CHAPTER III

Main Results

In this chapter, we prove the existence and uniqueness of the classical piecewise continuous almost periodic solution of the system (2.24) and then we investigate asymptotically stable of this solution.

3.1 The inhomogeneous initial value problem with impulses

Before proving the existence and uniqueness of the classical piecewise continuous almost periodic solution of the system (2.24), we first consider the impulsive differential system:

$$\begin{cases} u'(t) + Au(t) = g(t), & t \neq t_i \ ; \\ \Delta u(t_i) = h_i(t_i), & t = t_i, \ i \in \mathbb{N}; \\ u(t_0) = u_0 \end{cases} \tag{3.1}$$

where $\{t_i\}$ is a sequence such that $\{t_i^j\}$ is EAP, $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ satisfying exponential stability ,i.e., $||S(t)|| \leq Me^{-\beta t}$ for some $M, \ \beta > 0$ for all $t \geq 0$ on Banach space$X$, g is PCAP, h_i is PCAP for each $i \in \mathbb{N}$ and $\Delta u(t_i) = u(t_i^+) - u(t_i^-) = u(t_i^+) - u(t_i)$ denote the jump of u at t_i with the size of jump h_i and we assume that $\inf t_j^1 = \xi > 0$.

To get a solution of (3.1), we start constructing the solution by considering the differential equation:

$$\begin{cases} u'(t) + Au(t) = g(t), & t_0 < t < t_1 \\ u(t_0) = u_0. \end{cases} \tag{3.2}$$

Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ and let u_1 be a solution of (3.2). Then the X value function $w(s) = S(t-s)u_1(s)$ is differentiable for $t_0 < s < t$ and

$$\begin{aligned}\frac{dw}{ds} &= S(t-s)u_1'(s) + AS(t-s)u_1(s)\\ &= -S(t-s)Au_1(s) + S(t-s)g(s) + AS(t-s)u_1(s)\\ &= S(t-s)g(s) \qquad (3.3)\end{aligned}$$

If $g \in L^1(t_0, t_1)$, then $S(t-s)g(s)$ is integrable in the sense of Bochner and integrating (3.3) from t_0 to t yields

$$\begin{aligned} w(s)\mid_{t_0}^{t} &= \int_{t_0}^{t} S(t-s)g(s)ds\\ w(t) - w(t_0) &= \int_{t_0}^{t} S(t-s)g(s)ds\\ u_1(t) - S(t-t_0)u_0 &= \int_{t_0}^{t} S(t-s)g(s)ds.\end{aligned}$$

Therefore $u_1(t) = S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)g(s)ds$.

Now, we define

$$u(t_1) = S(t_1 - t_0)u_0 + \int_{t_0}^{t_1} S(t_1 - s)g(s)ds. \qquad (3.4)$$

So that u_1 is left continuous at t_1.

Next, we consider the equation:

$$\begin{cases} u'(t) + Au(t) = g(t), & t_1 < t < t_2\\ u(t_1) = u_1(t_1) + h_1(t_1). \end{cases} \qquad (3.5)$$

Let u_2 be a solution of (3.5). Then for similar process, we have

$$\begin{aligned} u_2(t) &= S(t-t_1)[u_1(t_1) + h_1(t_1)] + \int_{t_1}^{t} S(t-s)g(s)ds\\ &= S(t-t_1)[S(t_1-t_0)u_0 + \int_{t_0}^{t_1} S(t_1-s)g(s)ds + h_1(t_1)]\\ &+ \int_{t_1}^{t} S(t-s)g(s)ds\end{aligned}$$

$$= S(t-t_1)S(t_1-t_0)u_0 + \int_{t_0}^{t_1} S(t-t_1)S(t_1-s)g(s)ds$$

$$+ \int_{t_1}^{t} S(t-s)g(s)ds + S(t-t_1)h_1(t_1)$$

$$= S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)g(s)ds + S(t-t_1)h_1(t_1). \tag{3.6}$$

It is easy to see that this procedure can be repeated for $t_{k-1} < t < t_k$, for $k \in \mathbb{N}$ to get a solution

$$u_k(t) = S(t-t_{k-1})[u_{k-1}(t_{k-1}) + h_{k-1}(t_{k-1})] + \int_{t_{k-1}}^{t} S(t-s)g(s)ds. \tag{3.7}$$

Now, define

$$u(t) = \begin{cases} u_1(t), & t_0 \le t < t_1 \\ u_i(t), & t_{k-1} \le t < t_k, k = 2, 3, ..., p \\ u_{p+1}(t), & t_p \le t < t_{p+1} \\ \dots \\ \dots \end{cases} \tag{3.8}$$

Next, we use induction to show that (3.7) is satisfied on $[t_0, \infty)$. First, (3.7) is satisfied on $[t_0, t_1)$. If (3.7) is satisfied on (t_{k-1}, t_k), then for $t \in (t_k, t_{k+1})$,

$$u(t) = u_{k+1} = S(t-t_k)[u_k(t_k) + h_k(t_k)] + \int_{t_k}^{t} S(t-s)g(s)ds$$

$$= S(t-t_k)\left[S(t_k-t_0)u_0 + \int_{t_0}^{t_k} S(t_k-s)g(s)ds\right.$$

$$\left. + \sum_{t_0<t_i<t_k} S(t_k-t_i)h_i(t_i) + h_k(t_k)\right] + \int_{t_k}^{t} S(t-s)g(s)ds$$

$$u(t) = S(t-t_k)S(t_k-t_0)u_0 + \int_{t_0}^{t_k} S(t-t_k)S(t_k-s)g(s)$$

$$+ \sum_{t_0<t_i<t_k} S(t-t_k)S(t_k-t_i)h_i(t_i) + S(t-t_k)h_k(t_k) + \int_{t_k}^{t} S(t-s)g(s)ds$$

$$= S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)g(s)ds + \sum_{t_0<t_i<t} S(t-t_i)h_i(t_i).$$

Thus (3.7) is also true on $(t_k, t_{k+1}]$. Therefore (3.7) is true on $[t_0, \infty)$. Moreover, by using the exponential stability of analytic semigroup $\{S(t)\}_{t\geq 0}$ and a standard property of the almost periodicity for g and h_i for each $i \in \mathbb{N}$ we have $u(t)$ is bounded.

Definition 3.1. *A function $u(.) \in PC$ of the integral equation*

$$u(t) = S(t-t_0)u_0 + \int_{t_0}^{t} S(t-t_0)g(s)ds + \sum_{t_o<t_i<t} S(t-t_i)h_i(t_i) \qquad (3.9)$$

will be called a mild solution of system (3.1).

Definition 3.2. *A function $u(.) \in PC$ is a classical solution of system (3.1) if $u(t)$ is continuously differentiable for all $t \in \Re - \{t_i\}$, $u(.) \in D(A)$ and (3.1) is satisfied.*

Remark 3.3 ([2]). When $-A$ generates a semigroup with negative exponential, we have the fact that if $u(.)$ is a bounded mild solution of (3.1) on $\Re$. Then we can take the limit as $t_0 \to -\infty$ on the right hand of (3.9) to obtain

$$u(t) = \int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i). \qquad (3.10)$$

Conversely, if $u(.)$ is bounded and (3.10)is verified, then $u(.)$ is a mild solution of system (3.1).

The following lemmas are used in the proof of the the existence and uniqueness of the classical piecewise continuous almost periodic solution of the system (3.1).

Lemma 3.4 ([13]). *Let $f : \Re \to X$ be a PCAP function with $\{t_i\}$ be a sequence such that $\{t_i^j\}$ is EAP and*

$$inf t_j^1 = \xi > 0.$$

Then $\{f(t_i)\}$ is an almost periodic sequence.

Lemma 3.5. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ such that $||S(t)|| \leq Me^{-\beta t}$ for some M, $\beta > 0$ for all $t \geq 0$ on Banach space$X$. Let g be a PCAP function and h_i be almost periodic for each $i \in \mathbb{N}$. Assume that $inft_j^1 = \xi > 0$. For each $t \in \Re$ we define the integral equation by*

$$u(t) = \int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i). \tag{3.11}$$

Then $u(t)$ is bounded and PCAP.

Proof. To prove $u(t)$ is bounded, let $t \in (t_k, t_{k+1}]$ with a fixed k, we have

$$|u(t)| \leq \int_{-\infty}^{t} |S(t-s)||g(s)|ds + \sum_{t_i<t} |S(t-t_i)||h_i(t_i)|$$

By using a standard property of the almost periodicity for g and h_i for each $i \in \mathbb{N}$,

$$N \triangleq \sup_{t\in\Re} |g(t)| + \sup_{t\in\Re} |h_i(t)| < +\infty.$$

Then

$$\begin{aligned} |u(t)| &\leq NM \left[\int_{-\infty}^{t} e^{-\beta(t-s)}ds + \sum_{t_i<t} e^{-\beta(t-t_i)} \right] \\ &= NM \left[\frac{1}{\beta} + e^{-\beta(t-t_k)} \sum_{t_i<t} e^{-\beta\sum_{j=i}^{k-1}(t_{j+1}-t_j)} \right] \\ &\leq NM \left[\frac{1}{\beta} + \sum_{i=-\infty}^{k-1} e^{-\beta\xi(k-1-i)} \right] \\ &= NM \left[\frac{1}{\beta} + \frac{1}{1-e^{-\beta\xi}} \right] < +\infty. \end{aligned}$$

Therefore $u(t)$ is bounded.

Next, we show that $u(t)$ is PCAP. Since $g(t)$ is PCAP and $h_i(t)$ is almost periodic in i. Hence for each $\epsilon > o$ there exist a set P_ϵ relatively dense of $\epsilon -$ *almost periods* such that

$$|g(t+\tau) - g(t)| < \epsilon$$

and

$$|h_i(t+\tau) - h_i(t)| < \epsilon$$

for all $t \in \Re$, $\tau \in P_\epsilon$ and $|t - t_i| > \epsilon$ for all $i \in \mathbb{N}$. We obtain

$$\begin{aligned}
|u(t+\tau) - u(t)| &\leq \left| \int_{-\infty}^{t+\tau} S(t+\tau-s)g(s)ds - \int_{-\infty}^{t} S(t-s)g(s)ds \right| \\
&+ \left| \sum_{t_i < t+\tau} S(t+\tau-t_i)h_i(t_i) - \sum_{t_i<t} S(t-t_i)h_i(t_i) \right| \\
&= \left| \int_{-\infty}^{t} S(t-s)g(s+\tau)ds - \int_{-\infty}^{t} S(t-s)g(s)ds \right| \\
&+ \left| \sum_{t_i<t} S(t-t_i)h_i(t_i+\tau) - \sum_{t_i<t} S(t-t_i)h_i(t_i) \right| \\
&\leq \int_{-\infty}^{t} |S(t-s)| \, |g(s+\tau) - g(s)| \, ds \\
&+ \sum_{t_i<t} |S(t-t_i)| \, |h_i(t_i+\tau) - h_i(t_i)| \\
&\leq \epsilon M \left[\int_{-\infty}^{t} e^{-\beta(t-s)} ds + \sum_{t_i<t} e^{-\beta(t-t_i)} \right] \\
&\leq \epsilon \left[\frac{1}{\beta} + \frac{1}{1-e^{-\beta\xi}} \right] = \epsilon B
\end{aligned}$$

where $B = \left[\frac{1}{\beta} + \frac{1}{1-e^{-\beta\xi}}\right]$. This implies that $u(t)$ is PCAP. □

From the assumptions of Lemma3.5, we give some conditions for $g : \Re \to X$ and $h_i : \Re \to X$, $i \in \mathbb{N}$ such that lead $u(t)$ to a classical solution (regularity) of system (3.1).

Let I be an interval. A function $f : I \to X$ is hölder continuous with exponent ν, $0 < \nu < 1$ on I if there is a constant $L > 0$ such that

$$||f(t) - f(s)||_X \leq L|t-s|^\nu$$

for all s, $t \in I$. It is locally hölder continuous if every $t \in I$ has a neighborhood in which f is hölder continuous.

Lemma 3.6. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ such that $||S(t)|| \leq Me^{-\beta t}$ for some M, $\beta > 0$ for all $t \geq 0$ on Banach spaceX and let g be a locally Hölder continuous function on $\Re$. Assume that*

$0 \in \rho(A)$. *If*

$$v(t) = \int_{-\infty}^{t} S(t-s)g(s)ds$$

then $A^\alpha v(t)$ *is locally hölder continuous on* $\Re$ *for all* $0 < \alpha \leq 1$.

Proof. Since A^α is closed so is $A^\alpha S(t)$. By Theorem 2.16(a), $A^\alpha S(t)$ is every where defined and therefore by the closed graph theorem A^α is bounded.

Let $0 < \alpha \leq 1$. Then using (2.20) we have

$$\begin{aligned}
|A^\alpha v(t+h) - A^\alpha v(t)| &= \left| \int_{-\infty}^{t+h} A^\alpha S(t+h-s)g(s)ds - \int_{-\infty}^{t} A^\alpha S(t-s)g(s)ds \right| \\
&= \left| \int_0^\infty A^\alpha S(s)[g(t-s+h) - g(t-s)]ds \right| \\
&\leq \int_0^\infty |A^\alpha S(s)| \, |g(t-s+h) - g(t-s)| \, ds \\
&\leq Ch^r \int_0^\infty |A^\alpha S(s)| \, ds \\
&= Ch^r \int_0^\infty \left| A^{\alpha-1} AS(s) \right| ds \\
&\leq Ch^r \int_0^\infty \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \tau^{-\alpha} \left| AS(s+\tau) \right| d\tau ds \\
&\leq \frac{CM_1 h^r}{\Gamma(1-\alpha)} \int_0^\infty \int_0^\infty \tau^{-\alpha}(s+\tau)^{-1} e^{-\beta(s+\tau)} d\tau ds \\
&\leq \frac{CM_1 h^r}{\Gamma(1-\alpha)} \int_0^\infty e^{-\beta s} s^{-\alpha} \int_0^\infty u^{-\alpha}(1+u)^{-1} duds \\
&= \frac{CM_1 h^r}{\Gamma(1-\alpha)} \left[\int_0^\infty e^{-\beta s} s^{-\alpha} ds \right] \left[\int_0^\infty u^{-\alpha}(1+u)^{-1} du \right] \\
&= \left[\frac{C' M_\alpha \beta^{\alpha-1} \Gamma(1-\alpha)}{\Gamma(1-\alpha)} \right] h^r = C_\alpha h^r
\end{aligned}$$

where $C_\alpha = C' M_\alpha \beta^{\alpha-1}$. Therefore the proof is complete. □

Lemma 3.7. *Let* $-A$ *be the infinitesimal generator of an analytic semigroup* $\{S(t)\}_{t\geq 0}$ *such that* $||S(t)|| \leq Me^{-\beta t}$ *for some* M, $\beta > 0$ *for all* $t \geq 0$ *on Banach space*X, *sequence* $\{t_i\}$ *be such that* $\{t_i^j\}$ *is EAP and let* h_i. $i \in \mathbb{N}$ *be a PCAP function such that locally Hölder continuous with the same exponent on* $\Re$. *Assume that* $0 \in \rho(A)$ *and* $inf t_j^1 = \xi > 0$. *If*

$$w(t) = \sum_{t_i < t} S(t-t_i)h_i(t_i)$$

then $Aw(t)$ is locally Hölder continuous on $\Re$.

Proof. For fixed $k \in \mathbb{N}$, let $t, t+h \in [t_k, t_{k+1})$. Then

$$\begin{aligned}
|Aw(t+h) - Aw(t)| &= \left| \sum_{t_i<t+h} AS(t+h-t_i)h_i(t_i) - \sum_{t_i<t} AS(t-t_i)h_i(t_i) \right| \\
&= \left| \sum_{t_i<t} h_i(t_i+h) - \sum_{t_i<t} AS(t-t_i)h_i(t_i) \right| \\
&\leq \sum_{t_i<t} |AS(t-t_i)|\,|h_i(t_i+h) - h_i(t_i)| \\
&\leq CM_1 h^r \sum_{t_i<t} (t-t_i)^{-1} e^{-\beta(t-t_i)} \\
&\leq CM_1 \xi^{-1} h^r e^{-\beta t} \sum_{t_i<t} e^{\beta t_i} \\
&\leq CM_1 \xi^{-1} h^r e^{-\beta(t-t_k)} \sum_{t_i<t} e^{-\beta(t_k-t_i)} \\
&\leq CM_1 \xi^{-1} h^r \sum_{i=-\infty}^{i=k} e^{-\beta \sum_{j=1}^{j=i}(t_{j+1}-t_j)} \\
&\leq CM_1 \xi^{-1} h^r \sum_{i=-\infty}^{i=k} e^{-\beta\xi(k-i)} \\
&= CM_1 \xi^{-1} h^r \sum_{i=0}^{i=\infty} e^{-\beta\xi i} \\
&= CM_1 \xi^{-1} \left[\frac{1}{1-e^{-\beta\xi}} \right] h^r = C' h^r.
\end{aligned}$$

where $C' = CM_1\xi^{-1}\left[\frac{1}{1-e^{-\beta\xi}}\right]$. This implies that $Aw(t)$ is locally Hölder continuous on $\Re$. □

Theorem 3.8. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ such that $||S(t)|| \leq Me^{-\beta t}$ for some M, $\beta > 0$ for all $t \geq 0$ on Banach spaceX, sequence $\{t_i\}$ be such that $\{t_i^j\}$ is EAP. Assume that $0 \in \rho(A)$ and $inf t_j^1 = \xi > 0$. If $g : \Re \to X$ is PCAP and $h_i : \Re \to X$ is PCAP for each $i \in \mathbb{N}$ and both are locally Hölder continuous, then there exists a unique classical PCAP solution* *over $\Re$ of the system (3.1). Moreover, the solution is*

$$u(t) = \int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i). \tag{3.12}$$

Proof. Now, we show that $u(t)$ defining as (3.12) is a classical solution of system (3.1), let $t, t+h \in (t_k, t_{k+1}]$.
We obtain

$$\begin{aligned} u(t+h) - u(t) &= \left[\int_{-\infty}^{t+h} S(t+h-s)g(s)ds + \sum_{t_i<t+h} S(t+h-t_i)h_i(t_i)\right] \\ &- \left[\int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i)\right] \\ &= S(h)\left[\int_{-\infty}^{t+h} S(t-s)g(s)ds + \sum_{t_i<t+h} S(t-t_i)h_i(t_i)\right] \\ &- \left[\int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i)\right] \\ &= S(h)\left[\int_{-\infty}^{t} S(t-s)g(s)ds + \int_{t}^{t+h} S(t-s)g(s)ds \right. \\ &\left. + \sum_{t_i<t} S(t-t_i)h_i(t_i) + \sum_{t<t_i<t+h} S(t-t_i)h_i(t_i)\right] \\ &- \left[\int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i)\right] \end{aligned}$$

Since $t, t+h \in (t_k, t_{k+1}]$, $\sum_{t<t_i<t+h} S(t-t_i)h_i(t_i)$ is vanish. Then

$$\begin{aligned} \frac{u(t+h)-u(t)}{h} &= \frac{(S(h)-I)}{h}\left[\int_{-\infty}^{t} S(t-s)g(s)ds + \sum_{t_i<t} S(t-t_i)h_i(t_i)\right] \\ &+ \frac{S(h)}{h}\int_{t}^{t+h} S(t-s)g(s)ds \end{aligned} \tag{3.13}$$

From the continuity of g, it is clear that the second term on the right-hand sides of (3.13) has the limit $g(t)$ as $h \to 0$. Then by taking $h \to 0$, we have $u'(t) = -Au(t) + g(t)$ for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$ and $u(t) \in D(A)$. By assumption, Lemma

3.6 and Lemma 3.7 imply that $u'(t) = -Au(t) + g(t)$ is continuous on $\Re - \{t_i\}$. Next, we define

$$u(t_k) = \int_{-\infty}^{t_k} S(t_k - s)g(s)ds + \sum_{t_i < t_k} S(t_k - t_i)h_i(t_i).$$

By assumption, we have

$$\begin{aligned}
\Delta u(t_k) &= u(t_k^+) - u(t_k^-) = u(t_k^+) - u(t_k) \\
&= \lim_{h \to 0} \left[\int_{-\infty}^{t_k+h} S(t_k + h - s)g(s)ds + \sum_{t_i < t_k + h} S(t_k + h - t_i)h_i(t_i) \right] \\
&\quad - \left[\int_{-\infty}^{t_k} S(t_k - s)g(s)ds + \sum_{t_i < t_k} S(t_k - t_i)h_i(t_i) \right] \\
&= \lim_{h \to 0} \left[\int_{-\infty}^{t_k+h} [S(t_k + h - s) - S(t_k - s)]g(s)ds \right] \\
&\quad + \lim_{h \to 0} \left[\sum_{t_i < t_k + h} [S(t_k + h - t_i) - S(t_{k-t_i})h_i(t_i) \right] \\
&\quad + \lim_{h \to 0} S(h) \left[\int_{t_k}^{t_k+h} S(t_k - s)g(s)ds + \sum_{t_k \le t_i < t_k + h} S(t_k - t_i)h_i(t_i) \right] \\
&= \lim_{h \to 0} [S(h) - I] \int_{-\infty}^{t_k} S(t_k - s)g(s)ds \\
&\quad + \lim_{h \to 0} [S(h) - I] \sum_{t_i < t_k} S(t_k - t_i)h_i(t_i) \\
&\quad + \lim_{h \to 0} S(h) \left[\int_{t_k}^{t_k+h} S(t_k - s)g(s)ds + h_k(t_k) \right] \\
&= h_k(t_k).
\end{aligned}$$

Therefore the proof is complete. □

By Remark 3.3, we can see that the solution $u(t)$ which is defined by $u(t) = u(t, u_0)$ with initial value $u(t_0, u_0) = u_0$ of system (3.1) can be given by

$$u(t) = S(t - t_0)u_0 + \int_{t_0}^{t} S(t - t_0)g(s)ds + \sum_{t_0 < t_i < t} S(t - t_i)h_i(t_i).$$

By following from the proof of theorem 3.8, this mild solution is classical.

The proof of the existence and uniqueness of the solution for system (3.1) is finished and we use these results to prove the existence and uniqueness of the solution for the nonlinear impulsive system :

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)), \quad t \neq t_i \ ; \\ \Delta x(t_i) = J_i(x(t_i), \quad t = t_i \ , i \in \mathbb{N}; \\ x(t_0) = x_0 \end{cases} \tag{3.14}$$

where the sequence $\{t_i\}$ is such that $\{t_i^j\}$ is EAP and $\inf t_i^1 = \xi > 0$, $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ satisfying the exponential stability, i.e., $||S(t)|| \leq Me^{-\beta t}$ for some $M,\ \beta > 0$ for all $t \geq 0$ on Banach space$X$, $J_i(x)$ is PCAP for each $i \in \mathbb{N}$, $f(t, x)$ is PCAP at t.
Now, we assume that the function $f : \Re \times X_\alpha \to X$ and $J_i : X_\alpha \to X,\ i \in \mathbb{N}$ satisfy the condition:
(JF) There are constants $L > 0$ and $0 < \theta < 1$ such that

$$|f(t_1, x_1) - f(t_2, x_2)| \leq L(|t_1 - t_2|^\theta + |x_1 - x_2|_\alpha)$$

and

$$|J_i(x_1) - J_i(x_2)| \leq L\,|x_1 - x_2|_\alpha$$

for all $x_1, x_2 \in X_\alpha$ and $t_1, t_2 \in \Re$
where X is a real or complex Banach space with norm $|.|$, A^α is the fractional power and X_α is the Banach space $D(A^\alpha)$ endowed with the norm $|x|_\alpha = |A^\alpha x|$.

The initial value problem (3.14) does not necessarily have a solution of any kind. However, if it has a solution then the argument given at the beginning of the head of this section shows that this solution $x(t)$ satisfies the integral equation

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)f(s, x(s))ds + \sum_{t_0 < t_i < t} S(t - t_i)J_i(x(t_i)) \tag{3.15}$$

and this integral equation $x(t)$ is called a mild solution of system (3.14). Moreover, by using the exponential stability of analytic semigroup $\{S(t)\}_{t\geq 0}$ and a standard property of the almost periodicity for f and J_i for each $i \in \mathbb{N}$ we have $x(t)$ is bounded.

Definition 3.9. *A classical solution of system (3.14) is a function* $x(.) \in PC(\Re, X) \cap C^1(\Re - \{t_i\}, X)$ *and* $x(.) \in D(A)$ *which satisfies (3.14) on* $\Re$.

From the above assumptions, we prove following lemmas.

Lemma 3.10. *Let* $f : \Re \times \Omega \to X$, $\Omega \subseteq X$ *be a uniformly PCAP function and* $y : \Re \to \Omega$ *is PCAP such that* $\overline{R(y)} \subseteq \Omega$. *Then* $t \longmapsto f(t, x(t))$ *is PCAP.*

Proof. Given $\epsilon > 0$. Following the assumptions, there is a relatively dense set P_ϵ of $\Re$ such that

$$|f(t+\tau, y(t+\tau)) - f(t, y(t+\tau))| \leq \frac{\epsilon}{2}$$

for all $(t, \tau, y) \in \Re \times S_\epsilon \times \Omega$ and $|t - t_i| > \epsilon$, $i \in \mathbb{N}$.
Since y is PCAP and $\overline{R(y)} \subseteq \Omega$, there exist $\delta = \delta(\epsilon) > 0$ such that $|y(t+\tau) - y(t)| < \delta$ and

$$|f(t, y(t+\tau) - f(t, y(t)))| < \frac{\epsilon}{2}.$$

We obtain

$$\begin{aligned} |f(t+\tau, y(t+\tau)) - f(t, y(t))| &\leq |f(t+\tau, y(t+\tau)) - f(t, y(t+\tau))| \\ &\quad + |f(t, y(t+\tau)) - f(t, y(t))| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\leq \epsilon. \end{aligned}$$

Therefore the proof is complete. □

Corollary 3.11. *Let* $f : \Re \times \Omega \to X$, $\Omega \subseteq X$ *be a uniformly PCAP function which satisfies Lipchitz condition, i.e., there is a constant* $L > 0$ *such that* $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$ *for all* $x_1, x_2 \in X$ *and* $y : \Re \to \Omega$ *is PCAP. Then* $t \longmapsto f(t, x(t))$ *is PCAP.*

Proof. From the assumptions and following the proof of Lemma 3.10, we have

$$\begin{aligned}|f(t+\tau,y(t+\tau))-f(t,y(t))| &\leq |f(t+\tau,y(t+\tau))-f(t,y(t+\tau))| \\ &\quad + |f(t,y(t+\tau))-f(t,y(t))| \\ &\leq \epsilon + L\,|y(t+\tau)-y(t)| \\ &\leq \epsilon + L\delta \\ &\leq \epsilon(1+L), \text{ by choosing } \delta<\epsilon.\end{aligned}$$

Therefore the proof is complete. □

Remark 3.12. When $-A$ generates a semigroup with negative exponential, we have the fact that if $x(.)$ is a bounded mild solution of (3.14) on $\Re$, then we can take the limit as $t_0 \to -\infty$ on the right hand of (3.15) to obtain

$$x(t) = \int_{-\infty}^{t} S(t-s)f(t,x(s))ds + \sum_{t_i<t} S(t-t_i)J_i(x(t_i)). \tag{3.16}$$

Conversely, if $x(.)$ is bounded and (3.16) is verified, then $x(.)$ is a mild solution of (3.14).

We define the set

$$PCAP(X) \triangleq \{u(.) : \Re \to X \mid u(.) \text{ is PCAP}\}$$

with the usual supremum norm over $\Re$ denote by

$$|x|_\infty \triangleq \sup_{t\in\Re} |x(t)|.$$

Then $PCAP(X)$ is a Banach space.

From Remark 3.12, we define the operator F by

$$Fx(t) \triangleq \int_{-\infty}^{t} A^\alpha S(t-s)f(s,A^{-\alpha}x(s))ds + \sum_{t_i<t} A^\alpha S(t-t_i)J_i(A^{-\alpha}x(t_i)) \tag{3.17}$$

for all $x(.) \in PCAP(X)$.

We must show that F is well-defined.

Let $x(.) \in PCAP(X)$. Since $x(.)$ is PCAP, it follows from Lemma 3.4, the sequence

$\{x(t_i)\}$ is almost periodic. Therefore $\{J_i(x(t_i))\}, i \in \mathbb{N}$ is also almost periodic. It follows from Lemma 3.10. and assumptions of f that $t \longmapsto f(t, A^{-\alpha}x(t))$ is PCAP. By using a standard properties of the almost periodicity, we have

$$N \triangleq \sup_{t\in\Re} \left|f(t, A^{-\alpha}x(t))\right| + \sup_{t\in\Re} \left|J_i(A^{-\alpha}x(t))\right| < +\infty \tag{3.18}$$

We assume that $\inf t_j^1 = \xi > 0$ and for fixed k let $t \in (t_k, t_{k+1}]$.

By Theorem 2.16(c), we have

$$\begin{aligned} |Fx(t)| &\leq \left| \int_{-\infty}^{t} A^{\alpha}S(t-s)f(s, A^{-\alpha}x(s))ds \right| + \left| \sum_{t_i<t} A^{\alpha}S(t-t_i)J_i(A^{-\alpha}x(t_i)) \right| \\ &\leq M_\alpha N \left[\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\beta(t-s)}ds + \sum_{t_i<t}(t-t_i)^{-\alpha}e^{-\beta(t-t_i)} \right]. \end{aligned} \tag{3.19}$$

With changing of variable and $\inf t_j^1 = \xi > 0$, we have

$$\begin{aligned} |Fx(t)| &\leq M_\alpha N \left[\int_{0}^{\infty} s^{-\alpha}e^{-\beta s}ds + \xi^{-\alpha}e^{-\beta t}\sum_{t_i<t} e^{\beta t_i} \right] \\ &\leq M_\alpha N \left[\beta^{\alpha}\Gamma(1-\alpha) + \xi^{-\alpha}e^{-\beta(t-t_k)}\sum_{t_i<t} e^{-\beta(t_k-t_i)} \right] \end{aligned} \tag{3.20}$$

where $\Gamma(.)$ is the classical gamma function. We use the well known identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin\pi\alpha}$ for $0 < \alpha < 1$, we have

$$\begin{aligned} |Fx(t)| &\leq M_\alpha N \left[\frac{\beta^{\alpha}\pi}{\Gamma(\alpha)sin\pi\alpha} + \xi^{-\alpha} \sum_{i=-\infty}^{i=k} e^{-\beta\sum_{j=i}^{j=k-1}(t_{j+1}-t_j)} \right] \\ &\leq M_\alpha N \left[\frac{\beta^{\alpha}\pi}{\Gamma(\alpha)sin\pi\alpha} + \xi^{-\alpha} \sum_{i=-\infty}^{i=k} e^{-\beta\xi(k-i)} \right] \\ &\leq M_\alpha N \left[\frac{\beta^{\alpha}\pi}{\Gamma(\alpha)sin\pi\alpha} + \frac{\xi^{-\alpha}}{1-e^{-\beta\xi}} \right] < +\infty \end{aligned} \tag{3.21}$$

which show that $Fx(t)$ exists.

Lemma 3.13. *The operator F is well-defined and map $PCAP(X)$ into it self.*

Proof. For $x(.) \in PCAP(X)$, since $t \longmapsto f(t, A^{-\alpha}x(t))$ and $J_i(x(t)), i \in \mathbb{N}$ are PCAP. Hence for each $\epsilon > 0$ there exist a relatively dense set P_ϵ of $\epsilon-$*almost periods* such that

$$|f(t+\tau, A^{-\alpha}x(t+\tau)) - f(t, A^{-\alpha}x(t))| \leq \epsilon$$

and

$$|J_i(A^{-\alpha}x(t+\tau)) - J_i(A^{-\alpha}x(t))| \leq \epsilon$$

for all $t \in \Re$, $\tau \in P_\epsilon$ and $|t - t_i| > \epsilon$ for all $i \in \mathbb{N}$.

We obtain

$$\begin{aligned}
&|Fx(t+\tau) - Fx(t)| \\
&\leq \left| \int_{-\infty}^{t+\tau} A^\alpha S(t+\tau-s) f(s, A^{-\alpha}x(s))ds - \int_{-\infty}^{t} A^\alpha S(t-s) f(s, A^{-\alpha}x(s))ds \right| \\
&+ \left| \sum_{t_i<t+\tau} A^\alpha S(t+\tau-t_i) J_i(A^{-\alpha}x(t_i)) - \sum_{t_i<t} A^\alpha S(t-t_i) J_i(A^{-\alpha}x(t_i)) \right| \\
&= \left| \int_{-\infty}^{t} A^\alpha S(t-s) f(s+\tau, A^{-\alpha}x(s+\tau))ds - \int_{-\infty}^{t} A^\alpha S(t-s) f(s, A^{-\alpha}x(s))ds \right| \\
&+ \left| \sum_{t_i<t} A^\alpha S(t-t_i) J_i(A^{-\alpha}x(t_i+\tau)) - \sum_{t_i<t} A^\alpha S(t-t_i) J_i(A^{-\alpha}x(t_i)) \right| \\
&\leq \int_{-\infty}^{t} |A^\alpha S(t-s)| \left| f(s+\tau, A^{-\alpha}x(s+\tau)) - f(s, A^{-\alpha}x(s)) \right| ds \\
&+ \sum_{t_i<t} |A^\alpha S(t-t_i)| \left| J_i(A^{-\alpha}x(t_i+\tau)) - J_i(A^{-\alpha}x(t_i)) \right| \\
&\leq \epsilon M_\alpha \left[\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \sum_{t_i<t} (t-t_i)^{-\alpha} e^{-\beta(t-t_i)} \right] \\
&\leq \epsilon \left[\frac{\beta^\alpha \pi}{\Gamma(\alpha) sin \pi \alpha} + \frac{\xi^{-\alpha}}{1 - e^{-\beta\xi}} \right] = \epsilon B
\end{aligned} \tag{3.22}$$

Since ϵ is arbitrary, $Fx(.) \in PCAP(X)$. □

Corollary 3.14. *For each* $\epsilon > 0$, $0 < \alpha < 1$, *set*

$$B_\rho \triangleq \{x(.) \in PCAP(X) \mid |x|_\infty < \rho\}$$

if $\rho > M_\alpha N[\frac{\beta^\alpha \pi}{\Gamma(\alpha) sin \pi \alpha} + \frac{\xi^{-\alpha}}{1-e^{-\beta\xi}}]$, *then* $F(B_\rho) \subseteq B_\rho$, *i.e.*, $F : B_\rho \to B_\rho$.

Proof. The result is following in the proof of Lemma 3.13. □

Theorem 3.15. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ satisfying the exponential stability and let $\{t_i\}$ be a sequence such that $\{t_i^j\}$ is EAP and $inf t_j^1 = \xi > 0$. If $f : \Re \times X \to X$ is uniformly PCAP and $J_i : X \to X$ is uniformly PCAP for each $i \in \mathbb{N}$ such that both are satisfying the condition (JF). Then for positive constants L sufficiently small enough, there exists a unique a classical PCAP solution over $\Re$ of the system (3.14).*

Proof. Consider the operator $F : PCAP(X) \to PCAP(X)$
define by

$$Fu(t) \triangleq \int_{-\infty}^{t} A^{\alpha}S(t-s)f(s, A^{-\alpha}u(s))ds + \sum_{t_i<t} A^{\alpha}S(t-t_i)J_i(A^{-\alpha}u(t_i))$$

We will prove this theorem in the following steps;

1) To show the operator F has a fixed point, say y in $PCAP(X)$;

2) To show the fixed point y of F is locally Hölder continuous on $\Re$;

3) To show the map $t \longmapsto f(t, A^{-\alpha}u(t))$ and the map $t \longmapsto J_i(A^{-\alpha}u(t))$ are locally Hölder continuous on $\Re$.

Step 1). We show that F has a fixed point.
Let $x, y \in PCAP(X)$ and $t \in (t_k, t_{k+1}]$. Then

$$\begin{aligned} |Fx(t) - Fy(t)| \leq & \int_{-\infty}^{t} |A^{\alpha}S(t-s)| \left| f(s, A^{-\alpha}x(s)) - f(s, A^{-\alpha}y(s)) \right| ds \\ & + \sum_{t_i<t} |A^{\alpha}S(t-t_i)| \left| J_i(A^{-\alpha}x(t_i)) - J_i(A^{-\alpha}y(t_i)) \right|. \end{aligned} \tag{3.23}$$

From condition (JF) and Theorem2.16(c), we have

$$\begin{aligned} |Fx(t) - Fy(t)| &\leq LM_{\alpha} \left[\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \sum_{t_i<t} (t-t_i)^{-\alpha} e^{-\beta(t-t_i)} \right] \|x-y\|_{\alpha} \\ &\leq LM_{\alpha} \left[\frac{\beta\pi}{\Gamma(\alpha) sin\pi\alpha} + \frac{\xi^{-\alpha}}{1-e^{-\beta\xi}} \right] |x-y|_{\alpha} \\ &= \frac{LK(\alpha)}{N} |x-y|_{\alpha} \end{aligned} \tag{3.24}$$

where $K(\alpha) = NM_\alpha \left[\frac{\beta\pi}{\Gamma(\alpha) sin\pi\alpha} + \frac{\xi^{-\alpha}}{1-e^{-\beta\xi}}\right]$. Then we can deduce that F is a strictly contraction, provided L is sufficiently small, $L < \frac{N}{K(\alpha)}$. Then from the Banach fixed point theorem there exists $y \in PCAP(X)$ such that $Fy = y$.

Thus

$$y(t) = \int_{-\infty}^{t} A^\alpha S(t-s) f(s, A^{-\alpha} y(s)) ds + \sum_{t_i < t} A^\alpha S(t-t_i) J_i(A^{-\alpha} y(t_i)). \quad (3.25)$$

Since A^α is closed linear operator,

$$y(t) = A^\alpha \left[\int_{-\infty}^{t} S(t-s) f(s, A^{-\alpha} y(s)) ds + \sum_{t_i < t} S(t-t_i) J_i(A^{-\alpha} y(t_i))\right]. \quad (3.26)$$

Applying the operator $A^{-\alpha}$ on both sides of (3.26),

$$A^{-\alpha} y(t) = \int_{-\infty}^{t} S(t-s) f(s, A^{-\alpha} y(s)) ds + \sum_{t_i < t} S(t-t_i) J_i(A^{-\alpha} y(t_i)). \quad (3.27)$$

Step 2). We show that the fixed point y of F is locally Hölder continuous on $\Re$. By Theorem 2.16(d), we note that for every $h > 0$ and for every γ satisfying $0 < \gamma < 1 - \alpha$,

$$|(S(h) - I) A^\alpha S(t-s)| \leq C_\gamma h^\gamma |A^\alpha S(t-s)|. \quad (3.28)$$

Let $h > 0$ be such that $t + h \in (t_k, t_{k+1}]$ and $0 < \gamma < 1 - \alpha$. We obtain

$$\begin{aligned} |y(t) - y(t+h)| \leq & \left|\int_{-\infty}^{t} (S(h) - I) A^\alpha S(t-s) f(s, A^{-\alpha} y(s)) ds\right| \\ & + \left|\int_{t}^{t+h} A^\alpha S(t+h-s) f(s, A^{-\alpha} y(s)) ds\right| \\ & + \left|\sum_{t_i < t} (S(h) - I) A^\alpha S(t-t_i) J_i(A^{-\alpha} y(t_i))\right| \\ & + \left|\sum_{t < t_i < t+h} A^\alpha S(t+h-t_i) J_i(A^{-\alpha} y(t_i))\right| \\ = & |I_1| + |I_2| + |I_3| + |I_4|. \end{aligned} \quad (3.29)$$

It is easy to see that $|I_4|$ is vanish, since $t, t+h \in (t_k, t_{k+1}]$.

Using Theorem 2.16(c) and (3.28) we can estimate each of the terms of (3.29) separately:

$$|I_1| \le M_{\alpha+\gamma}NC_\gamma h^\gamma \int_{-\infty}^{t} (t-s)^{-(\alpha+\gamma)}e^{-\beta(t-s)}ds$$

$$= M_{\alpha+\gamma}NC_\gamma \left[\frac{\beta^{\alpha+\gamma}\pi}{\Gamma(\alpha+\gamma)sin\pi(\alpha+\gamma)}\right] h^\gamma = C_1 h^\gamma \tag{3.30}$$

$$|I_2| \le M_\alpha N \int_{t}^{t+h} (t+h-s)^{-\alpha}e^{-\beta(t+h-s)}ds$$

$$\le M_\alpha N \int_{t}^{t+h} (t+h-s)^{-\alpha}ds = M_\alpha N h^{1-\alpha} \le C_2 h^\gamma \tag{3.31}$$

$$|I_3| \le M_{\alpha+\gamma}NC_\gamma h^\gamma \sum_{t_i<t} (t-t_i)^{-(\alpha+\gamma)}e^{-\beta(t-t_i)}$$

$$\le M_{\alpha+\gamma}NC_\gamma \left[\frac{\xi^{-\alpha}}{1-e^{-\beta\xi}}\right] h^\gamma = C_3 h^\gamma. \tag{3.32}$$

Combining (3.29)-(3.32) with these estimates, we obtain y is locally Hölder continuous on $\Re$.

Let y be the solution of (3.25) and consider the system:

$$\begin{cases} x'(t) + Ax(t) = f(t, A^{-\alpha}y(t)), & t \ne t_i \\ \Delta x(t_i) = J_i(A^{-\alpha}y(t_i)), & t = t_i \end{cases} \tag{3.33}$$

Step 3). We show that $t \longmapsto f(t, A^{-\alpha}y(t))$ and $t \longmapsto J_k(A^{-\alpha}y(t))$ are locally Hölder continuous. From condition (JF) and y is locally Hölder continuous, there exists L_1, $L_2 > 0$ and $0 < \theta < 1$ such that

$$\left|f(t, A^{-\alpha}y(t)) - f(s, A^{-\alpha}y(s))\right| \le L_1(|t-s|^\theta + |y(t)-y(s)|_\alpha)$$

and

$$|y(t)-y(s)|_\alpha \le |y(t)-y(s)| \le L_2|t-s|^\gamma.$$

Then, we have

$$\left|f(t, A^{-\alpha}y(t)) - f(s, A^{-\alpha}y(s))\right| \le L_1(|t-s|^\theta + L_2|t-s|^\gamma)$$

$$\le L_3|t-s|^\lambda \tag{3.34}$$

where $\lambda = min\{\theta, \gamma\}$;

and

$$\begin{aligned} \left|J_k(A^{-\alpha}y(t)) - J_k(A^{-\alpha}y(s))\right| &\leq L_1 \left|y(t) - y(s)\right|_\alpha) \\ &\leq L_4 \left|t - s\right|^\gamma. \end{aligned} \tag{3.35}$$

Therefore $t \longmapsto f(t, A^{-\alpha}y(t))$ and $t \longmapsto J_k(A^{-\alpha}y(t))$ are locally Hölder continuous. By Theorem 3.8., the system (3.33) has a unique solution given by

$$x(t) = \int_{-\infty}^{t} S(t-s)f(s, A^{-\alpha}y(s))ds + \sum_{t_i<t} S(t-t_i)J_i(A^{-\alpha}y(t_i)). \tag{3.36}$$

Moreover, we have $x(.) \in D(A)$ for all $t \in \Re - \{t_i\}$ and a fortiori $x(.) \in D(A^\alpha)$. Applying the operator A^α on both sides of (3.36), we have

$$A^\alpha x(t) = \int_{-\infty}^{t} A^\alpha S(t-s)f(s, A^{-\alpha}y(s))ds + \sum_{t_i<t} A^\alpha S(t-t_i)J_i(A^{-\alpha}y(t_i)). \tag{3.37}$$

From (3.25) and (3.37) we already see that $x(t) = A^{-\alpha}y(t)$ is a solution of (3.14) as $t_0 \longrightarrow -\infty$. The uniqueness of y follows from the uniqueness of the solution of (3.33) and (3.15). Therefore the proof is complete. □

Moreover, by Remark 3.12 a solution $x(t)$ is defined by $x(t) = x(t, x_0)$ of system (3.14) with the initial value $x(t_0, x_0) = x_0$ can be given by

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)f(s, x(s))ds + \sum_{t_0<t_i<t} S(t-t_i)J_i(x(t_i)).$$

3.2 Asymptotic stability

In this section we study the behavior of the solutions for nonlinear impulsive differential equation system (3.14).

Lemma 3.16 ([13]). *For $t \geq t_0$ let a nonnegative piecewise continuous function $u(t)$ satisfy*

$$u(t) \leq c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0 \leq t_i < t} b_i u(t_i) \tag{3.38}$$

where $c \geq 0$, $b_i \geq 0$, $v(s) \geq 0$. $u(t)$ has discontinuous points of the first kind at $t = t_i$. Then we have

$$u(t) \leq c \left[\int_{t_0}^{t} v(s)ds \right] \prod_{t_0 \leq t_i < t} (1 + b_i). \tag{3.39}$$

Lemma 3.17 ([13]). *If $\{t_k^1\}$ is almost periodic then there exists the limit*

$$\lim_{T \to +\infty} \frac{N(t, t+T)}{T} = p \tag{3.40}$$

uniformly with respect to $t \in \Re$.

By Theorem 3.15. the solution $x(t) = x(t, x_0)$ of (3.14) can be given by

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^{t} S(t - s)f(s, x(s))ds + \sum_{t_0 < t_i < t} S(t - t_i)J_i(x(t_i)).$$

Then for any two solutions of system (3.14)

$$x(t) = x(t, x_0)$$

$$y(t) = y(t, y_0)$$

we have

$$\begin{aligned} |x(t) - y(t)| \leq\ & |S(t - t_0)|\,|x_0 - y_0| + \int_{t_0}^{t} |S(t - s)|\,|f(s, x(s)) - f(s, y(s))|\,ds \\ & + \sum_{t_0 < t_i < t} |S(t - t_i)|\,|J_i(x(t_i)) - J_i(y(t_i))| \\ & \leq Me^{-\beta(t - t_0)}\,|x_0 - y_0| + \int_{t_0}^{t} MLe^{-\beta(t-s)}\,|x(s) - y(s)|\,ds \\ & + \sum_{t_0 < t_i < t} MLe^{-\beta(t - t_i)}\,|x(t_i) - y(t_i)|\,. \end{aligned} \tag{3.41}$$

Set the transformation $u(t) \triangleq |x(t) - y(t)|\, e^{\beta}t$, we have

$$u(t) \leq \underbrace{Mu(t_0)}_{c} + \int_{t_0}^{t} \underbrace{ML}_{v(s)} u(s)ds + \sum_{t_0 < t_i < t} \underbrace{ML}_{b_i} u(t_i). \tag{3.42}$$

Since $c, v(s)$ and b_i are all positive and $u(s)$ is PCAP and by using Lemma 3.16. these imply that

$$u(t) \leq Mu(t_0) \left[\prod_{t_0 < t_i < t} (1 + ML) \right] e^{ML(t - t_0)} \tag{3.43}$$

that is

$$|x(t) - y(t)| \leq M\, |x_0 - y_0|\, e^{-\beta(t-t_0)} \left[\prod_{t_0<t_i<t} (1 + ML) \right] e^{ML(t-t_0)}. \quad (3.44)$$

Let $N(t, t_0)$ denote the number of impulses in the interval $[t_0, t)$. Then

$$|x(t) - y(t)| \leq M\, |x_0 - y_0|\, e^{-\beta(t-t_0)} (1 + ML)^{N(t,t_0)} e^{ML(t-t_0)}. \quad (3.45)$$

From Lemma 3.17. it follows that for any $\epsilon > 0$ there exists $K = K(\epsilon)$ such that

$$(1 + ML)^{N(t,t_0)} \leq K e^{(pln(1+ML)+\epsilon)(t-t_0)}. \quad (3.46)$$

Then

$$|x(t) - y(t)| \leq MKe^{-[\beta - pln(1+ML) - \epsilon - ML](t-t_0)} |x_0 - y_0|. \quad (3.47)$$

Since ϵ can be chosen arbitrary small and L can be chosen sufficiently small. Then for $t - t_0$ is big enough, the solutions of (3.14) are asymptotically stable.