CHAPTER I

INTRODUCTION

There are two sections in this chapter. We give the motivation that bring about this thesis in the first section. For the other, general facts for this research are given.

1.1 Motivation

It is known that the multiplicative structures of a ring, a semiring and a nearring are the same which are semigroups while the additive structures of a ring is an abelian group, of a semiring is a commutative semigroup and of a nearring is a group. Moreover, rings and semirings satisfy both the left and the right distributive laws; however, nearrings satisfy either the left or the right distributive law. One of generalized structures to those structures is a seminearring (or nearsemiring in another terminology) whose multiplicative and additive structures are merely semigroups; moreover, it satisfies one distributive law. The concept of seminearrings was first studied by B.V. Rootselaar in 1963. Seminearrings appear in a natural way in computer science see [7]. Some research papers concerning seminearrings are as follows: W.G.V. Hoorn and B.V. Rootselaar [8] and, recently, M. Shabir and I. Ahmed [17].

Γ-structures have long been studied since 1964 by N. Nobusawa or before that. To understand Γ-structures as a general rule, if R is a certain algebraic structure having one binary operation and Γ is simply a nonempty set, then R

can be formed a Γ-structure if there is a mapping from $R \times \Gamma \times R$ into R, sending (a, α, b) to $a\alpha b$, satisfying the property that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$ and some other specific properties according to what the algebraic structure of R is. There are various Γ-structures such as Γ-rings, Γ-nearrings and Γ-semirings. It turns out that Γ-nearrings and Γ-semirings are generalization of Γ-rings.

In this research, we define a new Γ-structure called a Γ-seminearring which is a generalization of seminearrings, Γ-nearrings, Γ-semirings and also Γ-rings. As we know, regular rings, weakly regular semirings, simple nearrings and 0-simple Γ-rings have been studied. This leads us to explore various aspects of Γ-seminearrings. The first purpose of this thesis is to study general properties of Γ-seminearrings such as some results involving sub Γ-seminearrings, ideals of a Γ-seminearring and quotient Γ-seminearrings. Another purpose of this work is to investigate regular Γ-seminearrings, weakly regular Γ-seminearrings, simple Γ-seminearrings and 0-simple Γ-seminearrings.

This thesis is separated into four chapters. We give precise definitions and some basic knowledge which will be used throughout this work in the rest of this chapter. In Chapter II, we give the definition of a Γ-seminearring and investigate general results concerning that new Γ-structure. Chapter III is devoted to regularity and weakly regularity of Γ-seminearrings. Finally, we study in Chapter IV simple Γ-seminearrings and 0-simple Γ-seminearrings. Moreover, necessary and sufficient conditions for being simple Γ-seminearrings and 0-simple Γ-seminearrings are provided.

1.2 Preliminaries

First, we reminisce Rees quotient semigroups.

Definition 1.2.1. Let A be a nonempty set. A relation ρ on A is called an *equivalence relation* on A if ρ is reflexive, symmetric and transitive. The *equivalence class* of $a \in A$ under ρ is denoted by $[a]_\rho$, i.e., $[a]_\rho = \{x \in A \mid x\rho a\}$. Moreover, $A/\rho = \big\{[x]_\rho \mid x \in A\big\}$ is the collection of all equivalence classes under ρ.

Definition 1.2.2. Let R be a semigroup. An equivalence relation ρ on R is called a *left (right) congruence* on R if $(a,b) \in \rho$ implies $(ca, cb) \in \rho$ $\big((ac, ab) \in \rho\big)$ for all $a, b, c \in R$. Furthermore, if ρ is both a left and a right congruence on R, then ρ is called a *congruence* on R.

In this thesis, unless otherwise state, when we say that R is a semigroup we means that R is a semigroup under $+$. Moreover, if ρ is an equivalence relation on a semigroup R, we use $a + \rho$ instead of $[a]_\rho$ for all $a \in R$.

Theorem 1.2.1. *Let ρ be a congruence on a semigroup R. Define a binary operation $*$ on $R/\rho = \{x + \rho \mid x \in R\}$ by*

$$(x+\rho)*(y+\rho) = (x+y)+\rho \quad \textit{for all } x, y \in R.$$

Then $(R/\rho, *)$ is a semigroup.

Conveniently, if ρ is a congruence on a semigroup R, then we say that R/ρ is a semigroup and write $(x+\rho)(y+\rho)$ in place of $(x+\rho)*(y+\rho)$ for all $x, y \in R$, i.e., the $*$ is omitted.

Definition 1.2.3. A nonempty subset I of a semigroup R is called a *left (right) ideal* of R if $RI \subseteq I$ $(IR \subseteq I)$. Furthermore, I is called an *ideal* if I is both a left and a right ideal of R.

Theorem 1.2.2. *Let I be an ideal of a semigroup R and ρ_I be the relation on R defined by*

$$x\rho_I y \text{ if and only if } x = y \text{ or } x, y \in I.$$

Then ρ_I is a congruence on R and R/ρ_I is also a semigroup.

Definition 1.2.4. The congruence ρ_I in Theorem 1.2.2 is called a *Rees congruence on R induced by I* and R/ρ_I a *Rees quotient semigroup* of R by I. Moreover, writing R/I is more preferable than R/ρ_I.

Next, we gather definitions of some Γ-structures mentioned in the previous section. W.E. Barnes [2] gave the definition of a Γ-ring which is a generalization of rings.

Definition 1.2.5. [2] Let R and Γ be additive abelian groups. Then R is called a Γ*-ring* if there exists a mapping from $R \times \Gamma \times R$ into R satisfying conditions $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$ and $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

The followings are examples of papers regarding Γ-rings: W.E. Barnes [2], S. Kyuno [10] and J. Luh [11].

Bh. Satyanarayana introduced Γ-nearrings in his doctoral thesis.

Definition 1.2.6. [16] Let R be an additive group and Γ a nonempty set. Then R is called a Γ*-nearring* if there exists a mapping from $R \times \Gamma \times R$ into R satisfying conditions $(a+b)\alpha c = a\alpha c + b\alpha c$ and $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

Unsurprisingly, Γ-nearrings are common generalization of nearrings and Γ-rings. Many results in Γ-nearrings have been developed:– see G.L. Booth [1], Y.B. Jun, M. Sapanci and M.A. Öztürk [9] and, lately, Bh. Satyanarayana and K.P. Syam [16].

T.K. Dutta and S.K. Sardar [5] set up Γ-semirings as generalization of semirings and Γ-rings.

Definition 1.2.7. [5] Let R and Γ be additive commutative semigroups. Then R is called a Γ*-semiring* if there exists a mapping from $R \times \Gamma \times R$ into R satisfying conditions $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + a\alpha c$ and $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

Further research involving Γ-semirings are found in T.K. Dutta and S.K. Sardar [5], R. Chinram [4] and S. Pianskool, S. Sangwirotjanapat and S. Tipyota [13].

Furthermore, from the above definitions, we see that Γ-nearrings and Γ-semirings are generalization of Γ-rings.

Finally, we gave examples of these Γ-structures.

Example 1.2.1. [2] Let X and Y be abelian groups, R and Γ be the set of all homomorphisms from X into Y and the set of all homomorphisms from Y into X, respectively. Define a mapping $R \times \Gamma \times R \to R$ by the usual composition. Then R is a Γ-ring.

Example 1.2.2. [5] Let R be the additive commutative semigroup of all $m \times n$ matrices over the set of all non-negative integers and Γ the additive commutative semigroup of all $n \times m$ matrices over the same set. Then R is a Γ-semiring if $a\alpha b$ denotes the usual matrix product of a, α, b where $a, b \in R$ and $\alpha \in \Gamma$.