CHAPTER I

PRELIMINARIES

The cardinality of a set X will be denoted by $|X|$.

Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all natural (positive) integers and the set of all real numbers, respectively.

An element a of a semigroup S is called an *idempotent* if $a^2 = a$. A *right* [*left*] *zero* of a semigroup S is an element $z \in S$ such that $xz = z$ [$zx = z$] for all $x \in S$. An element 0 of S is called a *zero* of S if $x0 = 0x = 0$ for all $x \in S$. If S has a right zero z_1 and a left zero z_2, then $z_1 = z_2$ which is the zero of S.

Note that right [left] zeroes of a semigroup S are idempotents of S. The identity of a group G is exactly one idempotent of G.

In a semigroup S, we can adjoin an extra element 0 and define $0x = x0 = 0$ for all $x \in S$. Then $S \cup \{0\}$ becomes a semigroup with zero 0. For a semigroup S, we let

$$S^0 = \begin{cases} S \cup \{0\} & \text{if } |S| = 1 \text{ or } S \text{ has no zero,} \\ S & \text{otherwise.} \end{cases}$$

Observe that if $|S| = 1$, then $S^0 \cong (\mathbb{Z}_2, \cdot)$.

A semigroup S is called a *right* [*left*] *zero semigroup* if $xy = y$ [$xy = x$] for all $x, y \in S$.

A *Kronecker semigroup* is a semigroup S with zero 0 such that for all $x, y \in S$,

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

A *right* [*left*] *nearring* is a triple $(N, +, \cdot)$ such that

(i) $(N, +)$ is a group,

(ii) $(N, \cdot)$ is a semigroup and

(iii) $(x + y) \cdot z = x \cdot z + y \cdot z\, [z \cdot (x + y) = z \cdot x + z \cdot y]$ for all $x, y, z \in N$.

A *subnearring* of a right [left] nearring is defined in the usual way.

Proposition 1.1 ([2], page 19). *Let* $(N, +, \cdot)$ *be a right* [*left*] *nearring with additive identity* 0. *Then*

(i) $0 \cdot x = 0\, [x \cdot 0 = 0]$ *for all* $x \in N$.

(ii) $(-x) \cdot y = -(x \cdot y)\, [x \cdot (-y) = -(x \cdot y)]$ *for all* $x, y \in N$.

A *zero* of a right [left] nearring $(N, +, \cdot)$ is an element $z \in N$ such that $zx = xz = z$ for all $x \in N$. If z is a zero of a right [left] nearring $(N, +, \cdot)$, then z is a right [left] zero of the semigroup $(N, \cdot)$. From Proposition 1.1(i), 0 is a left [right] zero of $(N, \cdot)$. Thus $z = 0$. Hence the nearring $(N, +, \cdot)$ has a zero if and only if $0x = x0 = 0$ for all $x \in N$ where 0 is the identity of $(N, +)$.

Some standard examples of nearrings which are not rings are the following ones.

Example 1.2. Let $M(\mathbb{R})$ be the set of all functions from $\mathbb{R}$ into itself,

$$C(\mathbb{R}) = \{f \in M(\mathbb{R}) \mid f \text{ is continuous on } \mathbb{R}\}$$

and

$$D(\mathbb{R}) = \{f \in M(\mathbb{R}) \mid f \text{ is differentiable on } \mathbb{R}\}.$$

Then $(M(\mathbb{R}), +, \circ)$, $(C(\mathbb{R}), +, \circ)$ and $(D(\mathbb{R}), +, \circ)$ are right nearrings which are not rings where $+$ and $\circ$ are the usual addition and composition of functions. For each $a \in \mathbb{R}$, let c_a be the constant function from $\mathbb{R}$ with range $\{a\}$. Then

$$\{c_a \mid a \in \mathbb{R}\} \subseteq D(\mathbb{R}) \subseteq C(\mathbb{R}) \subseteq M(\mathbb{R})$$

and

$$c_0 + f = f + c_0 = f \text{ and } c_0 f = c_0 \text{ for all } f \in M(\mathbb{R}).$$

That is, c_0 is the identity of the group $(M(\mathbb{R}), +)$ and a left zero of the semigroup $(M(\mathbb{R}), \circ)$. In fact,

$$c_1(c_0 + c_1) = c_1 \neq c_2 = c_1c_0 + c_1c_1.$$

Observe that the right nearrings $(M(\mathbb{R}), +, \circ)$, $(C(\mathbb{R}), +, \circ)$ and $(D(\mathbb{R}), +, \circ)$ are additively commutative. Since $c_1c_0 = c_1$, it follows that c_0 is not a zero of these nearrings. Hence $(M(\mathbb{R}), +, \circ)$, $(C(\mathbb{R}), +, \circ)$ and $(D(\mathbb{R}), +, \circ)$ are right nearrings without zero.

Throughout this research, right nearrings and left nearrings are assumed to be additively commutative. As in Example 1.2, we also have

Proposition 1.3 ([2], page 7). *If G is an abelian group and $M(G)$ is the set of all functions $f : G \to G$, then $(M(G), +, \circ)$ is a right nearring where $+$ and $\circ$ is the usual addition and composition of functions.*

A right [left] nearring $(N, +, \cdot)$ is called *zero-symmetric* if $0x = x0 = 0$ for all $x \in N$ where 0 is the identity of $(N, +)$.

Example 1.4. Let $M(\mathbb{R})$, $C(\mathbb{R})$ and $D(\mathbb{R})$ be as in Example 1.2 and let

$$M_0(\mathbb{R}) = \{f \in M(\mathbb{R}) \mid f(0) = 0\},$$
$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f(0) = 0\},$$
$$D_0(\mathbb{R}) = \{f \in D(\mathbb{R}) \mid f(0) = 0\}.$$

Then $(M_0(\mathbb{R}), +, \circ)$, $(C_0(\mathbb{R}), +, \circ)$ and $(D_0(\mathbb{R}), +, \circ)$ are clearly right nearrings having c_0 as their additive identity. Since for all $f \in M_0(\mathbb{R})$,

$$(c_0f)(x) = 0 = (fc_0)(x) \text{ for all } x \in \mathbb{R},$$

it follows that c_0 is the zero of $(M_0(\mathbb{R}), \circ)$. Hence $(M_0(\mathbb{R}), +, \circ)$, $(C_0(\mathbb{R}), +, \circ)$ and $(D_0(\mathbb{R}), +, \circ)$ are zero-symmetric right nearrings.

A semigroup S is said to *admit a ring structure* if S^0 is isomorphic to a multiplicative structure of some ring, or equivalently, there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is the operation on S^0.

Proposition 1.1 shows that if S has no left [right] zero, then there is no operation $+$ on S such that $(S, +, \cdot)$ is a right [left] nearring. Therefore the definition of semigroups admitting right [left] nearring structure is reasonably given as follows: We say that a semigroup S *admit a right [left] nearring structure* if S or S^0 is isomorphic to the multiplicative structure of some right [left] nearring, or equivalently,

(1) there is an operation $+$ on S such that $(S, +, \cdot)$ is a right [left] nearring where $\cdot$ is the operation on S or

(2) there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a right [left] nearring where $\cdot$ is the operation on S^0.

Denote by $\mathcal{R}$, $\mathcal{RNR}$ and $\mathcal{LNR}$ the classes of all semigroups which admit a ring structure, a right nearring structure and a left nearring structure, respectively. Then $\mathcal{R} \subseteq \mathcal{RNR} \cap \mathcal{LNR}$. By Example 1.2, the semigroups $(M(\mathbb{R}), \circ)$, $(C(\mathbb{R}), \circ)$ and $(D(\mathbb{R}), \circ)$ are all in $\mathcal{RNR}$. In fact, none of them belongs to $\mathcal{LNR}$ and hence they are not in $\mathcal{R}$. To see this, suppose that $(S(\mathbb{R}), \circ) \in \mathcal{LNR}$ where $S(\mathbb{R})$ is any of $(M(\mathbb{R}), \circ)$, $(C(\mathbb{R}), \circ)$ and $(D(\mathbb{R}), \circ)$. If $S(\mathbb{R})$ has a right zero, then c_0 must be a zero of $S(\mathbb{R})$, a contradiction (see Example 1.2). But $S(\mathbb{R}) \in \mathcal{LNR}$, so $(S^0(\mathbb{R}), \oplus, \cdot)$ is a left nearring for some operation $\oplus$ on $S^0(\mathbb{R})$ where $\cdot$ is the operation on $S^0(\mathbb{R})$. Let $f \in S^0(\mathbb{R})$ be such that $c_1 \oplus f = 0$. Then $f \neq 0$, so $c_1 \oplus c_1 = c_1(c_1 \oplus f) = 0$. It follows that $c_1 \oplus c_2 = g$ for some $g \in S(\mathbb{R})$. Thus $c_1 \oplus c_1 = c_1(c_1 \oplus c_2) = c_1 g = c_1$, so $c_1 = 0$, a contradiction.

We adopt the following notations when X is a nonempty set, F is a field and n is a positive integer.

dom f	:	the domain of the function f,
$G(X)$	:	the symmetric group on X
		(the group, under composition, of all bijections $f : X \to X$),
$T(X)$	:	the full transformation semigroup on X
		(the semigroup, under composition, of all functions $f : X \to X$),
$P(X)$	:	the partial transformation semigroup on X
		(the semigroup, under composition, of all functions from
		a subset of X into X),
$I(X)$	:	the 1-1 partial transformation semigroup on X
		(the symmetric inverse semigroup on X),
$M_n(F)$	:	the full $n \times n$ matrix semigroup over F under multiplication,
A_{ij}	:	the entry of $A \in M_n(F)$ in the $i^{\underline{th}}$ row and the $j^{\underline{th}}$ column,
I_n	:	the identity matrix in $M_n(F)$,
$G_n(F)$	:	the matrix group of all invertible matrices $A \in M_n(F)$,
$P_n(F)$	:	the matrix group of all permutation $n \times n$ matrices over F
		(the matrix group of all $A \in M_n(F)$ having the property that
		(i) $A_{ij} = 0$ or 1 for all $i, j \in \{1, \ldots, n\}$ and (ii) each row and
		each column of A contains exactly one 1),
$O_n(F)$	:	the matrix group of all orthogonal matrices $A \in M_n(F)$
		(the matrix group of all $A \in M_n(F)$ with $AA^t = I_n$),
$U_n(F)$	:	the matrix group of all invertible upper triangular matrices
	:	in $M_n(F)$,
$L_n(F)$	:	the matrix group of all invertible lower triangular matrices
	:	in $M_n(F)$,
$V_n(F)$	:	the matrix group $\{A \in M_n(F) \mid \det A = \pm 1\}$.

Notice that 1_X, the identity map on X, is the identity of $T(X)$, $P(X)$ and $I(X)$ and 0 (the empty transformation) is the zero of $P(X)$ and $I(X)$. Also, $G_n(F)$ is the group of units of $M_n(F)$ containing $P_n(F)$, $O_n(F)$, $U_n(F)$, $L_n(F)$ and $V_n(F)$ as subgroups and $P_n(F) \subseteq V_n(F)$.

Recall that the dihedral group of degree n is

$$D_n = \{1, a, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$$
$$\text{where } a^n = b^2 = 1 \text{ and } ba = a^{n-1}b,$$

and the alternating group of degree $n > 1$ is

$$A_n = \{f \in S_n \mid f \text{ is an even permutation}\}$$

where S_n is the symmetric group of degree n. Note that $|D_n| = 2n$ and $A_n = 1$ if $n = 1$, otherwise, $|A_n| = \dfrac{n!}{2}$. An infinite cyclic semigroup is an infinite semigroup S generated by an element $a \in S$, that is,

$$S = \{a^n \mid n \in \mathbb{N}\} \text{ and}$$
$$a^i \neq a^j \text{ for all distinct } i, j \in \mathbb{N}.$$