 1\right)\theta_0'^2\tan\theta_0 - 2\tilde{r}_*\theta'_0\left(48\tilde{r}_* + 8\tilde{r}_*^4 - 3\right) - 3\left(16\tilde{r}_*^8 - 1\right)\tan\theta_0 &= 0 \quad . \end{aligned}$$

It is just a binomial of the θ'_0 which we can solve for. Thus we obtain

$$\theta'_0 = \frac{(48\tilde{r}_* + 8\tilde{r}_*^4 - 3) - \sqrt{(48\tilde{r}_* + 8\tilde{r}_*^4 - 3)^2 + 9\,(16\tilde{r}_*^8 - 1)^2\tan^2\theta_0}}{3\tilde{r}_*\,(16\tilde{r}_*^8 - 1)\tan\theta_0} \quad . \tag{4.62}$$

By fixing the initial angle θ_0, the θ'_0 is determined. At the vanishing locus a solution is initiated by a pair of θ_0 and θ'_0. In order to obtain solution with range $\tilde{r} \in (1/\sqrt{2}, \infty)$, it is necessary to divide the solution into two parts according to ranges $\tilde{r} \in (\tilde{r}_*, \infty)$ and $\tilde{r} \in (1/\sqrt{2}, \tilde{r}_*)$. For the first part, solution starts at near vanishing locus (at $\tilde{r} = \tilde{r}_* + \epsilon$ as $\epsilon \to 0$) with a value of θ_0 and moves toward boundary $\tilde{r} \to \infty$. For the second part, solution starts at near vanishing locus (at $\tilde{r} = \tilde{r}_* - \epsilon$ as $\epsilon \to 0$) with the same θ_0 and moves backward to the black hole horizon $\tilde{r} \to 1/\sqrt{2}$.

4.4.2 Initial Condition of Minkowski Embedding

As we stated above, there exists a singular behavior on D7-brane world-volume, which is conical singularity. The singularity appears when coordinate θ reaches to $\pi/2$. The physical radial distance at that point seems to be blown up. The initial conditions for this type is given from a condition that guarantees the smoothness of D7-brane world-volume at $\theta \to \pi/2$ [17]. At the beginning, we will show the appearance of the singularity. Let us consider metric of the D7-brane world-volume, which is given by

$$\mathrm{d}S_{\mathrm{D7}}^2 = \frac{f^2}{4r^2R^2}\left(-\frac{f^2}{\tilde{f}}\mathrm{d}t^2 + \tilde{f}\mathrm{d}\vec{x}^2\right) + \left(\frac{R^2}{r^2} + R^2\theta'^2\right)\mathrm{d}r^2 + R^2\cos^2\theta\mathrm{d}\Omega_3^2. \quad (4.63)$$

This metric is derived from the induced metric where is metric of D7-brane world-volume. Remember that r is the radial coordinate of background AdS$_5$-Schwarzchild$\times$S^5. Therefore we have to define world-volume radial coordinate $\alpha = \cos\theta$ with used $\theta = \theta(r)$. Hence, $\mathrm{d}\alpha = -\sin\theta\theta'\mathrm{d}r$ and then

$$\mathrm{d}r^2 = \frac{\mathrm{d}\alpha^2}{\sin^2\theta\theta'^2} \quad .$$

Under the transformation, the physical distance on the S^3 becomes

$$\mathrm{d}l^2 = R^2\left(\frac{1}{r^2} + \theta'^2\right)\frac{\mathrm{d}\alpha^2}{\sin^2\theta\theta'^2} + R^2\alpha^2\mathrm{d}\Omega_3^2 \quad .$$

In order to remove the conical singularity, we impose a condition such that

$$\left.\frac{\left(\frac{1}{r^2} + \theta'^2\right)}{\sin^2\theta\theta'^2}\right|_{\theta=\frac{\pi}{2}} = 1 \quad ,$$
$$\frac{1}{r^2\theta'^2} = 0 \quad \text{only at } \theta = \pi/2.$$

Thus, the condition that removes the conical singularity at $\theta = \pi/2$ is $\theta'(r)|_{\theta=\pi/2} = \pm\infty$ but we may choose

$$\left.\theta'(r)\right|_{\theta=\frac{\pi}{2}} = -\infty \quad . \quad (4.64)$$

We can use the initial condition regardless of $r, \theta(r)$ or $\tilde{r}, \theta(\tilde{r})$. Solutions for Minkowski embedding initiate at $\theta = \pi/2$ with fixed $\theta' = -\infty$ at possible radial values, $\tilde{r} > \tilde{r}_*$. Each solution starts at different values of $\tilde{r}$, and then they smoothly move toward boundary.

4.5 Solutions and Discussion

4.5.1 Shapes of D7-brane

Solutions of the equation of motion Eq. (4.51) are $\theta\tilde{r}$. They are solved with choices of initial conditions by numerical method. Here, we are interested in positions of wrapping D7-branes at various $\tilde{r}$, so we plot the solved solutions by $\tilde{r}\sin\theta$ versus $\tilde{r}\cos\theta$ as shown in Figure 4.1. In the plot, we can see the separation along x^8, x^9 plane of the D7-branes on S^5 where r is varied, so called shapes of D7-branes. The collection of D7-branes shapes are also shown. The Minkowski

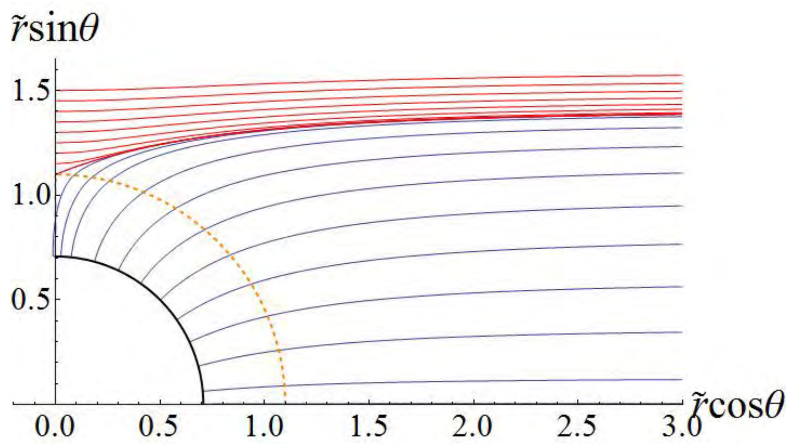

Figure 4.1: This figure shows collection of D7-branes shapes, with fixing $\tilde{r}_b = 1/\sqrt{2}$ and $\tilde{E} = 0.9$. The red lines represent the Minkowski embedding and the blue lines represent the black hole embedding. The black and orange dash circles show black hole horizon and vanishing locus, respectively.

embedding are represented by red lines. In this case, the D7-branes do not fill all ranges of radial direction, but end at $\tilde{r}|_{\theta=\pi/2}$. Assuming states of the D7-branes fluctuations propagate along radial direction, when the states reach the rim where D7-branes end. The states hit the rim; they reflect, and then propagate to boundary ($\tilde{r} \to \infty$). For this situation, modes of those states are quantized topologically by the rim and the boundary. Thus, modes of the states are discrete and stable. The discrete modes lead to discrete mesons mass spectrum which imply the stability of mesons, (see Ref. [18]). On the other hand, the blue lines are black hole embedding. For this case, the D7-branes are passing the vanishing locus (orange dash circle), and touch the black hole horizon (black circle). States on the D7-branes propagate and fall into black hole. They will not reflect [19]. Thus the D7-branes fluctuate by quasi-normal modes which implies instability and corresponds to unstable or melting mesons. However, there exists an edge where blue lines stretch very close to red lines (see Figure 4.1). The phase transition of mesons system is suggested by changing between two types of embedding through the edge.

Let us see shapes of the D7-branes with various strength of applied electric field. Figure 4.2 shows plotting of $\tilde{r}\sin\theta$ versus $\tilde{r}\cos\theta$ for $\tilde{E} = 0.05, 0.5, 0.9$ and 2.0, respectively. The value of $\tilde{r}\sin\theta$ represents quarks mass. Mesons binding

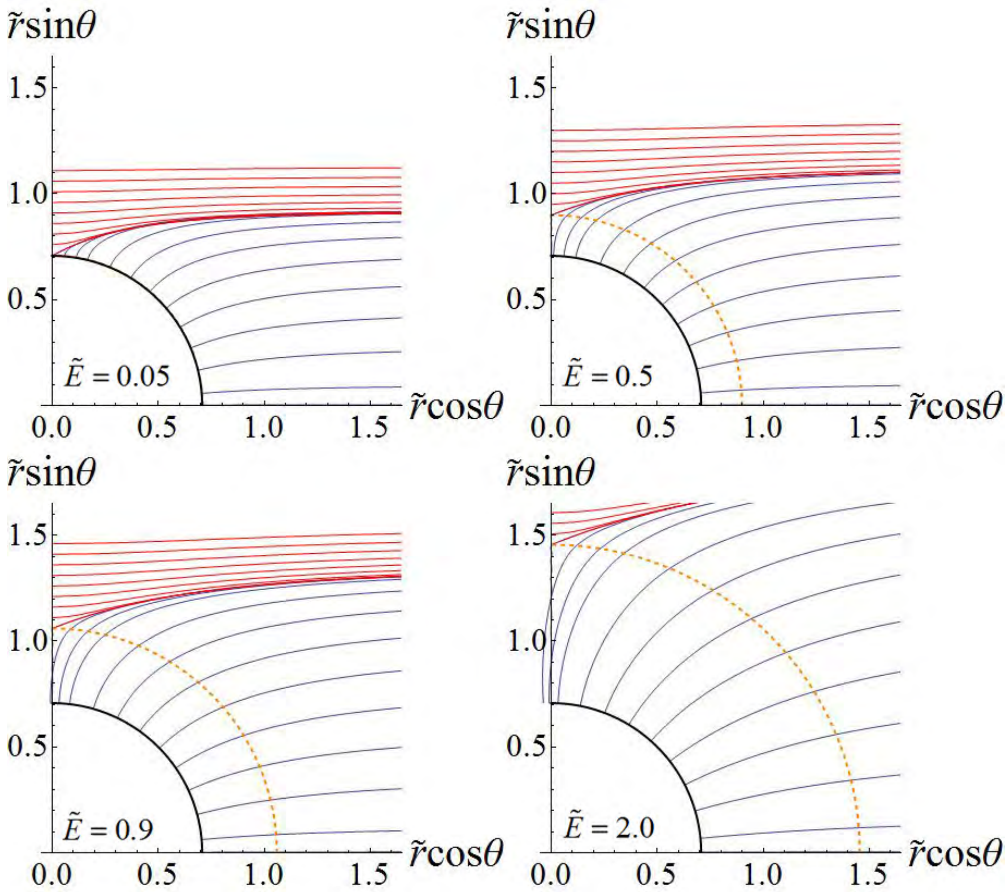

Figure 4.2: Shapes of wrapping D7-branes with different fixing of $\tilde{E}$. The radius orange dash circles are bigger as values of $\tilde{E}$ increase

energy is proportional to quarks mass. As the strength of electric field increases, size of vanishing locus also increases as a result of Eq. (4.38). Furthermore, the edge of phase transition is upper in $\tilde{r}\sin\theta$ direction as the $\tilde{E}$ increases. Thus the stronger electric field effects the instability of the stronger mesons binding energy. In other words, the strong electric field can overcome and melt mesons that bound pairs of quark and anti-quark by its strong binding energy.

4.5.2 Quarks mass Versus Chiral Condensate

Shapes of D7-branes as illustrated in the Figure (4.1) and (4.2) satisfy the exact equation of motion Eq. (4.51). However we also have the asymptotic equation of motion Eq. (4.56) with its asymptotic solution Eq. (4.57). As shown in the previous section, the $\tilde{m}$ and $\tilde{c}$ are actually constants of integration. They are fixed by collection of numerical solutions. The assumption of fixing $\tilde{m}$ and $\tilde{c}$ is that at far region ($\tilde{r} \gg \tilde{r}_*$) a numerical solution should be the same with an asymptotic solution with corrected $\tilde{m}$ and $\tilde{c}$. With this assumption, we can numerically extract for $\tilde{m}$ and $\tilde{c}$. A given initial condition, θ_0 for the black hole embedding or $\tilde{r}|_{\theta=\pi/2}$ for the Minkowski embedding, gives a pair of $\tilde{m}$ and $\tilde{c}$. In order to see relation between $\tilde{m}$ and $\tilde{c}$, we must extract them from all possible initial conditions. In

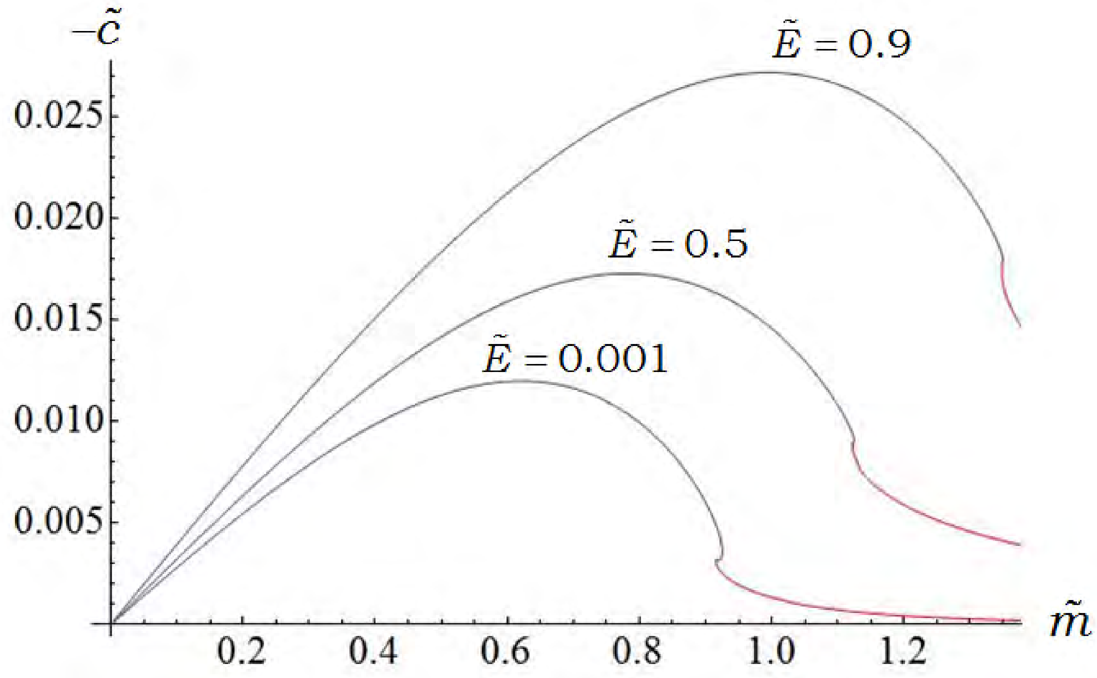

Figure 4.3: This figure shows relation of $\tilde{m}$ versus $\tilde{c}$ with various $\tilde{E}$. Again, the red lines represent the Minkowski embedding and the blue lines represent the black hole embedding. The top is for case of $\tilde{E} = 0.9$, the lowers are for $\tilde{E} = 0.5$ and 0.001 respectively.

Figure 4.3, we plot relation of $\tilde{m}$ and $\tilde{c}$ so called quark mass $\tilde{m}$ as a function of chiral condensate $\tilde{c}$. For a value of $\tilde{E}$, each point on whatever red or blue line is extracted from unique initial conditions. The red line represents the Minkowski embedding while the blue line represents the black hole embedding. The red and the blue lines join together and become multi-value relation, at some regions on $\tilde{m}, \tilde{c}$ plane. The multi-value behaviors also imply the first-order phase transition

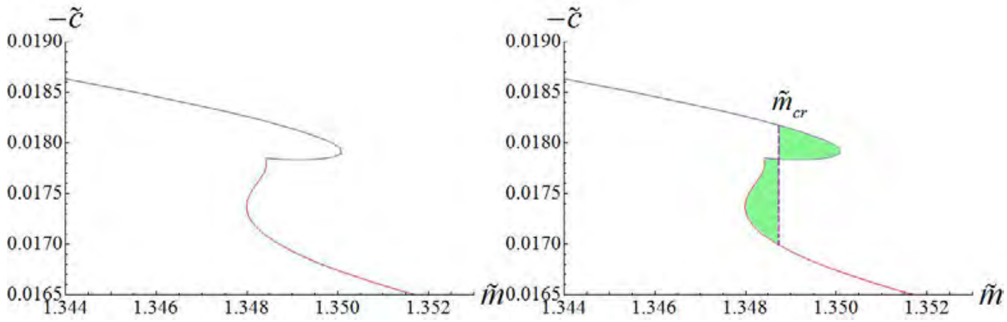

Figure 4.4: The figure shows multi-value behavior for case of $\tilde{E} = 0.9$. The right figure shows an example the critical mass value.

of mesons system. The accurate mass which mesons phase jumps from the red line to the blue line can be found by analyzing the free energy, which we do not perform in this thesis. That mass value is unique so called the critical mass. See Figure 4.4 for an example. The critical mass $\tilde{m}$ is positioned where the purple dash line intersects the $\tilde{m}$ axis. The line arise equality of two green area. This property follows the Maxwell equal area law of phase transition. For a situation that the mesons system reaches the critical mass, the system will jump upward or downward between red line and blue line through the purple dash line where the phase transition occurs.

4.6 Conductivity and Drag force

Here we consider conductivity for medium of our interest. We turn on the A_0 in D7-brane action that correspond with introducing electric charge carriers with finite density into system of mesons. After lighter mesons are melted due to high temperature and strong electric field, quarks and anti-quarks are free from mesons binding energy. The liberated quarks and anti-quark are also electric charge carriers. Heavier mesons are still stable mesons and we suppose that they are stationary in whole medium. The charge carriers respond to applied electric field and give rise to electric current on the medium. There exists a relation which is analogous to the Ohm's law, we obtain the conductivity.

4.6.1 Ohm's Law

At this time, we will calculate for Ohm's law for our medium and then read off the conductivity of this system (see also [9]). We recall

$$\tilde{T} = \sqrt{\frac{64\tilde{r}_*^4\tilde{D}^2 + \tilde{g}_*^3\cos^6\theta_0}{4\tilde{g}_*^2}}\tilde{E} \quad . \tag{4.65}$$

This equation is analogous to Ohm's law, since we can relate $\tilde{T}$ to the expectation value of electric current density, denoted by $\langle J^x\rangle$. Furthermore, the $\tilde{D}$ should be related to the expectation value of charge density which is $\langle J^t\rangle$. The said relations are given by the AdS/CFT dictionary, which is written as

$$Z_{\text{gauge}} = Z_{\text{SUGRA}}\Big|_{\phi_0} \quad , \tag{4.66}$$

in the other word

$$\langle \mathrm{e}^{\int \mathrm{d}^4x\phi_0(x)\mathcal{O}(x)}\rangle = \mathrm{e}^{S_{\text{D7}}}\Big|_{\phi=\phi_0} \quad . \tag{4.67}$$

The dictionary Eq. (4.67) is illustrated in general field $\phi(x)$ and operator $\mathcal{O}(x)$. We can find the expectation value of the operator by taking functional derivative with respect to $\phi(y)$. Following

$$\frac{\delta Z_{\text{gauge}}}{\delta\phi_0(y)} = Z_{\text{gauge}}\left\langle \int \mathrm{d}^4x\mathcal{O}(x)\delta(x-y)\right\rangle = Z_{\text{gauge}}\langle\mathcal{O}(y)\rangle \tag{4.68}$$

$$\frac{\delta Z_{\text{SUGRA}}}{\delta\phi_0(y)} = Z_{\text{SUGRA}}\lim_{r\to\infty}\frac{r^\Delta}{\sqrt{-g_{\text{AdS}}}}\frac{\delta S_{\text{D7}}}{\delta\phi_0} \quad , \tag{4.69}$$

we obtain

$$\langle\mathcal{O}(x)\rangle = \lim_{r\to\infty}\frac{r^\Delta}{\sqrt{-g}}\frac{\delta S_{\text{D7}}}{\delta\phi(x,r)} \quad . \tag{4.70}$$

The factor r^Δ is multiplicative counter term which is put by hand into above relation to cancel out the divergence. We substitute full space-time metric by $g \sim g_{\text{4D}}$, g_{4D} is determinant of metric tensor for only (3+1)-dimensional subspace of AdS$_5$-Schwarzchild. Because the AdS$_5$-Schwarzchild$\times$S^5 space becomes (3+1)-dimensional space-time at its boundary, the g_{4D} is

$$g_{\text{4D}} = -\frac{1}{256r^8R^8}f^2\tilde{f}^2 = -\frac{1}{256r^8R^8}(16r^8 - b^8)^2 \quad ,$$

so

$$\sqrt{-g_{\text{4D}}} = \frac{1}{16R^4}\left(r^4 - \frac{b^8}{r^4}\right) \quad .$$

Moreover we also put some counter terms and redefine $S_{\rm D7} = S_{\rm DBI} + S_{\rm counter}$ since we need the $S_{\rm counter}$ to cancel out some divergence terms when $r \to \infty$.

At this time, we will find a relation between $\tilde{D}$ and $\langle J^t \rangle$. We substitute the dictionary by $\langle \mathcal{O}(x) \rangle = \langle J^t(x) \rangle$ and $\phi(x, r) = A_t(x, r)$. Thus the dictionary now becomes

$$\langle J^t \rangle = \lim_{r\to\infty} \frac{r^4}{\sqrt{-g_{\rm 4D}}} \frac{\delta S_{\rm D7}}{\delta A_t(y, r)} \quad . \tag{4.71}$$

We substitute $r^4/\sqrt{-g_{\rm 4D}} \sim 1$ as $r = \Lambda \to \infty$, then

$$\begin{aligned}
\langle J^t \rangle &= \frac{\delta S_{\rm D7}}{\delta A_t(y, \Lambda)} = -N \int {\rm d}^4x{\rm d}r \frac{\partial \mathcal{L}_{\rm D7}}{\partial A'_t} \frac{\delta \partial_r A(y, r)}{\delta A_t(y, \Lambda)} \delta(x - y) \quad , \\
&= -ND \frac{\delta}{\delta A(x, \Lambda)} (A(x, \Lambda) - A(x, b)) \quad , \\
\langle J^t \rangle &= -2\pi\alpha' N b^3 \tilde{D} \quad \text{with} \quad N = 2\pi^2 N_f T_{\rm D7} \quad .
\end{aligned} \tag{4.72}$$

Next, we we will find a relation between $\tilde{T}$ and $\langle J^x \rangle$. We use the dictionary with $\langle \mathcal{O}(x) \rangle = \langle J^x(x) \rangle$ and $\phi(x, r) = A_x(x, r)$. The calculation is more subtle since $S_{\rm counter}$ depends on A'_x. The steps of calculations are

$$\begin{aligned}
\langle J^x \rangle &= \frac{\delta S_{\rm D7}}{\delta A_x(y, \Lambda)} \quad , \\
&= -N \int {\rm d}^4x{\rm d}r \left(\frac{\partial L_{\rm D7}}{\partial A'_x} \frac{\delta \partial_r A(y, r)}{\delta A_x(y, \Lambda)} \delta(x - y) + \frac{\partial L_{\rm D7}}{\partial \dot{A}_x} \frac{\delta \dot{A}(x, r)}{\delta A_x(y, \Lambda)} \right. \\
&\quad \left. + \frac{\partial L_c}{\partial \dot{A}_x} \frac{\delta \dot{A}(x, r)}{\delta A_x(y, \Lambda)} \right) \quad , \\
\langle J^t \rangle &= -2\pi\alpha' N b^3 \tilde{T} \quad .
\end{aligned} \tag{4.73}$$

L_c is counter Lagrangian depending on $\ln \dot{A}_x$. The terms $\int {\rm d}t \dot{A}_x$ are vanishes at $A_x(t_i) = A_x(t_f) = 0$. Thus we have relations as Eq. (4.72) and (4.73) by using AdS/CFT dictionary, they are substituted into Eq. (4.65).The result is precisely a relation between electric current density and electric field. The relation written in dimensionful parameters is

$$\langle J^x \rangle = \left(\frac{16 r_*^4 \langle J^t \rangle^2}{\tilde{f}_*^2} + \frac{(2\pi\alpha')^2 N^2 \tilde{f}_* \cos^6\theta_0}{4 r_*^2} \right)^{\frac{1}{2}} (2\pi\alpha') R^2 E \quad . \tag{4.74}$$

This relation is analogous to Ohm's law, $\langle J^x \rangle = \sigma_c E$. Thus we obtain conductivity,

$$\sigma_c = 2\pi\alpha' R^2 \left(\frac{16 r_*^4 \langle J^t \rangle^2}{\tilde{f}_*^2} + \frac{(2\pi\alpha')^2 N^2 \tilde{f}_* \cos^6\theta_0}{4 r_*^2} \right)^{\frac{1}{2}} \quad . \tag{4.75}$$

Moreover,

$$\frac{\tilde{f}_*}{4r_*^2} = \sqrt{(2\pi\alpha')^2 R^4 E^2 + b^4} \quad . \tag{4.76}$$

We may rewrite equation Eq. (4.77) to

$$\sigma_c = \sqrt{\sigma_{c,1}^2 + \sigma_{c,2}^2} \quad , \tag{4.77}$$

where

$$\begin{aligned}
\sigma_{c,1} &= \frac{2\pi\alpha' R^2 \langle J^t \rangle}{\sqrt{(2\pi\alpha')^2 R^4 E^2 + b^4}} \\
\sigma_{c,2} &= (2\pi\alpha')^2 R^2 N \left((2\pi\alpha')^2 R^4 E^2 + b^4\right)^{\frac{1}{4}} \cos^3\theta_0 \quad .
\end{aligned}$$

Thus the Eq. (4.74) is non-linear Ohm's law because the conductivity depends on E, charge density, $\cos\theta_0$ and temperature (as the conductivity is determined by b). The medium is better responsed to electric field when system has more charge density. When mesons are melted, moving quarks and anti-quarks give rise to current density with responsibility determined by $\cos\theta_0$.

4.6.2 Drag Force

The charge carriers are accelerated by electric field, and then they collide with stable mesons (if heavier mesons are included the system) and other charge carriers randomly. The collision makes the charge carriers decelerate against the electric force. Thus, the collision between charge carriers and other in the medium is effectively drag force. This phenomena is similar with Drude model of metal, where charge carriers experience the drag force due to their collision. Effectively, whole charge carriers experience total force as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \mu p + F \quad . \tag{4.78}$$

F determines force applied by electric field. By rescaling the Eq. (4.78) by electric charge, we substitute $F = E$. p is mean momentum of each charge carriers and μ is friction coefficient. We consider the metal at equilibrium $\mathrm{d}p/\mathrm{d}t = 0$ and employ the relativistic correlation of mass and momentum;

$$p = \frac{mv}{\sqrt{1-v^2}} \quad ,$$

where m is rest mass of all charge carriers. By substituting this momentum into Eq. (4.78) with $\mathrm{d}p/\mathrm{d}t = 0$, we reach

$$E = \frac{\mu m v}{\sqrt{1-v^2}} \quad . \tag{4.79}$$

The standard relation between charge density and electric current is $\langle J^x \rangle = \langle J^t \rangle v$ and we equate the $\langle J^x \rangle$ by using Ohm's law. Thus we have $\sigma_c E = \langle J^t \rangle v$ and then we obtain $v = \sigma_{c,1} E / \langle J^t \rangle$.

Let us look at the simplest case that whole mesons in the medium are stable, $\cos\theta_0 \sim 0$. We calculate the relativistic factor with relation as shown in Eq. (4.77), we obtain

$$\frac{v}{\sqrt{1-v^2}} = \frac{(2\pi\alpha')R^2 E}{b^2} \quad . \tag{4.80}$$

We substitute $v\sqrt{1-v^2}$ as obtained in Eq. (4.80) into the Eq. (4.79) with $R^4 = 4\pi\alpha'^2\lambda$ and $\mathcal{T} = b/4\pi R^2$. Then we obtain

$$\mu m = \sqrt{4\pi\lambda}\mathcal{T}^2 \quad . \tag{4.81}$$

This is the drag force which depends on 't Hooft coupling and temperature.

The other case of interest, the medium contains only mesons as a result of turning off A^t, corresponding to $\langle J^t \rangle = 0$. In conduction phase, charge carriers are free quarks and anti-quarks. We can denote the melting mesons density by ρ, so the charge density is 2ρ. We have $v = \sigma_{c,2} E / 2\rho$. Here we may perform the approximation that quarks and anti-quarks are massive so they move much slower than speed of light, $v \ll c$ ($\bar{h} = c = 1$). Under this approximation,

$$E \approx \mu m v \quad \text{or} \quad \mu m = \frac{2\rho}{\sigma_{c,2}} \quad .$$

Thus, we achieve drag force of system being in melting phase as

$$\mu m = \frac{64\rho\lambda}{(4\pi R^2)^4 N_f T_{\mathrm{D7}} \left(\frac{E^2}{256\pi^3\lambda} + \mathcal{T}^4\right)^{\frac{1}{4}} \cos^3\theta_0} \quad . \tag{4.82}$$

This drag force depends on the order of $\mathcal{T}^{-1}$ and $E^{-1/2}$.

Chapter V

Conclusion

In this thesis, we study mesons at finite temperature with external electric field by using the AdS/CFT correspondence. The correspondence originally suggests an equivalent of two different theories; the $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM theory) on flat (3+1)-dimensional space-time and the type IIB superstring theory on AdS$^5\times$S^5 space-time. Precisely, the $\mathcal{N} = 4$ SYM theory takes place on boundary of the AdS$^5\times$S^5 space-time. The correspondence is a strong/weak duality under a specific limit; large N_c limit and fixing the 't Hooft coupling at large value.

In this thesis, we discuss some information which is necessary to match up the two theories. The first information is D3-branes system that can be viewed by two descriptions. The $\mathcal{N} = 4$ SYM theory is constructed from a stack of N_c D3-branes with massless modes of 3-3 strings fluctuations. This is gauge description of D3-branes. On the other description, stack of D3-branes give rise to its gravity description governed by the type IIB supergravity which defined on AdS$^5\times$S^5 space-time. The second one we show that the coupling parameters of two descriptions are linked together by considering the loop correction of the two theories. The third one is matching between superconformal group and isometry group of the AdS$^5\times$S^5 space-time. And finally, we consider conformal transformation of AdS$^5\times$S^5 space-time and we find that the space-time is effective (3+1)-dimensional Minkowski space at its boundary. However, this thesis is lack of matching of entropy between two theories and discussion of the AdS/CFT dictionary is unclear.

Another stack of N_f D7-branes is added into the original D3-branes system under probe limit so called D3/D7 model. In gauge description, the adding corresponds to adding the quarks and anti-quarks and break half of supersymmetry. Quarks and anti-quarks gain their masses from separation between stacks of D3 and D7 branes. This is similar to spontaneous system breaking as appearing in the standard model. Now gauge description of D3/D7 model arises the $\mathcal{N} = 2$

$SU(N_c)$ SYM theory with global $SU(N_f)$ flavor where the quarks and anti-quarks are included into the theory. The quarks and anti-quarks are arisen from 3-7 and 7-3 string fluctuations. 7-7 strings fluctuations can be treated as probe mesons. For gravity description of D3/7D model, as probe limit, the D7-branes do not cause back-reaction to $\mathrm{AdS}^5\times\mathrm{S}^5$ space-time. The 7-7 string also exist in this description and their fluctuation can be thought as D7-branes fluctuations. Classical fluctuations of the D7-branes gives rise to mesons mass spectrum.

We modify the D3/D7 by exchange $\mathrm{AdS}^5\times\mathrm{S}^5$ for AdS_5-Schwarzchild$\times\mathrm{S}^5$. This corresponds to heating up the system by finite temperature. We also turn on background current and external electric field into our system. Thus we reach the model that we used to study the mesons system at finite temperature with external electric field. In gravity description, there exists a black hole in background AdS_5-Schwarzchild$\times\mathrm{S}^5$. The external electric field arises a virtual black hole (radius of its horizon is equal to vanishing locus) covering the real black hole. Shapes of the D7-branes consist of two types; Minkowski embedding and black hole embedding. The Minsowski embedding is a collection of D7-branes shapes which never touch the horizon of virtual or real black hole. The black hole embedding is a branch of D7-branes shapes which always pass through the virtual black hole and then fall into the real black hole. The physical interpretation of each embedding can be done by consideration of mesons spectroscopy, which we do not consider in this thesis.

We find that possibility of transition between two types of embedding corresponding to phase transition of mesons system. In particular, we see the possibility that stronger electric field can rip apart mesons with even stronger binding energy. The melted down of mesons system is analogous to metal/insulator phase transition. Constituent quarks and anti-quarks become electric charge carrier, the system of mesons is now conducting phase. In the conducting phase, there exists a formula between electric current and electric field which is identified with Ohm's law. And we can calculate the conductivity and drag force of the medium.

We find that Ohm's law is not precise, since the relation between electric current and electric field is non-linear. Thus, we may say that we obtain the conductivity which also depends on electric field. Moreover, the value of temperature, background current and quarks mass are contribute to value of conductivity.

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APPENDICES

Appendix A

Light-Cone Coordinates

We now discuss the light-cone coordinates which we use in entire Chapter 2 and Chapter 3. This coordinates system allows us to count physical degrees of freedom of closed string and open string in string theory or even in superstring theory. Here we suppose that we are in $D = (d+1)$-dimensional flat space-time. In stead of x^0 and x^1 directions included in $(x^0, x^1, x^2, ..., x^d)$, the light-cone coordinates use x^+ and x^- directions of $(x^+, x^-, x^2, ..., x^d)$. They are defined by

$$x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^1) \quad \text{and} \quad x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^1) \quad , \tag{A.1}$$

and we may conveniently denote the remaining directions by

$$x^I = (x^2, ..., x^d) \quad , \tag{A.2}$$

where I is called transverse index, $I = 2, 3, ..., d$. The x^+ and x^- are coincident with trajectory of light-ray moving along $+x^1$ and $-x^1$ directions. In other words, we use trajectory of light-ray as axes. The definitions of light-cone directions as defined in Eq. (A.1) and (A.2) can apply to any vectors. For example, $V^\mu = (V^0, V^1, V^2, ..., V^d)$ can be written in light-cone directions such that

$$V^+ = \frac{1}{\sqrt{2}}(V^0 + V^1) \quad \text{and} \quad V^- = \frac{1}{\sqrt{2}}(V^0 - V^1) \quad . \tag{A.3}$$

Thus, we write $V^\mu = (V^+, V^-, V^I)$ where $V^I = (V^2, V^3, ..., V^d)$.

It is necessary to consider dot product of two vectors. Here, we use the metric signature as follows: $g_{\mu\nu} = \text{drag}(-1, 1, ..., 1)$. For given vectors V^μ and W^μ, the Lorentzian dot product between of two vectors is

$$V_\mu W^\mu = -V^0W^0 + V^1W^1 + V^2W^2 + ... + V^dW^d \quad . \tag{A.4}$$

Let us calculate

$$\begin{aligned} V^+W^- &= \frac{1}{2}(V^0W^0 - V^0W^1 + V^1W^0 - V^1W^1) \quad , \\ V^-W^+ &= \frac{1}{2}(V^0W^0 + V^0W^1 - V^1W^0 - V^1W^1) \quad . \end{aligned}$$

We find that

$$\begin{aligned} V^+W^- + V^-W^+ &= V^0W^0 - V^1W^1 \\ V_IW^I &= V^2W^2 + ... + V^dW^d. \end{aligned}$$

Thus we conclude that dot product of two vectors written in light-cone coordinates, called light-cone dot product, takes form

$$V^\mu W_\mu = -V^+W^- - V^-W^+ + V^IW^I \quad . \tag{A.5}$$

Consequently, norm of vector V^μ is given by

$$|V|^2 = V^\mu V_\mu = -2V^+V^- + V^IV^I \quad . \tag{A.6}$$

This form of dot product leads to light-cone metric tensor, since the light-cone dot product can be illustrated as $V^\mu W_\mu = \tilde{g}_{\mu\nu}V^\mu W^\nu$. $\tilde{g}_{\mu\nu}$ denotes light-cone metric tensor. With help of the Eq. (A.5), we get

$$[\tilde{g}_{\mu\nu}] = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad . \tag{A.7}$$

We consider the Fourier transformation in convention of the light-cone coordinates. For a given function $f(x)$, generally, Fourier transformation of this function is

$$f(x) = \int \mathrm{d}^Dp \frac{1}{(2\pi)^D} \exp(ix^\mu p_\mu)\tilde{f}(p) \quad .$$

Since the light-cone dot product between a position vector and momentum is

$$x^\mu p_\mu = -x^+p^- - x^-p^+ + x^Ip^I \quad ,$$

the Fourier transformation is written by

$$f(x) = \int \mathrm{d}^Dp \frac{1}{(2\pi)^D} \exp(i(-x^+p^- - x^-p^+ + x^Ip^I))\tilde{f}(p) \quad . \tag{A.8}$$

Suppose X^μ and P^μ are operators which determine position and momentum, respectively. When the $f(x)$ is operated by X, we obtain

$$X^\mu f(x) = x^\mu f(x) \quad ,$$

where x^μ is position eigenvalue of the operator X^μ, with $\mu = +, -, I$. Form of the operator P^μ is more complicated. To see some hints, let us consider the Fourier transformation for Lorentzian coordinates;

$$f(x) = \int \mathrm{d}^D p \frac{1}{(2\pi)^D} \exp(i(x^0 p^0 - x^1 p^1 - ... - x^d p^d))\tilde{f}(p) \quad . \tag{A.9}$$

In convention of Lorentzian coordinates, the momentum operator takes the form

$$P^0 = i\frac{\partial}{\partial x^0} \quad \text{and} \quad P^i = -i\frac{\partial}{\partial x^i} \quad \text{for} \quad i = 1, ..., d \quad .$$

When the elements of momentum operator operate to $f(x)$ in Eq. (A.9), we obtain

$$P^0 f(x) = \int \mathrm{d}^D p \frac{p^0}{(2\pi)^D} \exp(i(-x^0 p^0 + x^i p^i))\tilde{f}(p) \tag{A.10}$$

$$P^i f(x) = \int \mathrm{d}^D p \frac{p^i}{(2\pi)^D} \exp(i(-x^0 p^0 + x^i p^i))\tilde{f}(p) \quad . \tag{A.11}$$

If $f(x)$ is a physical field, we can fix $p^0 =$ energy and $x^0 =$ time. Thus the operator P^0 correspond to the Hamiltonian. In order to find the forms of P^+, P^- and P^I, we begin from the fact that

$$P^+ f(x) = \int \mathrm{d}^D p \frac{p^+}{(2\pi)^D} \exp(i(-x^+ p^- - x^- p^+ + x^I p^I))\tilde{f}(p) \quad , \tag{A.12}$$

$$P^- f(x) = \int \mathrm{d}^D p \frac{p^-}{(2\pi)^D} \exp(i(-x^+ p^- - x^- p^+ + x^I p^I))\tilde{f}(p) \tag{A.13}$$

$$P^I f(x) = \int \mathrm{d}^D p \frac{p^i}{(2\pi)^D} \exp(i(-x^+ p^- - x^- p^+ + x^I p^I))\tilde{f}(p) \quad . \tag{A.14}$$

Thus, we can observe that components of momentum operator of the light-cone convention have to be

$$P^+ = i\frac{\partial}{\partial x^-} \quad , \quad P^- = i\frac{\partial}{\partial x^+} \quad \text{and} \quad P^I = -i\frac{\partial}{\partial x^I} \tag{A.15}$$

Again, we assume that $f(x)$ is a physical field and we choose to fix $x^+ =$ time. The fixing x^+ leads to an argument that $P^- =$ energy. Additionally, the light-cone convention Hamiltonian is P^-. In this thesis, we do not derive the light-direction of the Lornetzian generator $M^{\mu\nu} \rightarrow M^{+-}, M^{+I}, M^{-I}$ and M^{IJ}. In the book [5], many discussions about light-cone gauge are discussed.

Appendix B

Metric of Five-Dimensional Sphere

Let us consider a stack of D3-branes in (9+1)-dimensional space-time. We choose $x^4, x^5, ..., x^9$ to be transverse to the stack D3-branes. It is possible to compact transverse directions as a five-sphere S^5. The stack of D3-branes is viewed as a point at origin within a spherical symmetric space (coordinates of the space are $x^4, ..., x^9$). The five-sphere has $SO(6)$ isometry group, which is rotation group of the transverse directions. When we add another stack of D7-branes, the rotational symmetry is broken into $SO(4) \times SO(2)$. The rotation group $SO(4)$ is rotation of x^4, x^5, x^6 and x^7, while $SO(2)$ is rotation of x^8 and x^9. The x^4, x^5, x^6 and x^7 are compact to three-sphere S^3 subspace on the S^5. The line element which describes the S^5 for this case is more complicate. Let us look at the S^3, we suppose that the x^6 and x^7 form a circle S^1 subspace on the S^3. At this time, we may consider the case of S^3 with S^1 subspace. This case is easier and then we extend this consideration to the case of S^5 with S^3 subspace.

For the three-sphere of radius r_{S3}, the directions x^4, x^5, x^6 and x^7 have to satisfy the sphere equation such that

$$x_4^2 + x_5^2 + x_6^2 + x_7^2 = r_{\mathrm{S3}}^2 \quad . \tag{B.1}$$

Then, supposing that the subspace S^1 has radius equal to r_{S1}. Hence x^4 and x^5 directions need to satisfy $x_6^2 + x_7^2 = r_{\mathrm{S1}}^2$, so the Eq. (B.1) becomes

$$x_4^2 + x_5^2 + r_{\mathrm{S1}}^2 = r^2 \quad . \tag{B.2}$$

As following Eq. (B.2), the S^3 can be illustrated as a two-sphere on background $\mathbb{R}^3$ with axes x_4, x_5 and r_{S1}, like x, y and z (see Figure B.1). The compact directions x^6 and x^7 are now written in polar coordinates with an introduced angular coordinate, named θ_3. Thus,

$$x_6 = r_{\mathrm{S1}} \sin\theta_3 \quad \text{and} \quad x_7 = r_{\mathrm{S1}} \cos\theta_3 \quad . \tag{B.3}$$

Then, for S^3 as shown in Figure B.1, the radius r_{S3} can be projected onto axes x_4, x_5 and r_{S1}. With introducing other two angular coordinates named θ_1 and θ_2,

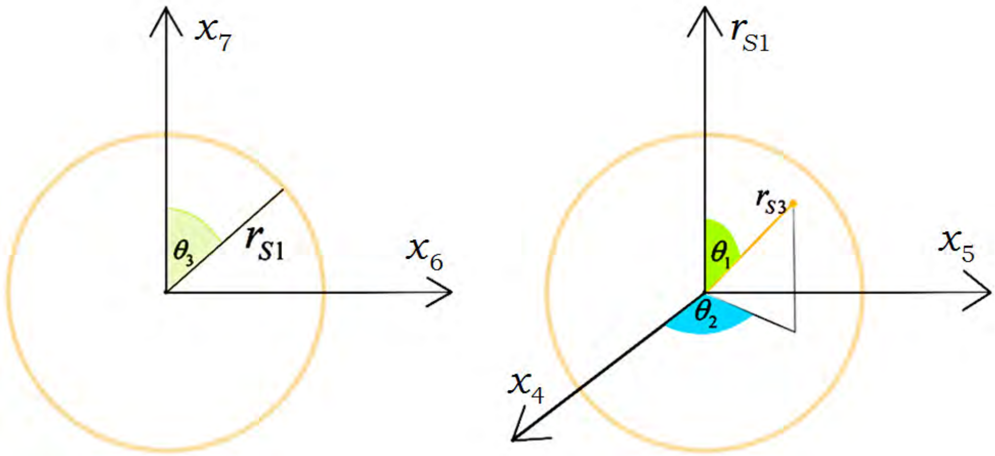

Figure B.1: Left is circle of radius r_{S1} and right is three-sphere of radius r_{S3}. In this view, the three-sphere can be thought as two-sphere on $x_4, x_5, r_{\mathrm{S1}}$ axes.

the projection of r_{S3} onto r_{S1} axis is $r_{S1} = r_{\mathrm{S3}} \cos\theta_1$. Together with projection of r_{S3} onto x^4 and x^5 axes, the x^4, x^5, x^6 and x^7 directions takes forms

$$x_4 = r_{\mathrm{S3}} \sin\theta_1 \cos\theta_2 \quad , \qquad x_5 = r_{\mathrm{S3}} \sin\theta_1 \sin\theta_2 \quad , \tag{B.4}$$

$$x_6 = r_{\mathrm{S3}} \cos\theta_1 \sin\theta_3 \quad \text{and} \quad x_7 = r_{\mathrm{S3}} \cos\theta_1 \cos\theta_3 \quad . \tag{B.5}$$

The value of θ_2 and θ_3 can take from 0 to 2π as usual, but only from 0 to $\pi/2$ for θ_1 since the r_{S1} being S^1 radius is positive definited. Thus we conclude that $\theta_2, \theta_3 \in [0, 2\pi]$ and $\theta_1 \in [0, \pi/2]$.

The line element on the S^3 may be written as $\mathrm{d}S^2 = \mathrm{d}x_4^2 + ... + \mathrm{d}x_7^2$. However we write this line element in terms of r_{S3} and those angular coordinates. We calculate the $(\mathrm{d}x^i)^2$ with $i = 4, ..., 7$, each x^i having forms as shown in Eq. (B.4) and (B.5). For simplicity, we set $r_{\mathrm{S3}} = 1$ making $\mathrm{d}S^2 = \mathrm{d}\Omega_3^2$. Steps of the calculations follow

$$\begin{aligned} \mathrm{d}x_4 &= -\sin\theta_1 \sin\theta_2 \mathrm{d}\theta_2 + \cos\theta_1 \cos\theta_2 \mathrm{d}\theta_1 \quad , \\ \mathrm{d}x_4^2 &= \sin^2\theta_1 \sin^2\theta_2 \mathrm{d}\theta_2^2 + \cos^2\theta_1 \cos^2\theta_2 \mathrm{d}\theta_1^2 \\ &\quad -2\sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \mathrm{d}\theta_1 \mathrm{d}\theta_2 \quad , \end{aligned} \tag{B.6}$$

$$\begin{aligned} \mathrm{d}x_5 &= \sin\theta_1 \cos\theta_2 \mathrm{d}\theta_2 + \cos\theta_1 \sin\theta_2 \mathrm{d}\theta_1 \quad , \\ \mathrm{d}x_5^2 &= \sin^2\theta_1 \cos^2\theta_2 \mathrm{d}\theta_2^2 + \cos^2\theta_1 \sin^2\theta_2 \mathrm{d}\theta_1^2 \\ &\quad +2\sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \mathrm{d}\theta_1 \mathrm{d}\theta_2 \quad . \end{aligned} \tag{B.7}$$

Now we get $\mathrm{d}x_4^2 + \mathrm{d}x_4^2 = \cos^2\theta_1 \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2$. Then,

$$
\begin{aligned}
\mathrm{d}x_6 &= \cos\theta_1 \cos\theta_3 \mathrm{d}\theta_3 - \sin\theta_1 \sin\theta_3 \mathrm{d}\theta_1 \quad , \\
\mathrm{d}x_6^2 &= \cos^2\theta_1 \cos^2\theta_3 \mathrm{d}\theta_3^2 + \sin^2\theta_1 \sin^2\theta_3 \mathrm{d}\theta_1^2 \\
&\quad -2\sin\theta_1 \cos\theta_1 \sin\theta_3 \cos\theta_3 \mathrm{d}\theta_1 \mathrm{d}\theta_3 \quad , \qquad \text{(B.8)} \\
\mathrm{d}x_7 &= -\cos\theta_1 \sin\theta_3 \mathrm{d}\theta_3 - \sin\theta_1 \cos\theta_3 \mathrm{d}\theta_1 \quad , \\
\mathrm{d}x_7^2 &= \cos^2\theta_1 \sin^2\theta_3 \mathrm{d}\theta_3^2 + \sin^2\theta_1 \cos^2\theta_3 \mathrm{d}\theta_1^2 \\
&\quad +2\sin\theta_1 \cos\theta_1 \sin\theta_3 \cos\theta_3 \mathrm{d}\theta_1 \mathrm{d}\theta_3 \quad . \qquad \text{(B.9)}
\end{aligned}
$$

Here, we reach $\mathrm{d}x_6^2 + \mathrm{d}x_7^2 = \cos^2\theta_1 \mathrm{d}\theta_3^2 + \sin^2\theta_1 \mathrm{d}\theta_1^2$. Finally, we achieve the line element of the S^3 such that

$$
\mathrm{d}\Omega_3^2 = \mathrm{d}\theta_1^2 + \cos^2\theta_1 \mathrm{d}\theta_3^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 \quad . \qquad \text{(B.10)}
$$

This line element is written in a sense of unit S^3. This can be generalized to line element of three-ball (three solid-sphere) by

$$
\mathrm{d}S_{\mathrm{B3}}^2 = \mathrm{d}r_{\mathrm{S3}}^2 + r_{\mathrm{S3}}^2 \mathrm{d}\Omega_3^2 \quad .
$$

We then consider line element of S^5 with S^3, and we extend the idea from the previous consideration. As we did for S^3, we say that the directions $x^4, x^5, ..., x^9$ have to satisfy

$$
x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 = r^2 \quad . \qquad \text{(B.11)}
$$

Because the direction $x_4^2 + x_5^2 + x_6^2 + x_7^2$ form S^3, they satisfy Eq. (B.1). Thus Eq. (B.11) becomes $r_{\mathrm{S3}}^2 + x_8^2 + x_9^2 = r^2$. In that notation, the S^5 can be viewed as a two-sphere on axes $x_8, x_9, r_{\mathrm{S3}}^2$ of background $\mathbb{R}^3$, see Figure B.2. By introducing other angular coordinates denoted by θ_4 and θ_5, projection of r to the axes are just

$$
r_{\mathrm{S3}} = r\cos\theta_4 \quad , \quad x_8 = r\sin\theta_4 \cos\theta_5 \quad \text{and} \quad x_9 = r\sin\theta_4 \sin\theta_5 \quad .
$$

The r_{S3} is again projected onto $x^4, ..., x^7$. Then we get

$$
x_4 = r\cos\theta_4 \sin\theta_1 \cos\theta_2 \quad , \quad x_5 = r\cos\theta_4 \sin\theta_1 \sin\theta_2 \quad , \qquad \text{(B.12)}
$$

$$
x_6 = r\cos\theta_4 \cos\theta_1 \sin\theta_3 \quad , \quad x_7 = r\cos\theta_4 \cos\theta_1 \cos\theta_3 \quad , \qquad \text{(B.13)}
$$

$$
x_8 = r\sin\theta_4 \cos\theta_5 \quad \text{and} \quad x_9 = r\sin\theta_4 \sin\theta_5 \quad . \qquad \text{(B.14)}
$$

For more convenient, we may write $x_i = (r\cos\theta_4)y_i$ for $i = 4, ..., 7$ where y_i are coordinate of unit there-sphere, changing x_i in equation from Eq. (B.1) to (B.9) to be y_i.

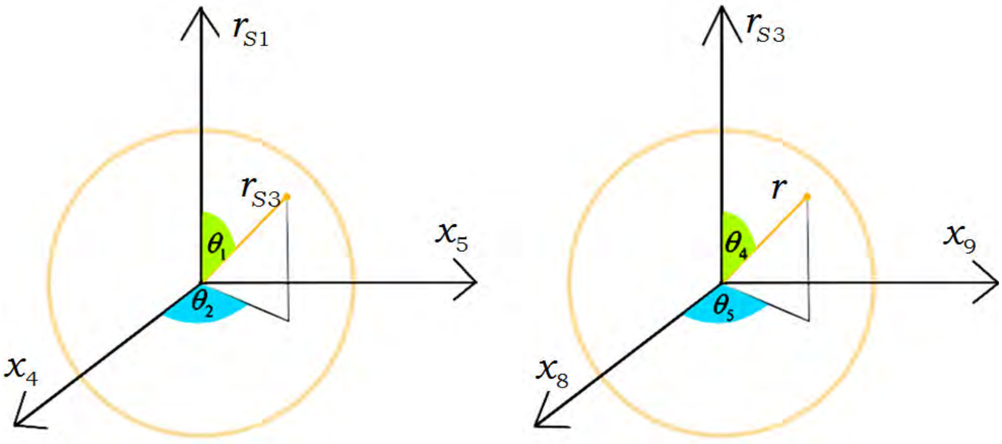

Figure B.2: Left is three-sphere of radius r_{S3} and right is five-sphere of radius r. In this view, the five-sphere can be seen as two-sphere on x_8, x_9, r_{S3} axes.

Line-element on the S^5 is $\mathrm{d}S^2 = \mathrm{d}x_4^2 + ... + \mathrm{d}x_9^2$ thus we need to find forms of $\mathrm{d}x_a^2$ for $a = 4, ..., 9$. For simplicity, we consider unit S^5 that is $r = 1$ and $\mathrm{d}S^2 = \mathrm{d}\Omega_5^2$. By performing derivative to x_i;

$$\mathrm{d}x_i = \cos\theta_4 \mathrm{d}y_i - y_i \sin\theta_4 \mathrm{d}\theta_4 \quad ,$$

we obtain

$$\mathrm{d}x_i^2 = \cos^2\theta_4 \mathrm{d}y_i^2 + y_i^2 \sin^2\theta_4 \mathrm{d}\theta_4^2 - 2y_i \cos\theta_4 \sin\theta_4 \mathrm{d}\theta_4 \mathrm{d}y_i \quad . \tag{B.15}$$

Thus,

$$\begin{aligned}\mathrm{d}x_4^2 + \mathrm{d}x_5^2 + \mathrm{d}x_6^2 + \mathrm{d}x_7^2 = {} & \cos^2\theta_4 \left(\mathrm{d}y_4^2 + \mathrm{d}y_5^2 + \mathrm{d}y_6^2 + \mathrm{d}y_7^2\right) \\ & + \left(y_4^2 + y_5^2 + y_6^2 + y_7^2\right) \sin^2\theta_4 \mathrm{d}\theta_4 \\ & -2\left(y_4 \mathrm{d}y_4 + y_5 \mathrm{d}y_5 + y_6 \mathrm{d}y_6 + y_7 \mathrm{d}y_7\right) \\ & \times \sin\theta_4 \cos\theta_4 \mathrm{d}\theta_4 \quad .\end{aligned} \tag{B.16}$$

We calculate the Eq. (B.16) term by term, the first term is

$$\mathrm{d}y_4^2 + \mathrm{d}y_4^2 + \mathrm{d}y_6^2 + \mathrm{d}y_7^2 = \mathrm{d}\Omega_3^2 \quad . \tag{B.17}$$

The second term is

$$\begin{aligned} y_4^2 + y_5^2 + y_6^2 + y_7^2 &= \sin^2\theta_1 \cos^2\theta_2 + \sin^2\theta_1 \sin^2\theta_2 + \cos^2\theta_1 \sin^2\theta_3 \\ &\quad + \cos^2\theta_1 \cos^2\theta_3 \quad , \\ y_4^2 + y_5^2 + y_6^2 + y_7^2 &= 1 \quad . \end{aligned} \tag{B.18}$$

The third term of Eq. (B.16) is reached by following the way

$$\begin{aligned}
-y_4\mathrm{d}y_4 &= \sin^2\theta_1\sin\theta_2\cos\theta_2\mathrm{d}\theta_2 - \sin\theta_1\cos\theta_1\cos^2\theta_2\mathrm{d}\theta_1 \quad ,\\
-y_5\mathrm{d}y_5 &= -\sin^2\theta_1\sin\theta_2\cos\theta_2\mathrm{d}\theta_2 - \sin\theta_1\cos\theta_1\sin^2\theta_2\mathrm{d}\theta_1 \quad ,\\
-y_6\mathrm{d}y_6 &= -\cos^2\theta_1\sin\theta_3\cos\theta_3\mathrm{d}\theta_3 + \sin\theta_1\cos\theta_1\sin^2\theta_3\mathrm{d}\theta_1 \quad ,\\
-y_7\mathrm{d}y_7 &= \cos^2\theta_1\sin\theta_3\cos\theta_3\mathrm{d}\theta_3 + \sin\theta_1\cos\theta_1\cos^2\theta_3\mathrm{d}\theta_1 \quad .
\end{aligned}$$

Then we find that

$$y_4\mathrm{d}y_4 + y_5\mathrm{d}y_5 + y_6\mathrm{d}y_6 + y_7\mathrm{d}y_7 = 0 \quad . \tag{B.19}$$

So, the Eq. (B.16) now becomes

$$\mathrm{d}x_4^2 + \mathrm{d}x_4^2 + \mathrm{d}x_6^2 + \mathrm{d}x_7^2 = \sin^2\theta_4\mathrm{d}\theta_4^2 + \cos^2\theta_4\mathrm{d}\Omega_3^2 \quad . \tag{B.20}$$

For x_8 and x_9, calculations for them follow

$$\begin{aligned}
\mathrm{d}x_8 &= -\sin\theta_4\sin\theta_5\mathrm{d}\theta_5 + \cos\theta_4\cos\theta_5\mathrm{d}\theta_4 \quad ,\\
\mathrm{d}x_8^2 &= \sin^2\theta_4\sin^2\theta_5\mathrm{d}\theta_5^2 + \cos^2\theta_4\cos^2\theta_5\mathrm{d}\theta_4^2\\
&\quad -2\sin\theta_4\cos\theta_4\sin\theta_5\cos\theta_5\mathrm{d}\theta_4\mathrm{d}\theta_5 \quad , \qquad (\mathrm{B.21})\\
\mathrm{d}x_9 &= \sin\theta_4\cos\theta_5\mathrm{d}\theta_5 + \cos\theta_4\sin\theta_5\mathrm{d}\theta_4 \quad ,\\
\mathrm{d}x_9^2 &= \sin^2\theta_4\cos^2\theta_5\mathrm{d}\theta_5^2 + \cos^2\theta_4\sin^2\theta_5\mathrm{d}\theta_4^2\\
&\quad +2\sin\theta_4\cos\theta_4\sin\theta_5\cos\theta_5\mathrm{d}\theta_4\mathrm{d}\theta_5 \quad . \qquad (\mathrm{B.22})
\end{aligned}$$

So we reach $\mathrm{d}x_8^2 + \mathrm{d}x_9^2 = \cos^2\theta_4\mathrm{d}\theta_4^2 + \sin^2\theta_4\mathrm{d}\theta_5^2$. Finally, line-element of S^5 is written by

$$\mathrm{d}\Omega_5^2 = \mathrm{d}\theta_4^2 + \cos^2\theta_4\mathrm{d}\Omega_3^2 + \sin^2\theta_4\mathrm{d}\theta_5^2 \quad , \tag{B.23}$$

where $\mathrm{d}\Omega_3^2$ may or may not write as in Eq. (B.10). We can see that the S^3 subspace on unit S^5 has radius $\cos^2\theta_4$, so we conclude that size of the S^3 is depending on the θ_4 which determines position of S^3 on the S^5. For five-ball, its line-element is just

$$\mathrm{d}S_{\mathrm{B5}}^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2 \quad .$$

The value of θ_5 is from 0 to 2π while for θ_4 is from 0 to $\pi/2$, or $\theta_5 \in [0, 2\pi]$ and $\theta_4 \in [0, \pi/2]$.

In Chapter 4, the angular coordinates $\theta_1, ..., \theta_5$ are renamed by $\theta_1 \to \Psi$, $\theta_2 \to \gamma$, $\theta_3 \to \beta$, $\theta_4 \to \theta$ and $\theta_5 \to \phi$. Thus, the $\mathrm{d}\Omega_5^2$ takes form as

$$\mathrm{d}\Omega_5^2 = \mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\Omega_3^2 + \sin^2\theta\mathrm{d}\phi^2 \quad , \tag{B.24}$$

where

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\Psi^2 + \cos^2\Psi \mathrm{d}\beta^2 + \sin^2\Psi \mathrm{d}\gamma^2 \quad . \tag{B.25}$$

Here, the value of angular coordinates are $\phi, \gamma, \beta \in [0, 2\pi]$ and $\theta, \Psi \in [0, \pi/2]$.

VITAE

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Presentations

1. Holographic Mesons in External Electric Field: The 19$^{\text{th}}$ National Graduate Research Conference, Rajabhat Rajanagarindra University, Chachoengsao, Thailand, 23-24 December 2010.

International Schools

1. 1$^{\text{st}}$ CERN School Thailand, Chulalongkorn University, Bangkok, 4-13 October 2010.
2. Attened Siam Physics Congress 2010, Sai Yok, Kanchahaburi, Thailand, 25-27 March 2010.
3. The Siam GR+HEP+COSMO Symposium IV, Naresuan University, Thailand, 26-28 July 2009.