CHAPTER II

Preliminaries

In this chapter, we present those aspects of some definitions and theorems which are required in subsequent chapters.

2.1 Fixed point Theorems

Consider a function $\varphi : \Re \rightarrow \Re$ and suppose that we require to solve the equation $\varphi(x) = 0$. This is equivalent to solving the equation

$$\psi(x) = x$$

where $\psi(x) = \varphi(x) + x$ for all $x \in \Re$.
Thus x is a zero of φ if and only if x is a fixed point of ψ, i.e. a point which is left unaltered after the application of ψ.
More generally, many problems are equivalent to solving

$$Af = f$$

where $A : X \supseteq D(A) \rightarrow R(A) \subseteq X$ is an operator (not necessarily linear), acting in some normed vector spaces i.e. we seek a fixed point $f \in D(A)$ of the operator A (for simplicity, we write Af rather than $A(f)$).

There are many fixed point theorems which guarantee existence and/or uniqueness of fixed points. We state here what is used in this thesis.

Definition 2.1. *Let X be a normed vector space and let*
$A : X \supseteq D(A) \rightarrow R(A) \subseteq X$ be an operator (not necessarily linear). Then

1) A is a contraction if there exists a constant κ with $0 \leq \kappa \leq 1$ such that

$$\|Af_1 - Af_2\| \leq \kappa \|f_1 - f_2\| \text{ for all } f_1, f_2 \in D(A). \tag{2.1}$$

2) A is strictly contraction if there exists a constant κ with $0 \leq \kappa < 1$ such that (2.1) holds.

Theorem 2.2 ([8]). *(The contraction mapping theorem;Banach fixed point theorem) Let X be a Banach space and let $A : X \to X$ be a strictly contraction. Then the equation $Af = f$ has a unique solution in X.i.e. A has a unique fixed point f.*

The result of this theorem can be easily generalized as follows:

Corollary 2.3 ([8]). *Let X_0 be a closed subset of the Banach space X and assume that the operator A map X_0 into itself and a strictly contraction on X_0. Then the equation $Af = f$ has a unique solution $f \in X_0$.*

2.2 Bochner Integral

A Banach space setting of evolution equations requires taking the derivative in the Banach space. Hence, integration of Banach space valued function is an important tool of this setting. We define the Bochner integral of such functions and derive its basic properties.

In the following, a subset of $\Re^n$ is said to be measurable iff it is Lebesgue measurable. The functions will be defined on the nonempty measurable set $S \subseteq \Re^n$, with range in a Banach space X.

$x : S \to X$ is called weakly measurable if $s \longmapsto \ell(x(s))$ is Lebesgue measurable function for each $\ell \in X^*$.

$x : S \to X$ is called almost separably-valued if there exists $\{y_1, y_2, \ldots\} \subseteq X$ such that $\inf_i ||x(s) - y_i|| = 0$ for almost all $s \in S$.

$x : S \to X$ is called strongly measurable if it is weakly measurable and almost separably-valued.

$x : S \to X$ is said to be Bochner integrable if x is strongly measurable and the functions $s \longmapsto ||x(s)||$ is Lebesgue integrable. The set of all such functions x is a vector space and will be denoted by $L(S, X)$. The following Theorem 2.4 enables us to define the Bochner integral $\int_S x$ of $x \in L(S, X)$ to be $y \in X$ which satisfies (2.2).

Theorem 2.4 ([9]). *If $x \in L(S, X)$, then there exists a unique $y \in X$ such that*

$$\ell(y) = \int_S \ell(x(s))ds, \qquad \text{for all } \ell \in X^*. \tag{2.2}$$

Moreover, $||y|| \leq \int_S ||x(s)||ds$.

2.3 Semigroup of linear operators

We are now ready to make the following formal definition. Throughout this section X be a Banach space.

Definition 2.5. *A one-parameter family $\{S(t)\}_{t \geq 0}$ of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

1) $S(0) = I$, *(I is the identity operator on X) ;*

2) $S(t + s) = S(t)S(s)$ *for every $t, s \geq 0$ (the semigroup property) .*

A semigroup of bounded linear operators $\{S(t)\}_{t \geq 0}$ is uniformly continuous if

$$\lim_{t \to 0^+} \|S(t) - I\|_{\mathbb{L}(X)} = 0.$$

The linear operator A defined by

$$D(A) = \{x \in X \mid \lim_{t \to 0^+} \frac{S(t)x - x}{t} \ \text{ exists}\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt}\bigg|_{t=0} \qquad \text{for all } \ x \in D(A)$$

is called the infinitesimal generator of semigroup $\{S(t)\}_{t \geq 0}$, $D(A)$ is the domain of A.

From Definition (2.5), we have a semigroup $\{S(t)\}_{t \geq 0}$ with a unique infinitesimal generator. If $S(t)$ is uniformly continuous, its infinitesimal generator is a bounded operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $\{S(t)\}_{t \geq 0}$ and this semigroup is unique.

Definition 2.6. *A semigroup $\{S(t)\}_{t\geq 0}$ of a bounded linear operator on X is a strongly continuous semigroup of a bounded linear operators if*

$$\lim_{t\to 0^+} S(t)x = x, \qquad \textit{for every } \; x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of a C_0 or C_0 − *semigroup*.

Example 2.7. Let $X = L^p(\Re)$ with $1 \leq p < +\infty$. Define $S(0) = I$ and for $t > 0$ define $S(t)$ on X by

$$(S(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4t}} f(y)dy \tag{2.3}$$

for all $f \in X$ and $x \in \Re$.

Then $\{S(t)\}_{t\geq 0}$ is C_0 − *semigroup* on X called the *Gauss-Weierstrass semigroup*. The right hand side of (2.3) represents the Fourier convolution of the function $f \in X$ with the function k defined by

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}, \qquad \text{for all } x \in \Re.\ t > o. \tag{2.4}$$

This function k is the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{for all } x \in \Re,\ t > o. \tag{2.5}$$

See more detail [8], Example 18.

Example 2.8. Another important partial differential equation is the wave equation. For one dimension space, this equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{for all } x \in \Re,\ t > o. \tag{2.6}$$

we now take for simplicity $v = 1$.

The analogue of the *Gauss-Weierstrass semigroup* is the *Poisson semigroup*.

Let $X = L^p(\Re)$ with $1 \leq p < +\infty$. For $t > 0$ define $S(t)$ on X by

$$(S(t)f)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t}{t^2 + (x-y)^2} f(y)dy \tag{2.7}$$

for all $f \in X$ and $x \in \Re$ and define $S(0) = I$.

Then $\{S(t)\}_{t\geq 0}$ is the $C_0 - semigroup$ on X. We can see that (2.7) represents the Fourier convolution of the function $f \in X$ with the function k defined by

$$k(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2}, \qquad \text{for all } x \in \Re,\ t > o. \tag{2.8}$$

For more detail see [8], Example 1.10.

Theorem 2.9 ([10]). *Let $-A$ be an infinitesimal generator of the $C_0 - semigroup$ $\{S(t)\}_{t\geq 0}$.*
Then

a) for all $x \in X$

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x; \tag{2.9}$$

b) for all $x \in X$, $\int_0^t S(s)xds \in D(A)$ and

$$A\int_0^t S(s)xds = x - S(t)x; \tag{2.10}$$

c) for all $x \in D(A)$, $S(t)x \in D(A)$ and

$$\frac{d}{dt}S(t)x = -AS(t)x = -S(t)Ax; \tag{2.11}$$

d) for all $x \in D(A)$

$$S(s)x - S(t)x = \int_s^t S(\tau)Axd\tau = \int_s^t AS(\tau)xd\tau. \tag{2.12}$$

Theorem 2.9 have some simple consequences which we now state.

Corollary 2.10 ([10]). *If $-A$ is the infinitesimal generator of a $C_0 - semigroup$ $\{S(t)\}_{t\geq 0}$ then $D(A)$ is dense in X and A is a closed linear operator.*

2.4 Analytic Semigroup

We dealt with semigroups whose domain was the real nonnegative axis. We will consider the possibility of extending the domain of the parameter to regions in the complex plane that include the nonnegative real axis. Now we set

$$\Delta \triangleq \{z \in \mathbb{C} \mid \theta_1 < argz < \theta_2 \text{ and } \theta_1 < 0 < \theta_2\}.$$

Definition 2.11. *For $z \in \Delta$, let $S(z)$ be a bounded linear operator on a Banach space X. The family $\{S(z)\}_{z\in\Delta}$ is an analytic semigroup in Δ if*

i) $z \to S(z)$ is analytic in Δ;

ii) $S(0) = I$ and $\lim_{z\to 0} S(z)x = x$ for every $x \in X$, $z \in \Delta$;

iii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for every $z_1, z_2 \in \Delta$.

A semigroup $\{S(t)\}_{t\geq 0}$ will be called analytic if it is analytic in some sector Δ containing nonnegative real axis.

Recall that if A is a linear (not necessarily bounded) operator in X the resolvent set $\rho(A)$ is the set of all complex number λ for which $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is bounded linear operator in X. The family $R(\lambda : A) \triangleq (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators, is called the resolvent of A.

For convenience we also assume that $0 \in \rho(A)$ where A is the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$.

Theorem 2.12 ([10]). *A densely defined linear operator A in X satisfying*

$$\rho(A) \supseteq \Xi \triangleq \{\lambda \mid |arg\lambda| > \frac{\pi}{2} + \delta\} \cup \{0\} \quad \textit{for some } 0 < \delta < \frac{\pi}{2} \tag{2.13}$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \quad \textit{for } \lambda \in \Xi,\ \lambda \neq 0 \tag{2.14}$$

is the infinitesimal generator of a uniformly bounded C_0 − semigroup $\{S(t)\}_{t\geq 0}$ satisfying $\|S(t)\| \leq C$ for some constant C.

From Theorem 2.12 we are interested in how to extend the semigroup $\{S(t)\}_{t\geq 0}$ generated by a densely defined A satisfying (2.13) and (2.14) to an analytic semigroup in the sector $\Delta_\delta = \{z \mid |argz| < \delta\}$ and in very closed subsector $\overline{\Delta}_{\delta'} = \{z \mid |argz| \leq \delta' < \delta\}$ and $||S(z)||$ is uniformly bounded.

Theorem 2.13 ([10]). *Let A be an infinitesimal generator of the uniformly bounded C_0 – semigroup $\{S(t)\}_{t\geq 0}$ and assume that $0 \in \rho(A)$. The following statement are equivalent:*

a) The semigroup $\{S(t)\}_{t\geq 0}$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \mid |argz| < \delta\}$ and in very closed subsector $\overline{\Delta}_{\delta'} = \{z \mid |argz| \leq \delta' < \delta\}$ and $\|S(z)\|$ is uniformly bounded.

b) There exist $0 < \delta < \frac{\pi}{2}$ and $M > 0$ such that satisfying (2.13) and (2.14).

c) $S(t)$ is differentiable for $t > 0$ and there is a constant C such that

$$\|AS(t)\| \leq \frac{C}{t} \quad \text{for } t > 0.$$

2.5 Fractional Powers of Closed Operators

In this section, we start by brief outline of the definition of fractional powers of closed operators and their some properties. We concentrate mainly on fractional powers of operator A for which $-A$ is the infinitesimal generator of an analytic semigroup. The result of this section will be used to study some properties of solutions for nonlinear impulsive initial value problems.

Assumption 2.14. *Let A be a densely defined closed linear operator for which*

$$\rho(A) \supseteq \Xi^+ = \{\lambda \mid 0 < \omega < |arg\lambda| \leq \pi\} \cup V \tag{2.15}$$

where V is a neighborhood of zero and

$$\|R(\lambda : A)\| \leq \frac{M}{1 + |\lambda|}. \tag{2.16}$$

The assumption that $0 \in \rho(A)$ and therefore a whole neighborhood V of zero is in $\rho(A)$ was mainly for convenience. For an operator A satisfying Assumption (2.14) and $\alpha > 0$ we define

$$A^{-\alpha} \triangleq \frac{1}{2\pi i} \int_C z^{-\alpha}(A - zI)^{-1} dz$$

where path C runs in the resolvent set of A from $-\infty e^{iv}$ to $+\infty e^{iv}$, $\omega < v < \pi$ avoid the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z.

For $0 < \alpha < 1$ we can deform the path of integration C into the upper and lower sides of negative real axis and obtain

$$A^{-\alpha} = \frac{sin\alpha\pi}{\pi} \int_0^{+\infty} t^{-\alpha}(tI + A)^{-1} dt,\ 0 < \alpha < 1. \tag{2.17}$$

If $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$, i.e,$\omega < \pi/2$, we obtain still another representation of $A^{-\alpha}$. In this case since by Assumption 2.14 and $0 \in \rho(A)$, there exist $\delta > 0$ such that $A + \delta$ is still an infinitesimal generator of an analytic semigroup. This implies the following estimates;

$$||S(t)|| \leq Me^{-\beta t}, \quad \text{for some} M,\ \beta > 0 \tag{2.18}$$

$$||AS(t)|| \leq M_1 t^{-1} e^{-\beta t}, \quad \text{for some} M_1,\ \beta > 0 \tag{2.19}$$

$$||A^m S(t)|| \leq M_m t^{-m} e^{-\beta t}., \quad \text{for some} M_m,\ \beta > 0 \tag{2.20}$$

By using the fact that

$$(tI + A)^{-1} = \int_0^{+\infty} e^{-st} S(t) ds \tag{2.21}$$

converges uniformly for $t \geq 0$ in the uniform operator topology, (2.18). Substituting (2.21) into (2.17) and using Fubini 's theorem, we finally obtain

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} S(t) dt,\ 0 < \alpha < 1 \tag{2.22}$$

and we define $A^{-0} = I$.

Since $A^{-\alpha}$ is one to one, $A^{\alpha} = (A^{-\alpha})^{-1}$. For $0 < \alpha \leq 1$, A^{α} is a closed linear operator whose domain $D(A^\alpha) \supseteq D(A)$ is dense in X. The closeness of $A^{-\alpha}$ implies that $D(A^\alpha)$ endowed with the graph norm

$$|x|_{D(A)} = |x| + |A^\alpha x|, \qquad x \in D(A^\alpha)$$

is a Banach space. Since $0 \in \rho(A)$, A^α is invertible and its graph norm is equivalent to the norm $|x|_\alpha = |A^\alpha x|$. Thus $D(A^\alpha)$ equipped with the norm $|.|_\alpha$ is a Banach space which we denote X_α.

Lemma 2.15 ([11]). *There exists a constant C such that*

$$||A^{-\alpha}|| \leq C \quad \textit{for} \quad 0 \leq \alpha \leq 1$$

Theorem 2.16 ([11]). *Let $-A$ be the infinitesimal generator of analytic semigroup $\{S(t)\}_{t\geq 0}$.*
If $0 \in \rho(A)$ then

a) $S(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$;

b) for every $x \in D(A^\alpha)$, we have $S(t)A^\alpha x = A^\alpha S(t)x$;

c) for every $t > 0$ the operator $A^\alpha S(t)$ is bounded and $|A^\alpha S(t)|_{L(X)} \leq M_\alpha t^{-\alpha} e^{-\beta}$, for some M_α, $\beta > 0$;

d) for $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, we have $|S(t)x - x| \leq C_\alpha t^\alpha |A^\alpha x|$, for some $C_{\alpha>0}$.

Lemma 2.17 ([11]). *Let $0 < \alpha < 1$. If $x \in D(A)$, then*

$$A^\alpha = \frac{sin\pi\alpha}{\pi} \int_0^\infty t^{\alpha-1} A(tI + A)^{-1} x$$

2.6 Almost periodicity and Piecewise continuous almost periodicity

The concept of almost periodicity was introduced by H.Bohr[3] and plays a significant rule in the theory of oscillations.D.Wexler[12] started to investigate piecewise continuous almost periodic(PCAP) functions. Below we introduce the description of the PCAP functions.

Definition 2.18. *A subset S of $\Re$ is said to be a relatively dense set if there exists $\ell > 0$ such that*

$$[a, a+\ell] \cap S \neq \phi, \qquad \text{for all } \ a \in \Re.$$

Definition 2.19. *Let $\epsilon > 0$ and $\{t_i\}$ be a sequence of $\Re$. An integer p is called an $\epsilon -$ almost period of the sequence $\{t_i\}$ if for each k, we have $|t_{k+p} - t_k| < \epsilon$.*

Definition 2.20. *A sequence $\{t_i\}$ is almost periodic if for any $\epsilon > 0$ there exists a relatively dense set of its $\epsilon -$ almost periods.*

Let $C_b(\Re, X)$ denote the usual Banach space of bounded continuous functions from $\Re$ into X under supremum norm $|.|_\infty$. Given a function $f : \Re \to X$ and $\omega \in \Re$, we denote the $\omega -$ *translate* of f as $f_\omega(t) = f(\omega + t)$, $t \in \Re$. We will denote the set of all translate of f by $H(f) \triangleq \{f_\omega \mid \omega \in \Re\}$.

Definition 2.21. *(Bochner's characterization of almost periodicity) A function $f : \Re \to X$ is called almost periodic if $H(f)$ is relatively compact in $C_b(\Re, X)$.*

Definition 2.22. *A function $f : \Re \to X$ called almost periodic if*

1) f is continuous;

2) for each $\epsilon > 0$ there is $\ell(\epsilon) > 0$ such that every interval I of length $\ell(\epsilon)$ contain a number τ such that

$$|f(t+\tau) - f(t)| < \epsilon, \qquad \text{for all } t \in \Re.$$

Let $\{t_i\}$ be a sequence of real such that $t_i < t_{i+1}$ and $|t_i| \to +\infty$ as $|i| \to +\infty$. Denote $t_i^j \triangleq t_{i+j} - t_i$ for all $i, j \in \mathbb{Z}$.

Definition 2.23. *The family of sequence $\{t_i^j\}$ is said to be equipotentially almost periodic (EAP) if for any $\epsilon > 0$ there is a relatively dense set of $\epsilon -$almost periods, that are common to all sequence $\{t_i^j\}$.*

Let PC be the space of piecewise continuous functions, defined on $\Re$ with discontinuous of the first kind at the point t_i.

Definition 2.24. *We say that a function $u(.) \in PC$ is a piecewise continuous almost periodic (PCAP) if*

a) the sequence $\{t_i\}$ is such that the derived sequence $\{t_i^j\}$ are EAP;

b) for $\epsilon > 0$ there exists $\delta > 0$ such that if t_1, t_2 belong to the same interval of continuity and $|t_1 - t_2| < \delta$, then $|u(t_1) - u(t_2)| < \epsilon$;

c) for $\epsilon > 0$ there exists a relatively dense set P_ϵ of $\epsilon -$almost periods such that if $\tau \in P_\epsilon$, then $|u(t+\tau) - u(t)| < \epsilon$ for all $t \in \Re$ which satisfy the condition $|t - t_i| > \epsilon$, $i \in \mathbb{Z}$.

Definition 2.25. *Let Ω be an open subset of X and $f : \Re \times \Omega \to Y$ be a PCAP in $t \in \Re$. Then f is called uniformly piecewise continuous almost periodic, if for every $\epsilon > 0$ and every compact $K \subseteq \Omega$ there exists a relatively P_ϵ of $\epsilon -$ almost periods such that*

$$||f(t+\tau, y) - f(t, y)||_Y \leq \epsilon$$

for all $(t, \tau, y) \in \Re \times P_\epsilon \times Y$ and $|t - t_i| > \epsilon, i \in \mathbb{Z}$.

The number τ in Definition 2.25 is called the almost period of f relative to ϵ.

Example 2.26. Define

$$f(t) = \begin{cases} e^{it} + e^{i\sqrt{2}t}, & t \neq t_i; \\ -1, & t = t_i,\ i \in \mathbb{Z} \end{cases}$$

for all $t \in \Re$ and sequence $\{t_i\}$ is such that $\{t_i^j\}$ is EAP. Then $f(t)$ is PCAP.

Proof. Let $\epsilon > 0$. Choose $l(\epsilon) = \epsilon$. Let $a \in \Re$. Then $I = (a, a+\epsilon]$ is an interval of length $l(\epsilon)$. If we define $h : I \to (0, \epsilon]$ by

$$h(x) = x - a \quad \text{for all} \quad x \in I,$$

then h is continuous, 1-1 and onto function. Hence, it suffice to show that there is $\tau \in (0, \epsilon]$ such that $|f(t+\tau) - f(t)| < \epsilon$. Let $t \in \Re - \{t_i\}$.

Now, we choose $\tau < \frac{\epsilon}{1+\sqrt{2}}$. We obtain

$$\begin{aligned}
|f(t+\tau)-f(t)| &\leq |e^{it}(e^{i\tau}-1)| + |e^{i\sqrt{2}t}(e^{i\sqrt{2}\tau}-1)| \\
&= \sqrt{(cos\tau-1)^2+sin^2\tau} + \sqrt{(cos\sqrt{2}\tau-1)^2+sin^2\sqrt{2}\tau} \\
&= \sqrt{2(1-cos\tau)} + \sqrt{2(1-cos\sqrt{2}\tau)} \\
&= \sqrt{4sin^2\frac{\tau}{2}} + \sqrt{4sin^2\frac{\sqrt{2}\tau}{2}} \\
&= 2|sin\frac{\tau}{2}| + 2|sin\frac{\sqrt{2}\tau}{2}| \\
&\leq 2(\frac{\tau}{2}) + 2(\frac{\sqrt{2}\tau}{2}) = (1+\sqrt{2})\tau < \epsilon.
\end{aligned}$$

This implies that $f(t)$ is PCAP. □

Lemma 2.27. *Every almost-periodic function f is bounded and uniformly continuous. Indeed, the following holds for f:*

- *f^* (the congugate) is almost periodic.*
- *$f(t+a)$ and $f(at)$ are almost periodic for all $a \in \Re$.*
- *$Kf(t)$ and $f(t)+K$ are almost periodic for all $K \in \mathbb{C}$.*
- *If $|f(t)| \geq a > 0$, then $\frac{1}{f(t)}$ is almost periodic.*
- *If $\frac{df}{dt}$ is uniformly continuous, then it is also almost periodic.*
- *If $\int_c^t f(s)ds$ is almost periodic, then it is bounded, and vice versa.*

We have also for a sequence of almost periodic functions $\{f_k\}$:

- *$f_1 + f_2$ is almost periodic.*
- *$f_1 f_2$ is almost periodic.*
- *If $f_k \to f$ uniformly, then f is almost periodic.*

2.7 Impulsive differential equations

We begin this section by describing a set of relations which characterize an evolution process subject to impulsive effects.
Let us consider an evolution process described by

i) a system of differential equation

$$x'(t) + Ax(t) = f(t, x) \tag{2.23}$$

where $f : \Re \times \Omega \to X$, Ω is an open subset of a Banach space X, $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$.

ii) the set $M(t)$, $N(t) \subseteq \Omega$ for each $t \in \Re$

iii) the operator $B(t) : M(t) \to N(t)$ for each $t \in \Re$.

Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.23) starting at (t_0, x_0). The evolution process behaves as follows: the point $P_t = (t, x(t))$ begins its motion from the initial point $P_{t_0} = (t_0, x_0)$ and move along a curve $\{(t, x) \mid t \geq t_0, x = x(t)\}$ until the time $t_1 > t_0$ at which point P_t meets the set $M(t)$. At $t = t_1$ the operator $B(t)$ transfers the point $P_{t_1} = (t_1, x(t_1))$ into $P_{t_1^+} = (t_1, x_1^+) \in N(t_1)$ where $x_1^+ = B(t_1)x(t_1)$. Then the point P_t continues to move further along the curve with $x(t) = x(t, t_1, x_1^+)$ as a solution of (2.23) starting at $P_{t_1^+} = (t_1, x_1^+)$ until it hit the set $M(t)$ at the moment $t_2 > t_1$. Then once again the point $P_{t_2} = (t_2, x_2)$ is transfered to the point $P_{t_2^+} = (t_2, x_2^+) \in N(t_2)$ where $x_2^+ = B(t_2)x(t_2)$.

As before, the point P_t continues to move forward with $x(t) = x(t, t_2, x_2^+)$ as the solution of (2.23) starting at (t_2, x_2^+). Thus the evolution process continues forward as long as the solution of (2.23) exists.

The set of relations i), ii) and iii) is called the characterize the above mentioned evolution process an impulsive differential system, the curve which described by the point P_t the integral curve and the function that defines the integral curve a solution of the impulsive differential system.

A solution of an impulsive differential system maybe

a) a continuous function, if the integral curve does not intersect the set $M(t)$ or hit at the fixed point of operator $B(t)$;

b) a piecewise continuous function having finite number of discontinuous of the first kind if the integral curve meets $M(t)$ at a finite number of points which are not the fixed point of the operator $B(t)$;

c) a piecewise continuous function having a countable number of discontinuous of the first kind if the integral curve encounters the set $M(t)$ at a countable number of points that are the fixed point of the operator $B(t)$.

The moment t_i at which the point P_t hits the set $M(t)$ are called moments of impulsive effect. We will assume that the solution $x(t)$ of the impulsive differential system is left continuous at $t_i, i \in \mathbb{N}$, that is

$$x(t_i^-) = \lim_{h \to 0^+} x(t_i - h) = x(t_i).$$

The meaning of the impulsive differential systems gives rise to several types of systems such as

1) Systems with impulses at fixed times;
2) Systems with impulses at variable times;
3) Autonomous systems with impulses.

Now, we will give description for only one type say, type 1 that we use in this thesis.

Let $M(t)$ be a set to represent a sequence of planes $t = t_i$ such that $\{t_i\}$ is a sequence of time such that $t_i \to +\infty$ as $i \to +\infty$. Let us define the operator $B(t)$ for $t = t_i$ only so that the sequence of operator $B(i)$ is given by

$$B(i) : \Omega \to \Omega, \qquad B(i)(x) = x + J_i(x)$$

where $J_i : \Omega \to \Omega$. As a result, the set $N(t)$ is also defined for $t = t_i$ and therefore $N(i) = B(i)M(i)$.

With this choice of $M(i), N(i)$ and $B(i)$, the differential equation with impulses at fixed times may be described by

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)), & t \neq t_i \\ \Delta x(t_i) = J_i(x(t_i)), & t = t_i, i \in \mathbb{N} \\ x(t_0) = x_0. \end{cases} \tag{2.24}$$

Example 2.28. Consider the impulsive differential equation

$$\begin{cases} x'(t) = 1 + [x(t)]^2, & t \neq t_i, \\ \Delta x(t_i) = -1, & t_i = \frac{i\pi}{4},\ i \in \mathbb{N}. \end{cases} \tag{2.25}$$

The solution $x(t)$ with $x(0) = 0$ is continual for all $t \geq 0$. In fact, we have $x(t) = tan(t - \frac{i\pi}{4})$, $t \in (\frac{i\pi}{4}, \frac{(i+1)\pi}{4}]$ which is periodic with period $\frac{\pi}{4}$. However, the corresponding differential equation has the solution $x(t) = tant$ whose interval of existence is $[0, \frac{\pi}{2})$ since $\lim_{t \to \frac{\pi}{2}^-} x(t) = +\infty$. This means that we can control blow-up system to periodic bounded solution by using an impulsive control.

2.8 Definition of Stability

In general, the stability notions of a given solution $y(t)$ of the impulsive differential system (2.24) can not be transfered to the stability notions of the trivial solution by change of variables. This is because of the fact that moments of impulse effect of $y(t)$ need not the same as that of a neighboring solution $x(t)$ of (2.24) and consequently demanding that the difference $|x(t) - y(t)|$ be small for all $t \geq 0$ seems unnatural. For convenience, we will define the notation by

$$x(t) = x(t, t_0, x_0) \qquad \text{and} \qquad y(t) = x(t, t_0, y_0).$$

Definition 2.29. *Let $y(t)$ be a given solution of (2.24) existing for $t \geq t_0$ and suppose that $y(t)$ has impulses at $t = t_i$ such that $t_i < t_{i+1}$ and $t_i \to +\infty$ as $i \to +\infty$. Then the solution $y(t)$ is said to be asymptotically stable if for each $\epsilon > 0$, then exists $\delta > 0$ such that $|x_0 - y_0| < \delta$ implies $|x(t) - y(t)| < \epsilon$ for $t \geq t_0$ and $|t - t_i| > \epsilon$.*