ขอบเขตก่อนประสบการณ์สำหรับผลเฉลยของสมการคลื่นน้ำท่วมโดยวิธีความหนืดใกล้ศูนย์

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A PRIORI BOUNDS FOR SOLUTIONS OF FLOOD WAVE EQUATIONS BY VANISHING VISCOSITY METHOD

Ms. Piraya Thipakorn

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By	Ms. Piraya Thipakorn
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Anusorn Chonwerayuth, Ph.D.)

. . . . Thesis Advisor

(Sujin Khomrutai, Ph.D.)

. . . . Examiner

(Ratinan Boonklurb, Ph.D.)

. . . . External Examiner

(Sawanya Sakuntasathien, Ph.D.)

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วิทยานิพนธ์ฉบับนี้ใช้วิธีความหนืดใกล้ศูนย์เพื่อประมาณค่าผลเฉลยของระบบสมการแบบกฎอนุรักษ์ สำหรับการเคลื่อนที่ของคลื่นน้ำท่วมดังต่อไปนี้

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = g'S - C_f u^2 v \end{cases}$$

โดยที่ v และ u เป็นตัวแปรเชิงกายภาพ และ g', S และ C_f เป็นค่าคงตัว พร้อมด้วยเงื่อนไขเริ่มต้น $(v(x,0),u(x,0)) = (v_0(x),u_0(x))$ ขั้นตอนในการประยุกต์วิธีความหนืดใกล้ศูนย์คือการได้ L^∞-bound ของผลเฉลย ในงานนี้ได้พิสูจน์ค่าประมาณก่อนประสบการณ์ของผลเฉลยของระบบสมการนี้โดยใช้วิธีการบริเวณแผ่ขยายยืนยง

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In this thesis, we use the vanishing viscosity method to approximate solutions of the system of conservation laws for a motion of flood wave

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = g'S - C_f u^2 v \end{cases}$$

where v and u are physical variables and g', S and C_f are constants with initial data $(v(x,0), u(x,0)) = (v_0(x), u_0(x))$. A key step in applying the vanishing viscosity method is to obtain L^∞-bounds for solutions. In this work, we prove a priori estimate of solutions to this system by using the method of expanding invariant region.

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Student's Signature ....

Advisor's Signature ....

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CONTENTS

CHAPTER I

INTRODUCTION

In 2004, Tao Luo and Tong Yang [1] studied some properties of solutions to the flood wave equation in the space of BV functions. Let $u > 0$ and $h > 0$ be the velocity and depth of the water at a point $x \in \mathbb{R}$ and time t, respectively, $v = \frac{1}{h}$ and g be the gravitational acceleration. Given that $g' = g \cos\alpha$ and $S = \tan\alpha$, where $0 < \alpha < \frac{\pi}{2}$ is the angle between the river and the ground, and $p(v) = \frac{1}{2}g'v^{-2}$. Let ξ be the space variable. Let $x = \int_{\xi(t)}^{\xi} h(y,t)dy$, where $\xi(t)$ is an arbitrary particle path satisfying $\dot{\xi}(t) = u(\xi(t), t)$. The motion of flood wave can be modelled by the system of conservation laws:

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = \dfrac{g'S - C_f u^2 v}{\delta} \end{cases} \tag{1.1}$$

where $C_f > 0$ is a constant frictional coefficient and $\delta > 0$ is a small relaxation parameter. Normally, there is no general method for solving system of the form (1.1) with a given initial data.

In [2], the vanishing viscosity method is explained. This method is an indirect way to solve for solutions of a hyperbolic system of conservation laws in general. It approximates the solution of a problem with a parabolic system by adding viscosity terms. An important key step in applying the vanishing viscosity method is to obtain L^∞-bounds for solutions. The obtained L^∞-bounds of the parabolic system then lead to existence of weak solution to system (1.1).

In 1977, K.N. Chueh, C.C. Conley and J.A. Smoller [10] introduced the concept of positively invariant regions for the study of reaction-diffusion systems. Existence of such regions lead to a priori bounds of solution. The obtained a priori bounds can be used to derive the existence of solutions of the original conservation laws systems using standard techniques. The method of invariant regions however cannot be directly applied to the nonhomogeneous system (1.1) due to the source term. In 1997, Weifu Fang and Kazufumi Ito [11] extended the arguments from [9] and [10] to the concept of expanding invariant region for general nonhomogeneous system. The method of expanding invariant region yields L^∞-bound of the solution to nonhomogeneous system. The method of positively invariant region and expanding invariant region will be introduced in chapter 2.

In this work, we consider the model of the motion to flood wave (1.1) when $\delta = 1$. Hence we consider the equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = g'S - C_f u^2 v. \end{cases} \tag{1.2}$$

The purpose of this thesis is to prove a priori L^∞-bound of solution to the problem consisting of the viscosity approximate parabolic system associating with (1.2) with initial condition.

The thesis is organized into five chapters as follows.

In Chapter 2, we introduce basic knowledge. Moreover, the positively invariant region and the expanding invariant region are explained. In Chapter 3, we briefly explain some characteristics of the flood wave equations (1.2). For convenient of the reader, we prove that this system is strictly hyperbolic and give the right and left eigenvectors of its Jacobian matrix. In addition, we show that the system (1.2) is genuinely nonlinear. The Riemann invariant and a pair of an entropy-entropy flux are given. In Chapter 4, we explain the vanishing viscosity method which considers a viscosity parabolic system having solution $(v^\varepsilon(x,t), u^\varepsilon(x,t))$

associating with the system (1.2). Our main results will be presented in the final chapter, Chapter 5. Here, we prove an a priori L^∞-bound for the solution v^ε and also prove the positivity of the solution u^ε to the viscosity parabolic system. We prove that the trajectory of the solution $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ is inside an expanding invariant region. Moreover, a priori bound which is an upper bound of solution $u^\varepsilon(x,t)$ and a lower bound of solution $v^\varepsilon(x,t)$ are obtained.

Flood Wave Equation

We close the introduction with the physical reasoning behind the flood wave equation.

Physically, in [12], the author explained a flood as a large body of water increases and overflows onto a dry land, or generally, according to hydrologists a flood may be characterized as a discharge rate in a stream or a river which exceeds an acceptable threshold value and is often an expression of the variability of rainfall in the river.

In [13], the mathematical reasoning behinds the flood wave equation is given. For a steady flow model in a river, the frictional force of the river bed and the gravitational force are balanced. In an unsteady flow, we consider the case of a broad rectangular channel of constant inclination α. Define x to be the space variable and t the time variable. Let h be the depth of water and u the velocity of water where both are functions of (x,t). The conservation of fluid in a unit breadth is

$$\frac{d}{dt}\int_{x_2}^{x_1} h dx + [hu]_{x_2}^{x_1} = 0. \tag{1.3}$$

Since the frictional force and the gravitational force may not balance in this case, we must add a condition of the conservation of momentum. The appropriate equation in hydraulic theory is

$$\frac{d}{dt}\int_{x_2}^{x_1} hudx + [hu^2]_{x_2}^{x_1} + \left[\frac{1}{2}gh^2\cos\alpha\right]_{x_2}^{x_1} = \int_{x_2}^{x_1} gh\sin\alpha dx - \int_{x_2}^{x_1} C_f u^2 dx \quad (1.4)$$

where $x_2 < x < x_1$ and C_f is a constant frictional coefficient. Assume that h and u are continuously differentiable. Dividing by $x_1 - x_2$ and taking the limit $x_1 - x_2 \to 0$ to (1.3) and (1.4), we obtain

$$\begin{cases} h_t - (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}g'h^2\right)_x = g'hS - C_f u^2 \end{cases} \quad (1.5)$$

where $g' = g\cos\alpha$ and $S = \tan\alpha$. The system (1.1) is obtained from changing a form of (1.5) in Lagrangian coordinates.

CHAPTER II

PRELIMINARIES

In this chapter, we give basic knowledge which will be used in this thesis. The concept of positively invariant region and the expanding invariant region are explained.

2.1 Basic knowledge and notations

In this section, we give necessary definitions and important theorems for studying this work. Examples of strictly hyperbolic equations and systems will also be given.

Definition 1. *([2], Systems of Conservation Laws)*

Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ a smooth function. A 1-dimensional system of conservation laws has the following form

$$u_t + f(u)_x = 0 \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{2.1}$$

where $u = (u_1, u_2, ..., u_n)^T$ is a vector-valued function from $\mathbb{R} \times [0,\infty)$ into Ω and the flux-functions are $f(u) = (f_1(u), f_2(u), ..., f_n(u))^T$. When the right hand side of (2.1) *is a nonzero function, it is called a nonhomogeneous system. The system* (2.1) *can be written in a nonconservative form:*

$$u_t + A(u)u_x = 0 \tag{2.2}$$

where

$$A(u) = \left(\frac{\partial f_i}{\partial u_k}(u) \right)_{1 \le i,k \le n} \tag{2.3}$$

is the Jacobian matrix of the map $u \mapsto f(u)$*.*

Definition 2. *([2], Cauchy problem) Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $u : \mathbb{R} \times [0, \infty) \to \Omega$*. The problem, consisting of the system* (2.1) *with an initial condition*

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R} \tag{2.4}$$

where u_0 *is a given function from* $\mathbb{R}$ *into* Ω*, is called a Cauchy problem or an initial value problem (IVP).*

Definition 3. *([3], Hyperbolic Systems of Conservation Laws)*

The system (2.2) *is called hyperbolic if the matrix* $A(u)$ *has* n *real eigenvalues*

$$\lambda_1(u) \leq \lambda_2(u) \leq ... \leq \lambda_n(u)$$

and n *linearly independent corresponding right eigenvectors* $r_j(u)$ *and left eigenvectors* $l_j(u)$*, i.e., for any* $1 \leq j \leq n$*,*

$$A(u)r_j(u) = \lambda_j(u)r_j(u) \tag{2.5}$$

and

$$l_j(u)A(u) = \lambda_j(u)l_j(u) \tag{2.6}$$

respectively. It is called strictly hyperbolic if, for any $1 \leq j \leq n$*, the eigenvalues* $\lambda_j(u)$ *are all distinct. If* $\lambda_i(u) = \lambda_j(u)$ *for some* $1 \leq i, j \leq n$*, then* (2.2) *is called nonstrictly hyperbolic or hyperbolically degenerate.*

Example 1. ([2], The p-system)

A model for one-dimentional isentropic gas dynamics in Lagrangian coordinates is given by

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \tag{2.7}$$

where v is the specific volume, u is the velocity and the pressure $p(v) = Av^{-\gamma}$ for some constants $A > 0$ and $\gamma \geq 1$. The system (2.7) is a system of conservation laws because it can be written as

$$U_t + F(U)_x = 0$$

where $U = \begin{pmatrix} v \\ u \end{pmatrix}$, $F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$ and $\Omega = \{(v,u) \in \mathbb{R}^2 : v > 0\}$. The Jacobian matrix of $F(U)$ is

$$A(U) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$

The characteristic equation for the matrix $A(U)$ is

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ p'(v) & -\lambda \end{pmatrix} = \lambda^2 + p'(v) = 0.$$

Thus the two eigenvalues are $\lambda_1 = -\sqrt{(-p'(v))}$ and $\lambda_2 = \sqrt{(-p'(v))}$. If $p'(v) < 0$, then $\lambda_1 < \lambda_2$. Hence (2.7) is strictly hyperbolic when $p'(v) < 0$.

Definition 4. *(Gradients)*

Let Ω be an open subset of $\mathbb{R}^n$ and let f be a function from Ω into $\mathbb{R}^n$. The gradient of a function f is defined by

$$\nabla f = \frac{\partial f}{\partial u_1} e_1 + ... + \frac{\partial f}{\partial u_n} e_n$$

where each $f = (f_1, \ldots, f_n)$ is expressed in a row matrix and e_j, $1 \leq j \leq n$, denotes the column vector with a 1 at the j^{th} coordinate and 0 elsewhere. We denote $Df = \nabla f^T$.

Definition 5. *([4], Genuinely nonlinear systems)*

The system (2.2) *is called genuinely nonlinear in the λ_j characteristic field if*

$$\nabla \lambda_j^T \cdot r_j \neq 0.$$

If $\nabla\lambda_j^T \cdot r_j = 0$ *for some* $1 \leq j \leq n$*, then the system* (2.2) *is called linearly degenerate in the* λ_j *characteristic field.*

Definition 6. *([4], Riemann invariants)*

Let Ω *be an open subset of* $\mathbb{R}^n$ *and* w_j *be functions from* Ω *into* $\mathbb{R}$ *for* $j = 1, 2, \ldots, n$*. Then the function* $w_j = w_j(u)$ *is called Riemann invariants of system* (2.2) *corresponding to* λ_j*, if the following equation holds:*

$$\nabla w_j^T \cdot r_j = 0.$$

Definition 7. *([2], Entropy-entropy fluxes)*

Suppose Ω *is a convex open subset of* $\mathbb{R}^n$*. Then a convex function* $\eta : \Omega \to \mathbb{R}$ *is called an entropy for the system of conservation laws* (2.1) *if there exists a function* $q : \Omega \to \mathbb{R}$ *called an entropy flux such that*

$$\nabla\eta^T(u)A(u) = \nabla q^T(u) \tag{2.8}$$

where $A(u)$ *is the Jacobian matrix* (2.3)*. In particular, if* (2.8) *holds, then all classical solution of* (2.1) *satisfies the following equation*

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}q(u) = 0. \tag{2.9}$$

Definition 8. *([6],* L^p *Spaces)*

Let $(X, \mathfrak{M}, \mu)$ *be a measure space. Suppose* f *is a measurable function on* X *and* $0 < p < \infty$*. Define*

$$||f||_p = \left[\int |f|^p d\mu\right]^{1/p}.$$

The L^p *space is defined by*

$$L^p(X) = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_p < \infty\}.$$

Define

$$||f||_\infty = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}.$$

The L^∞ space is defined by

$$L^\infty(X) = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_\infty < \infty\}.$$

Proposition 1. *([6]) If $f \in L^1(X)$, then*

$$\left|\int f d\mu\right| \leq \int |f| d\mu.$$

Theorem 1. *([6], Hölder's inequality)*

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose $1 \leq p, q \leq \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If f and g are measurable functions on X, then

$$||fg||_1 \leq ||f||_p ||g||_q.$$

In particular, if $f \in L^p(X)$ and $g \in L^q(X)$, then $fg \in L^1(X)$.

Theorem 2. *([7], Fubini's theorem)*

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be $\sigma-$finite measure spaces. Suppose $f : X \times Y \to \mathbb{C}$ is an $\mathfrak{M} \times \mathfrak{N}-$measurable function. If

$$\int_X \int_Y |f_x| d\nu d\mu < \infty \qquad \text{or} \qquad \int_Y \int_X |f^y| d\mu d\nu < \infty,$$

then

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f_x d\nu \right) d\mu = \int_Y \left(\int_X f^y d\mu \right) d\nu$$

where $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.

Definition 9. *([8], Sobolev spaces)*

The space $W^{k,p}(\Omega)$ consists of all locally summable functions $w : \Omega \to \mathbb{R}$ such that $D^\alpha w$ exists in the weak sense and belongs to $L^p(\Omega)$ for each multiindex α with $|\alpha| \leq k$. It is called a Sobolev space. If $p = 2$, we denote

$$H^k(\Omega) = W^{k,2}(\Omega) \qquad (k = 0, 1, 2, ...)$$

and it is a standard result that $H^k(\Omega)$ are Hilbert space for all integers k. In particular, if $u \in H^1(\mathbb{R} \times (0, T])$, then

$$\int_{\mathbb{R}\times(0,T]} |u|^2 dtdx, \int_{\mathbb{R}\times(0,T]} |\partial_t u|^2 dtdx \quad and \quad \int_{\mathbb{R}\times(0,T]} |\partial_x u|^2 dtdx$$

are finite.

Lemma 1. *([8], Cauchy's inequality with ε)*

Let $a, b > 0$ and $\varepsilon > 0$. Then

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 2. *([8], Gronwall's inequality; integral form)*

Let ξ be a nonnegative and continuous function on $[0, T]$. If

$$\xi(t) \le C \int_0^t \xi(s) ds$$

for some constant $C \ge 0$ for all $0 \le t \le T$, then

$$\xi(t) = 0 \qquad for\ all\ 0 \le t \le T.$$

2.2 Positively invariant regions and expanding invariant regions

In this section, we give definitions of positively invariant regions and quasi-convexity. We also explain the concept of an expanding invariant region and state the main theorem.

In [9], they consider the smooth solution $u(x,t) \in \mathbb{R}^n$ of the following nonlinear systems of reaction-diffusion

$$u_t = \varepsilon D(u,x)u_{xx} + M(u,x)u_x + H(u,t) \qquad (x,t) \in \Omega \times [0,\infty), \tag{2.10}$$

with the initial data

$$u(x,0) = u_0(x) \qquad x \in \Omega \tag{2.11}$$

where $\varepsilon > 0$, Ω is an open interval in $\mathbb{R}$, $D(u, x)$ and $M(u, x)$ are matrix-valued functions defined on an open subset $U \times V \subset \mathbb{R}^n \times \Omega$, $D \geq 0$ and $H(u, t)$ is a smooth mapping from $U \times [0, \infty)$ into $\mathbb{R}^n$.

Assume that this problem has a local (in time) solution on some set X of smooth functions from Ω to $\mathbb{R}^n$. In other words, given a function $u_0(x)$ in the set X, there is a positive number δ and a smooth solution $u(x, t)$ of (2.10) and (2.11) defined for $x \in \Omega$ and $t \in [0, \delta)$ such that $u(\cdot, t) \in X$ for all $0 \leq t < \delta$.

Definition 10. *([9], Positively invariant regions)*

Let Σ be a closed subset of $\mathbb{R}^n$. If a solution $u(x, t)$ of (2.10) *and* (2.11) *having initial and boundary values in Σ, satisfies $u(x, t) \in \Sigma$ for all $x \in \Omega$ and $t \in [0, \delta)$, then Σ is called a positively invariant region for the local solution defined by* (2.10) *and* (2.11).

According to [9], the invariant regions Σ of (2.10) and (2.11) has the form

$$\Sigma = \bigcap_{j=1}^{m} \{u \in U : G_j(u) \leq 0\} \tag{2.12}$$

where G_j are certain smooth real-valued functions depending on D, M, and H.

Now, we recall the definition of quasi-convexity.

Definition 11. *([9], Quasi-convexity)*

The smooth function $G : \mathbb{R}^n \to \mathbb{R}$ is called quasi-convex at v if whenever $dG_v(\eta) = 0$, then $d^2G_v(\eta, \eta) \geq 0$.

Theorem 3. *([9]) Let Σ be defined by* (2.12)*. Suppose that for all $t \in [0, \infty)$ and for every $u_0 \in \partial\Sigma$ (so $G_j(u_0) = 0$ for some j), the following conditions hold:*

(1) ∇G_j^T is a left eigenvector of $D(u_0, x)$ and $M(u_0, x)$ for all $x \in \Omega$;

(2) If $\nabla G_j^T D(u_0, x) = \mu \nabla G_j^T$ with $\mu \neq 0$, then G_j is quasi-convex at u_0;

(3) $\nabla G_j^T \cdot (H) \leq 0$ at u_0, for all $t \in \mathbb{R}^+$.

Then Σ is invariant for (2.10)*, for every $t \in [0, \infty)$.*

This invariant region enables us to get an L^∞-bound which constitutes an important key step in applying the vanishing viscosity method.

In [11], they study a problem consisting of a system of a form

$$u_t = \varepsilon u_{xx} + M(u,x)u_x + H(u,x,t) \qquad (x,t) \in \mathbb{R} \times [0,T), \tag{2.13}$$

with the initial data

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R} \tag{2.14}$$

where $\varepsilon > 0$, $M(u,x)$ is matrix-valued function defined on $\mathbb{R}^n \times \mathbb{R}$ and $H(u,x,t)$ is a smooth mapping from $\mathbb{R}^n \times \mathbb{R} \times [0,T)$ into $\mathbb{R}^n$. Since the method of invariant regions cannot be applied to this nonhomogeneous system, [11] extended the concept of invariant region (2.12) from [9] and [10] to the expanding invariant region which depending on t of general nonhomogeneous system

$$\Sigma(t) = \bigcap_{j=1}^{k} \{u \in \mathbb{R}^n : G_j(u) \leq Ae^{\omega t}\} \tag{2.15}$$

where ω and A are two constants.
The following theorem states under certain circumstances that the solution of (2.13) with initial data (2.14) stay inside the expanding invariant region (2.15). This theorem will be used to prove our main theorem.

Theorem 4. *([11]) Let $u(x,t)$ be a smooth solution to* (2.13) *with initial data* (2.14)*, and $\Sigma(t)$ be defined by* (2.15) *for some constants A and ω. Suppose for (t_0, u_0) such that $u_0 \in \partial\Sigma(t_0)$ (i.e. $G_j(u_0) = Ae^{\omega t_0}$ for some j), the following conditions hold:*

(1) ∇G_j^T at u_0 is a left eigenvector of $M(u_0, x)$ for all x;

(2) G_j is quasi-convex at u_0;

(3) $\nabla G_j^T \cdot (H) \leq \omega A e^{\omega t_0}$ for all $x \in \mathbb{R}$ and at $(t,u) = (t_0, u_0)$.

Then the trajectory $u(x,t)$ is inside $\Sigma(t)$ for all time $t > 0$; that is,

$$G_j(u(x,t)) = Ae^{\omega t} \text{ for all } j = 1, 2, .., k.$$

CHAPTER III

CHARACTERIZATION OF THE SYSTEM OF HYPERBOLIC CONSERVATION LAWS FOR A MOTION OF FLOOD WAVE

In this chapter, we characterize the system of conservation laws (1.2) which is the main system of this work. We prove that the flood wave equations are strictly hyperbolic and give the right and left eigenvectors of the Jacobian matrix to this system expressed in nonconservative form. We use the right eigenvectors to prove genuine nonlinearity and to find the Riemann invariants of the our system. In addition, a pair of entropy-entropy flux is given. These results are standard and can be found in the lituratures. However, for completeness and for convenience of the reader we shall provide the derivations.

3.1 Main system

The system (1.2) can be written in the vector equation form as follow:

$$U_t + F(U)_x = H(U) \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{3.1}$$

where $U = \begin{pmatrix} v \\ u \end{pmatrix}$, $F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$ and $H(U) = \begin{pmatrix} 0 \\ g'S - C_f u^2 v \end{pmatrix}$.

The nonconservative form of the system (3.1) is

$$U_t + A(U)U_x = H(U) \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{3.2}$$

where

$$A(U) = \begin{pmatrix} \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial u} \\ \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -g' v^{-3} & 0 \end{pmatrix} \tag{3.3}$$

is the Jacobian matrix of $F(U) = \begin{pmatrix} F_1(v,u) \\ F_2(v,u) \end{pmatrix} = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$.

3.2 Strict hyperbolicity

We show that the system (3.2) is strictly hyperbolic.

Proposition 2. *The system* (3.2) *is strictly hyperbolic if and only if* $0 < v < \infty$.

Proof. We want to find the eigenvalues of the Jacobian matrix $A(U)$. From (3.3), we have

$$A - \lambda I = \begin{pmatrix} 0 & -1 \\ -g' v^{-3} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ -g' v^{-3} & -\lambda \end{pmatrix}.$$

Then

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ -g' v^{-3} & -\lambda \end{pmatrix} = \lambda^2 - g' v^{-3}.$$

Hence the two eigenvalues λ_j, $j = 1, 2$, which satisfy $\det(A - \lambda_j I) = 0$, are

$$\lambda_1 = -\sqrt{g'} v^{-3/2} \tag{3.4}$$

and

$$\lambda_2 = \sqrt{g'} v^{-3/2} \tag{3.5}$$

where $g' = g \cos \alpha$ is a positive constant. Since the assumption $0 < v < \infty$ and a constant $\sqrt{g'} > 0$, λ_1 and λ_2 are two real distinct eigenvalues. Hence the system (3.2) is strictly hyperbolic. Conversely, if the system (3.2) is strictly hyperbolic, then $\lambda_1 \neq \lambda_2$. It is clear that $0 < v < \infty$. □

3.3 Eigenvectors

In this section, we want to find the right and left eigenvectors of the Jacobian matrix (3.3) corresponding to the eigenvalues λ_1 and λ_2. The right eigenvectors will be used to prove genuine nonlinearity and to find Riemann invariants of the system (3.2) in the next section.

Proposition 3. *Let* $U = \begin{pmatrix} v & u \end{pmatrix}^T$. *The two right eigenvectors of the Jacobian matrix* (3.3) *are*

$$r_1(U) = \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} \tag{3.6}$$

and

$$r_2(U) = \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix}. \tag{3.7}$$

Proof. For the eigenvalue $\lambda_1 = -\sqrt{g'}v^{-3/2}$ by (2.5), the right eigenvector r_1 satisfies

$$\begin{pmatrix} 0 & -1 \\ -g'v^{-3} & 0 \end{pmatrix} r_1(U) = -\sqrt{g'}v^{-3/2} r_1(U).$$

That is

$$\frac{1}{-\sqrt{g'}v^{-3/2}} \begin{pmatrix} 0 & -1 \\ -g'v^{-3} & 0 \end{pmatrix} r_1(U) = I_{2\times 2} r_1(U).$$

Thus

$$\begin{pmatrix} -1 & 1/\sqrt{g'}v^{-3/2} \\ g'v^{-3}/\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} r_1(U) = 0. \tag{3.8}$$

Let $a, b \in \mathbb{R}$ and $r_1(U) = \begin{pmatrix} a \\ b \end{pmatrix}$. (3.8) becomes

$$\begin{pmatrix} -1 & 1/\sqrt{g'}v^{-3/2} \\ g'v^{-3}/\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Then we get $b = \sqrt{g'}v^{-3/2}a$. Hence

$$r_1(U) = \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix}$$

is a right eigenvector corresponding λ_1. For the eigenvalue $\lambda_2 = \sqrt{g'}v^{-3/2}$, we repeat the same procedure to get a right eigenvector

$$r_2(U) = \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix}$$

corresponding λ_2. □

Proposition 4. *Let* $U = \begin{pmatrix} v & u \end{pmatrix}^T$. *The two left eigenvector of the Jacobian matrix* (3.3) *are*

$$l_1(U) = \begin{pmatrix} 1 & 1/\sqrt{g'}v^{-3/2} \end{pmatrix}$$

and

$$l_2(U) = \begin{pmatrix} 1 & -1/\sqrt{g'}v^{-3/2} \end{pmatrix}.$$

Proof. For the eigenvalue $\lambda_1 = -\sqrt{g'}v^{-3/2}$ by (2.6), the right eigenvector l_1 satisfies

$$l_1(U) \begin{pmatrix} 0 & -1 \\ -g'v^{-3} & 0 \end{pmatrix} = -\sqrt{g'}v^{-3/2} l_1(U).$$

That is

$$l_1(U) \begin{pmatrix} 0 & 1/\sqrt{g'}v^{-3/2} \\ g'v^{-3}/\sqrt{g'}v^{-3/2} & 0 \end{pmatrix} = l_1(U) I_{2\times 2}.$$

Thus

$$l_1(U) \begin{pmatrix} -1 & 1/\sqrt{g'}v^{-3/2} \\ g'v^{-3}/\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} = 0 \tag{3.9}$$

Let $a, b \in \mathbb{R}$ and $l_1(U) = \begin{pmatrix} a & b \end{pmatrix}$. (3.9) becomes

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} -1 & 1/\sqrt{g'}v^{-3/2} \\ g'v^{-3}/\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} = 0.$$

Hence we get $b = \dfrac{a}{\sqrt{g'}v^{-3/2}}$. Therefore,

$$l_1(U) = \begin{pmatrix} 1 & 1/\sqrt{g'}v^{-3/2} \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = \sqrt{g'}v^{-3/2}$, we repeat the same procedure to get a left eigenvector

$$l_2(U) = \begin{pmatrix} 1 & -1/\sqrt{g'}v^{-3/2} \end{pmatrix}$$

corresponding λ_2. □

3.4 Genuine nonlinearity

In this section, we prove that the system (3.2) is genuinely nonlinear in the λ_1 and λ_2 characteristic field where λ_1 and λ_2 satisfy (3.4) and (3.5), respectively.

Proposition 5. *The strictly hyperbolic system* (3.2) *is genuinely nonlinear in the λ_1 and λ_2 characteristic field where λ_1 and λ_2 satisfy* (3.4) *and* (3.5), *respectively.*

Proof. If $\lambda_1 = -\sqrt{g'}v^{-3/2}$, consider

$$\nabla\lambda_1 = \frac{\partial\lambda_1}{\partial v}e_1 + \frac{\partial\lambda_1}{\partial u}e_2 = \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} \\ 0 \end{pmatrix}.$$

Then

$$\nabla\lambda_1^T \cdot r_1 = \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} = \frac{3}{2}\sqrt{g'}v^{-5/2}.$$

Since $g' = g\cos\alpha$ where $0 < \alpha < \frac{\pi}{2}$,

$$0 < g' < g. \tag{3.10}$$

We have

$$\sqrt{g'} \neq 0. \tag{3.11}$$

Since the system (3.2) is strictly hyperbolic, by Proposition 2, $0 < v < \infty$. Hence $\nabla\lambda_1^T \cdot r_1 = \frac{3}{2}\sqrt{g'}v^{-5/2} \neq 0$. Therefore, the system (3.2) is genuinely nonlinear in the λ_1 characteristic field.

If $\lambda_2 = \sqrt{g'}v^{-3/2}$, consider

$$\nabla\lambda_2 = \frac{\partial\lambda_2}{\partial v}e_1 + \frac{\partial\lambda_2}{\partial u}e_2 = \begin{pmatrix} \frac{-3}{2}\sqrt{g'}v^{-5/2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-3}{2}\sqrt{g'}v^{-5/2} \\ 0 \end{pmatrix}.$$

Then

$$\nabla\lambda_2^T \cdot r_2 = \begin{pmatrix} \frac{-3}{2}\sqrt{g'}v^{-5/2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{-g'}v^{-3/2} \end{pmatrix} = \frac{-3}{2}\sqrt{g'}v^{-5/2}.$$

From (3.11) and $0 < v < \infty$, $\nabla\lambda_2^T \cdot r_2 = \frac{-3}{2}\sqrt{g'}v^{-5/2} \neq 0$. Hence (3.2) is also genuinely nonlinear in the λ_2 characteristic field. $\square$

3.5 Riemann invariants

The Riemann invariants are useful for obtaining the expanding invariant region which leads to obtaining the $L^\infty-$bound. In this section, we will find the two Riemann invariants of the system (3.2) corresponding to the eigenvalues λ_1 and λ_2.

Proposition 6. *The two Riemann invariants of the system* (3.2) *corresponding to λ_1 and λ_2 are*

$$r(U) = u + 2\sqrt{g'}v^{-1/2}$$

and

$$s(U) = u - 2\sqrt{g'}v^{-1/2},$$

respectively.

Proof. Since the Jacobian matrix (3.3) have two eigenvalues, we will consider two cases.

If $\lambda_1 = -\sqrt{g'}v^{-3/2}$, we have $r_1(U) = \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix}$. By Definition 6, the Riemann invariant r which corresponds to λ_1 satisfying

$$\nabla r^T \cdot r_1 = \begin{pmatrix} \frac{\partial r}{\partial v} & \frac{\partial r}{\partial u} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} = 0.$$

That is

$$\frac{\partial r}{\partial v} + \sqrt{g'}v^{-3/2}\frac{\partial r}{\partial u} = 0.$$

From [2], we have

$$r(U) = u - \int^v \sqrt{g'}m^{-3/2}dm = u + 2\sqrt{g'}v^{-1/2}. \tag{3.12}$$

If $\lambda_2 = \sqrt{g'}v^{-3/2}$, we have $r_2(U) = \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix}$. By Definition 6, the Riemann invariant s which corresponds to λ_2 satisfying

$$\nabla s^T \cdot r_2 = \begin{pmatrix} \frac{\partial s}{\partial v} & \frac{\partial s}{\partial u} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix} = 0.$$

That is

$$\frac{\partial s}{\partial v} - \sqrt{g'}v^{-3/2}\frac{\partial s}{\partial u} = 0.$$

From [2], we have

$$s(U) = u + \int^v \sqrt{g'}m^{-3/2}dm = u - 2\sqrt{g'}v^{-1/2}. \tag{3.13}$$

□

3.6 Entropy-entropy fluxes

In [2], having a pair of an entropy-entropy flux for the system of conservation laws (2.1) is a required condition of the vanishing viscosity theorem. It is a main theorem used to prove existence of weak solution to the system of conservation laws (2.1). In this section, we give a pair of an entropy-entropy flux for our system (3.2).

Proposition 7. *A pair of an entropy-entropy flux of the system* (3.2) *is*

$$\eta(U) = \frac{u^2}{2} + \frac{g'}{2v} \quad \textit{and} \quad q(U) = \frac{g'u}{2v^2}. \tag{3.14}$$

Proof. We want to find $(\eta(U), q(U))$ such that $\nabla\eta^T(U)A(U) = \nabla q^T(U)$ where $A(U)$ is the Jacobian matrix (3.3). We multiply the first equation of (1.2) by $-p(v)$ and the second equation of (1.2) by u. Then we have

$$\begin{cases} -p(v)v_t + p(v)u_x = 0 \\ uu_t + up(v)_x = u\left(g'S - C_f u^2 v\right). \end{cases}$$

Adding these two equations together, we obtain

$$-p(v)v_t + p(v)u_x + uu_t + up(v)_x = u\left(g'S - C_f u^2 v\right).$$

That is

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2} - P(v)\right) + \frac{\partial}{\partial x}(p(v)u) = u\left(g'S - C_f u^2 v\right) \tag{3.15}$$

where $P(v)$ is a primitive of $p(v)$. We see that (3.15) is nonhomogeneous form of (2.9). Since $p(v) = \frac{1}{2}g'v^{-2}$,

$$P(v) = \int p(v) = \int \frac{1}{2}g'v^{-2}dv = \frac{-g'}{2v}.$$

Let $\eta(U) = \frac{u^2}{2} - P(v) = \frac{u^2}{2} + \frac{g'}{2v}$ and $q(U) = p(v)u$. We have

$$\nabla\eta^T(U) = \begin{pmatrix} \frac{g'}{2}v^{-2} & u \end{pmatrix} \tag{3.16}$$

and

$$\nabla q^T(U) = \begin{pmatrix} -g'uv^{-3} & \frac{g'}{2}v^{-2} \end{pmatrix}. \tag{3.17}$$

By (3.16) and (3.17), we have

$$\begin{aligned}\nabla\eta^T(U)A(U) &= \begin{pmatrix}\frac{g'}{2}v^{-2} & u\end{pmatrix}\begin{pmatrix}0 & -1\\ -g'v^{-3} & 0\end{pmatrix}\\ &= \begin{pmatrix}-g'uv^{-3} & \frac{g'}{2}v^{-2}\end{pmatrix}\\ &= \nabla q^T(U).\end{aligned}$$

Hence a pair of an entropy-entropy flux of (3.2) is

$$\eta(U) = \frac{u^2}{2} + \frac{g'}{2v} \quad \text{and} \quad q(U) = \frac{g'u}{2v^2}.$$

□

CHAPTER IV

VANISHING VISCOSITY METHOD

In this chapter, the vanishing viscosity method is introduced. In addition, we give the viscosity parabolic system of flood wave equations.

4.1 Introduction

The vanishing viscosity method is explained in [2]. This method is useful for studying weak solutions of hyperbolic systems. For this method we shall consider a parabolic system which is obtained from the hyperbolic system by adding an additional viscosity term with a small parameter $\varepsilon > 0$. Explicitly, given a small parameter $\varepsilon > 0$, a parabolic system associating with (2.1), (2.4) is

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{4.1}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \tag{4.2}$$

where $\varepsilon u_{xx}^\varepsilon$ is called a viscosity term and $u_0^\varepsilon(x) \to u_0(x)$ as $\varepsilon \to 0$.

The following theorem is taken from [2].

Theorem 5. *[2] Assume that* (2.1) *admits an entropy V with entropy fluxes F. Let $(u^\varepsilon)_\varepsilon$ be a sequence of sufficiently smooth solutions of* (4.1) *with*

$$||u^\varepsilon||_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq C, \tag{4.3}$$

$$u^\varepsilon \to u \quad as \quad \varepsilon \to 0 \quad a.e. \quad in \ \mathbb{R}\times\mathbb{R}^+,$$

where $C > 0$ is a constant independent of ε. Then u is a weak solution of (2.1) *and it satisfies the entropy condition*

$$\frac{\partial}{\partial t}V(u) + \frac{\partial}{\partial x}F(u) \leq 0$$

in the sense of distributions on $\mathbb{R} \times \mathbb{R}^+$.

According to the theorem, there are three basic ingredients for the vanishing viscosity method

(1) Establishing a priori bound for viscosity system (4.1), (4.2),

(2) Proving existence of smooth solutions for (4.1), (4.2) and

(3) Studying structure properties of entropy pairs for the hyperbolic system (2.1).

4.2 The Viscosity Parabolic System of Flood Wave Equations

The viscosity parabolic system associating with the flood wave equations (1.2) is

$$\begin{cases} v_t^\varepsilon - u_x^\varepsilon = \varepsilon v_{xx}^\varepsilon \\ u_t^\varepsilon + p(v^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon + g'S - C_f(u^\varepsilon)^2 v^\varepsilon \end{cases} \tag{4.4}$$

where $\varepsilon > 0$ and $(x,t) \in \mathbb{R} \times [0,\infty)$. In a vector equation form it is

$$U_t^\varepsilon + F(U^\varepsilon)_x = \varepsilon U_{xx}^\varepsilon + H(U^\varepsilon) \tag{4.5}$$

where $U^\varepsilon = \begin{pmatrix} v^\varepsilon \\ u^\varepsilon \end{pmatrix}$, $F(U^\varepsilon) = \begin{pmatrix} -u^\varepsilon \\ p(v^\varepsilon) \end{pmatrix}$ and $H(U^\varepsilon) = \begin{pmatrix} 0 \\ g'S - C_f(u^\varepsilon)^2 v^\varepsilon \end{pmatrix}$.

Since we will apply the expanding invariant region method, we write the system (4.5) in the form (2.13) of Theorem 4 as

$$U_t^\varepsilon = \varepsilon U_{xx}^\varepsilon + M(U^\varepsilon)(U^\varepsilon)_x + H(U^\varepsilon) \tag{4.6}$$

where

$$M(U^\varepsilon) = \begin{pmatrix} 0 & 1 \\ g'(v^\varepsilon)^{-3} & 0 \end{pmatrix}. \tag{4.7}$$

In the next chapter we shall prove an a priori bound for the solution U^ε independent of ε.

CHAPTER V

MAIN RESULTS

This chapter contain our main results. Let (v, u) be solution of (4.6). We will consider the parabolic system (4.6) with initial data

$$(v, u)(x, 0) = (v_0(x), u_0(x)) \qquad x \in \mathbb{R} \tag{5.1}$$

where v_0 and $u_0 \in L^\infty(\mathbb{R})$. We prove an L^∞-bound of solutions $v(x, t)$ and prove that $u(x, t)$ is a nonnegative solution. Then we show that a trajectory of the solution $(v(x, t), u(x, t))$ is inside the expanding invariant region. In addition, we give a priori bound of solution $(v(x, t), u(x, t))$.

5.1 L^∞-bound of the solution $v(x, t)$

In this section, we prove an L^∞-bound of the solution $v(x, t)$ to the viscosity system (4.6) with initial condition. The obtaining L^∞-bound of the solution $v(x, t)$ will be used in the next section.

Theorem 6. *Suppose $v, u \in H^1(\mathbb{R} \times (0, T])$ and the solution (v, u) satisfies*

$$v_t - u_x = \varepsilon v_{xx} \tag{5.2}$$

for all $(x, t) \in \mathbb{R} \times (0, T]$ and $\varepsilon > 0$ with initial condition (5.1). Then $v(x, t)$ has the following L^∞-bound:

$$|v(x, t)| \ \le \ ||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{y,s}}. \tag{5.3}$$

Proof. Let $\varepsilon > 0$ and $(x, t) \in \mathbb{R} \times (0, T]$. Define

$$f_1(x,t) = u_x(x,t). \tag{5.4}$$

From (5.2) and (5.1), we have the heat problem

$$v_t = \varepsilon v_{xx} + f_1(x,t)$$

with the initial condition

$$v(x,0) = v_0(x)$$

where $v_0 \in L^\infty(\mathbb{R})$. By the Green's function, we obtain the solution $v(x,t)$ of this heat problem to be

$$v(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} v_0(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} u_x(y,s)dyds.$$

By the triangle inequality, we have

$$|v(x,t)| \leq \left| \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} v_0(y)dy \right| + \left| \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} u_x(y,s)dyds \right|. \tag{5.5}$$

Now, we consider $\frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} v_0(y)dy$. Since $v_0 \in L^\infty(\mathbb{R})$,

$$|v_0(y)| \leq ||v_0||_\infty \qquad (y \in \mathbb{R}). \tag{5.6}$$

Let $w_1 = \frac{x-y}{\sqrt{4\varepsilon t}}$. Then we have $-w_1^2 = \frac{-(x-y)^2}{4\varepsilon t}$ and $dw_1 = \frac{-1}{\sqrt{4\varepsilon t}}dy$. Thus

$$\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} dy = -\sqrt{4\varepsilon t} \int_{\infty}^{-\infty} e^{-w_1^2} dw_1.$$

By the gaussian integral, we have $\int_{\infty}^{-\infty} e^{-w_1^2} dw_1 = -\sqrt{\pi}$. So, we obtain that

$$\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} dy = \sqrt{4\pi\varepsilon t}. \tag{5.7}$$

By (5.6) and (5.7), we have

$$\left| \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\varepsilon t}} v_0(y)dy \right| \leq \frac{||v_0||_\infty}{\sqrt{4\pi\varepsilon t}} \sqrt{4\pi\varepsilon t} = ||v_0||_\infty. \tag{5.8}$$

Next, we consider $\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} u_x(y,s)dyds$. Let

$$f_2(x,y,t,s) = \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} \qquad (x, y \in \mathbb{R}, 0 < t, s \le T). \tag{5.9}$$

By (5.4) and (5.9), we can write

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} u_x(y,s)dyds = \int_0^t \int_{-\infty}^{\infty} f_2(x,y,t,s) f_1(y,s)dyds. \tag{5.10}$$

We show that $f_1(y,s)$ and $f_2(x,y,t,s) \in L^2(\mathbb{R} \times [0,t))$. Since $u \in H^1(\mathbb{R} \times (0,T])$,

$$\int_{-\infty}^{\infty} \int_0^t |u_x(y,s)|^2 dsdy < \infty.$$

By Theorem 2,

$$\int_0^t \int_{-\infty}^{\infty} |f_1(y,s)|^2 dyds = \int_0^t \int_{-\infty}^{\infty} |u_x(y,s)|^2 dyds < \infty.$$

Then

$$\left[\int_0^t \int_{-\infty}^{\infty} |f_1(y,s)|^2 dyds\right]^{1/2} = ||u_x||_{L^2(\mathbb{R}\times[0,t))} < \infty. \tag{5.11}$$

Thus

$$f_1 \in L^2(\mathbb{R} \times [0,t)).$$

Consider

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} |f_2(x,y,t,s)|^2 dyds &= \int_0^t \int_{-\infty}^{\infty} f_2^2(x,y,t,s)dyds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon(t-s)} e^{\frac{-(x-y)^2}{2\varepsilon(t-s)}} dyds.\end{aligned}$$

Let $w_2 = \frac{x-y}{\sqrt{2\varepsilon(t-s)}}$. Then $-w_2^2 = \frac{-(x-y)^2}{2\varepsilon(t-s)}$ and $dw_2 = \frac{-1}{\sqrt{2\varepsilon(t-s)}}dy$. Hence

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon(t-s)} e^{\frac{-(x-y)^2}{2\varepsilon(t-s)}} dyds = \int_0^t \frac{\sqrt{2\varepsilon(t-s)}}{4\pi\varepsilon(t-s)} \left(\int_{-\infty}^{\infty} e^{-w_2^2} dw_2\right) ds.$$

Since $\int_{-\infty}^{\infty} e^{-w_2^2} dw_2 = \sqrt{\pi}$, we get that

$$\begin{aligned}\int_0^t \frac{\sqrt{2\varepsilon(t-s)}}{4\pi\varepsilon(t-s)} \left(\int_{-\infty}^{\infty} e^{-w_2^2} dw_2\right) ds &= \int_0^t \frac{\sqrt{2\pi\varepsilon(t-s)}}{4\pi\varepsilon(t-s)} ds \\ &= \frac{1}{\sqrt{8\pi\varepsilon}} \int_0^t \frac{1}{\sqrt{t-s}} ds \\ &= \sqrt{\frac{t}{2\pi\varepsilon}}.\end{aligned}$$

Thus

$$\int_0^t \int_{-\infty}^{\infty} |f_2(x,y,t,s)|^2 dyds = \sqrt{\frac{t}{2\pi\varepsilon}}.$$

Since $0 < t \leq T$, $\sqrt{\frac{t}{2\pi\varepsilon}} \leq \sqrt{\frac{T}{2\pi\varepsilon}}$. Hence

$$\left[\int_0^t \int_{-\infty}^{\infty} |f_2(x,y,t,s)|^2 dyds\right]^{1/2} = \sqrt[4]{\frac{T}{2\pi\varepsilon}} < \infty. \tag{5.12}$$

Thus

$$f_2 \in L^2(\mathbb{R} \times [0,t)).$$

Since f_1 and $f_2 \in L^2(\mathbb{R} \times [0,t))$, by Theorem 1, $f_2 f_1 \in L^1(\mathbb{R} \times [0,t))$ and

$$\int_0^t \int_{-\infty}^{\infty} |f_2 f_1| dyds \leq \left[\int_0^t \int_{-\infty}^{\infty} |f_2|^2 dyds\right]^{1/2} \left[\int_0^t \int_{-\infty}^{\infty} |f_1|^2 dyds\right]^{1/2}. \tag{5.13}$$

Since $f_2 f_1 \in L^1(\mathbb{R} \times [0,t))$, by Proposition 1,

$$\left|\int_0^t \int_{-\infty}^{\infty} f_2 f_1 dyds\right| \leq \int_0^t \int_{-\infty}^{\infty} |f_2 f_1|\, dyds. \tag{5.14}$$

By (5.14) and (5.13), hence

$$\left|\int_0^t \int_{-\infty}^{\infty} f_2 f_1 dyds\right| \leq \left[\int_0^t \int_{-\infty}^{\infty} |f_2|^2 dyds\right]^{1/2} \left[\int_0^t \int_{-\infty}^{\infty} |f_1|^2 dyds\right]^{1/2}. \tag{5.15}$$

By (5.15), (5.12) and (5.11), we obtain that

$$\left|\int_0^t \int_{-\infty}^{\infty} f_2(x,y,t,s) f_1(y,s) dyds\right| \leq \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{y,s}}.$$

From (5.10), that is

$$\left|\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\varepsilon(t-s)}} e^{\frac{-(x-y)^2}{4\varepsilon(t-s)}} u_x(y,s) dyds\right| \leq \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{y,s}}. \tag{5.16}$$

By (5.5), (5.8) and (5.16), hence

$$|v(x,t)| \ \leq \ ||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{y,s}}.$$

□

5.2 The non-negativity of the solution $u(x,t)$

In this section, we prove that the solution $u(x,t)$ of the viscosity system (4.6) is nonnegative.

Theorem 7. *Suppose the solution* $v, u \in C^2([0,T];\mathbb{H}^1(\mathbb{R}))$ *satisfy*

$$u_t + p(v)_x = \varepsilon u_{xx} + f(v,u) \qquad (x,t) \in \mathbb{R} \times (0,T] \tag{5.17}$$

where $\varepsilon > 0$, $p(v) = \frac{1}{2}g'v^{-2}$ *and* $f(v,u) = g'S - C_f u^2 v$ *with initial condition* (5.1). *Suppose that* $u(x,t) = 0$ *at* $x = \pm\infty$, $u(x,0) \geq 0$, $u(x,0)$ *has compact support and* $K = \max_{t\in[0,T]} \int_{-\infty}^{\infty} p(v(x,t))^2 dx$ *is finite. There is* $T_0 > 0$ *such that* $u(x,t) \geq 0$ *for all* $(x,t) \in \mathbb{R} \times (0,T_0]$.

Proof. Let

$$\phi(x,s) = \min\{0, u(x,s)\}$$

for all $(x,s) \in \mathbb{R} \times (0,T]$. Since $u(x,0) \geq 0$,

$$\phi(x,0) = 0 \quad \text{ for all } \ x \in \mathbb{R}. \tag{5.18}$$

Suppose for any $T_0 > 0$, there is $(x,s) \in \mathbb{R} \times (0,T_0]$ such that $u(x,s) < 0$. We will consider as two cases.

Case1 : $u(x_0,0) = 0$ and $u_s(x_0,0) < 0$ for some $x_0 \in \mathbb{R}$.

Since $u(x_0,0) = 0$,

$$\phi(x_0,0) = 0 \tag{5.19}$$

and there is $0 < T_1 \leq T$ such that

$$\phi(x_0,s) = u(x_0,s) \qquad (s \in (0,T_1]). \tag{5.20}$$

From (5.19) and (5.20), we have

$$\phi(x_0, s) = u(x_0, s) \qquad (s \in [0, T_1]).$$

Thus

$$\phi_s(x_0, s) = u_s(x_0, s) \qquad (s \in [0, T_1]). \tag{5.21}$$

Hence $\phi_s(x_0, 0) = u_s(x_0, 0)$. Since $u_s(x_0, 0) < 0$, $\phi_s(x_0, 0) < 0$. That is

$$\phi_s(x_0, 0) \neq 0. \tag{5.22}$$

Let $(x, s) \in \mathbb{R} \times [0, T_1]$ where $0 < T_1 \leq T$ and

$$\Phi(s) = \int_{-\infty}^{\infty} \phi^2(x, s)dx.$$

By (5.18), for each $x \in \mathbb{R}$, we have

$$\Phi(0) = \int_{-\infty}^{\infty} \phi^2(x, 0)dx = 0$$

and

$$\Phi'(0) = \left[\int_{-\infty}^{\infty} 2\phi(x, s)\phi_s(x, s)dx\right]_{s=0} = \int_{-\infty}^{\infty} 2\phi(x, 0)\phi_s(x, 0)dx = 0.$$

In addition, we have

$$\begin{aligned}
\Phi''(0) &= \left[\int_{-\infty}^{\infty} 2\phi(x, s)\phi_{ss}(x, s) + 2\phi_s^2(x, s)dx\right]_{s=0} \\
&= \int_{-\infty}^{\infty} 2\phi(x, 0)\phi_{ss}(x, 0) + 2\phi_s^2(x, 0)dx \\
&= \int_{-\infty}^{\infty} 2\phi(x, 0)\phi_{ss}(x, 0)dx + \int_{-\infty}^{\infty} 2\phi_s^2(x, 0)dx.
\end{aligned}$$

From (5.18), we obtain $\int_{-\infty}^{\infty} 2\phi(x, 0)\phi_{ss}(x, 0)dx = 0$. By (5.22), we get

$$\Phi''(0) = \int_{-\infty}^{\infty} 2\phi_s^2(x, 0)dx \neq 0.$$

By the Taylor's theorem, then there exists a function $h_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(s) = \Phi(0) + \Phi'(0)s + \frac{\Phi''(0)s^2}{2!} + h_2(s)s^2$$

and $\lim_{s\to 0} h_2(s) = 0$. Since $\Phi(0) = 0$ and $\Phi'(0) = 0$, it follows that

$$\Phi(s) = \frac{\Phi''(0)s^2}{2!} + R_2(s). \tag{5.23}$$

where $\Phi''(0) = \int_{-\infty}^{\infty} 2\phi_s^2(x,0)dx \neq 0$ and $R_2(s) = h_2(s)s^2$ is the remainder term.

Let $\varepsilon > 0$. Choose λ to be a finite negative number satisfying $\lambda < \dfrac{-2K}{\varepsilon\Phi''(0)}$. Let $\bar{\varepsilon} > 0$ and $t \in (0, T_1]$. Multiplying the equation (5.17) with $\phi(x,s)\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}$ where $0 < s \leq T_1 \leq T$ and then integrating over $\mathbb{R} \times (0,t)$ yields

$$\int_0^t \int_{-\infty}^{\infty} u_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds + \int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$$

$$= \varepsilon \int_0^t \int_{-\infty}^{\infty} u_{xx}(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds + \int_0^t \int_{-\infty}^{\infty} f(v(x,s),u(x,s))\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds. \tag{5.24}$$

Step 1 : Consider $\int_0^t \int_{-\infty}^{\infty} u_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$.

We have

$$\int_0^t \int_{-\infty}^{\infty} u_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds = \int_0^t \int_{-\infty}^{\infty} \phi_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds \tag{5.25}$$

Consider $\int_0^t \int_{-\infty}^{\infty} \phi_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$. We will show that $\phi_s(x,s), \phi(x,s) \in L^2(\mathbb{R} \times [0,t))$ and $e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \in L^\infty(\mathbb{R} \times [0,t))$.

(i) To show that $\phi_s(x,s) \in L^2(\mathbb{R} \times [0,t))$, since $u(x,s) \in H^1(\mathbb{R} \times (0,T_1])$,

$$\int_{-\infty}^{\infty} \int_0^t |u_s(x,s)|^2 dsdy < \infty.$$

By Theorem 2, we have

$$\int_0^t \int_{-\infty}^{\infty} |u_s(x,s)|^2 dxds < \infty.$$

Since $\phi(x,s) = \min\{0, u(x,s)\}$,

$$\int_0^t \int_{-\infty}^{\infty} |\phi_s(x,s)|^2 dxds < \infty$$

and hence

$$||\phi_s||_2 = \left[\int_0^t \int_{-\infty}^{\infty} |\phi_s(x,s)|^2 dxds\right]^{1/2} < \infty.$$

That is

$$\phi_s(x,s) \in L^2(\mathbb{R} \times [0,t)).$$

(ii) To show that $\phi(x,s) \in L^2(\mathbb{R} \times [0,t))$, since $u \in H^1(\mathbb{R} \times (0,T_1])$,

$$\int_{-\infty}^{\infty} \int_0^t |u(x,s)|^2 dsdx < \infty.$$

By Theorem 2,

$$\int_0^t \int_{-\infty}^{\infty} |u(x,s)|^2 dxds < \infty.$$

Since $\phi(x,s) = \min\{0, u(x,s)\}$,

$$\int_0^t \int_{-\infty}^{\infty} |\phi(x,s)|^2 dyds < \infty$$

and hence

$$||\phi||_2 = \left[\int_0^t \int_{-\infty}^{\infty} |\phi(x,s)|^2 dxds\right]^{1/2} < \infty.$$

That is

$$\phi(x,s) \in L^2(\mathbb{R} \times [0,t)).$$

From (i) and (ii), $\phi_s(x,s), \phi(x,s) \in L^2$. By Theorem 1, we obtain

$$\phi_s(x,s)\phi(x,s) \in L^1(\mathbb{R} \times [0,t)).$$

Then

$$\int_0^t \int_{-\infty}^{\infty} |\phi_s(x,s)\phi(x,s)| dxds < \infty. \tag{5.26}$$

(iii) To show that $e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \in L^\infty$, since $\bar{\varepsilon}$ is fixed and $0 < s \le T_1$,

$$\bar{\varepsilon} < s + \bar{\varepsilon} \le T_1 + \bar{\varepsilon}.$$

Then

$$\frac{1}{\bar{\varepsilon}} > \frac{1}{s+\bar{\varepsilon}} \ge \frac{1}{T_1 + \bar{\varepsilon}}.$$

Since $-\lambda$ is positive,

$$\frac{-\lambda}{\bar{\varepsilon}} > \frac{-\lambda}{s+\bar{\varepsilon}} \geq \frac{-\lambda}{T_1+\bar{\varepsilon}}.$$

Hence

$$e^{\frac{-\lambda}{T_1+\bar{\varepsilon}}} \leq e^{\frac{-\lambda}{s+\bar{\varepsilon}}} < e^{\frac{-\lambda}{\bar{\varepsilon}}}. \tag{5.27}$$

Since $-\lambda > 0$ is finite and $\bar{\varepsilon}$ is fixed, $e^{\frac{-\lambda}{\bar{\varepsilon}}}$ is a constant. Thus

$$e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \in L^\infty.$$

By (5.27) and (5.26), we have

$$\int_0^t \int_{-\infty}^{\infty} |\phi_x(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}|dxds < e^{\frac{-\lambda}{\bar{\varepsilon}}} \int_0^t \int_{-\infty}^{\infty} |\phi_x(x,s)\phi(x,s)|dxds$$
$$< \infty.$$

By Theorem 2, we get that

$$\int_0^t \int_{-\infty}^{\infty} \phi_x(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds = \int_{-\infty}^{\infty} \int_0^t \phi_x(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx \tag{5.28}$$

We have

$$\int_{-\infty}^{\infty} \int_0^t \phi_x(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx = \int_{-\infty}^{\infty} \int_0^t (\frac{1}{2}\phi^2(x,s))_s e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx. \tag{5.29}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_0^t (\frac{1}{2}\phi^2(x,s))_s e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx \\
&\quad= \int_{-\infty}^{\infty} \left[\frac{1}{2}e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\phi^2(x,s)|_{s=0}^{s=t} - \int_0^t \frac{1}{2}\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}ds\right]dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t) - \frac{1}{2}e^{\frac{-\lambda}{\bar{\varepsilon}}}\phi^2(x,0) - \int_0^t \frac{1}{2}\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}ds\right]dx.
\end{aligned} \tag{5.30}$$

By (5.18) and (5.30), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_0^t (\frac{1}{2}\phi^2(x,s))_s e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx \\
&\quad= \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t)dx - \frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx.
\end{aligned} \tag{5.31}$$

Now, consider $\displaystyle\int_{-\infty}^{\infty}\int_0^t \phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx$.

(iv) To show that $\frac{\lambda}{(s+\bar{\varepsilon})^2}\in L^\infty$, since $\bar{\varepsilon}>0$ is fixed and $0<s\le T_1$,

$$\bar{\varepsilon}^2<(s+\bar{\varepsilon})^2\le(T_1+\bar{\varepsilon})^2.$$

Then

$$\frac{1}{\bar{\varepsilon}^2}>\frac{1}{(s+\bar{\varepsilon})^2}\ge\frac{1}{(T_1+\bar{\varepsilon})^2}.$$

Since λ is negative,

$$\frac{\lambda}{\bar{\varepsilon}^2}<\frac{\lambda}{(s+\bar{\varepsilon})^2}\le\frac{\lambda}{(T_1+\bar{\varepsilon})^2}. \tag{5.32}$$

Therefore,

$$\frac{\lambda}{(s+\bar{\varepsilon})^2}\in L^\infty(\mathbb{R}\times[0,t)).$$

From (i), we have $\phi(x,s)\in L^2$. By Theorem 1, $\phi^2(x,s)\in L^1(\mathbb{R}\times[0,t))$. That is

$$\int_{-\infty}^{\infty}\int_0^t|\phi^2(x,s)|dsdx<\infty. \tag{5.33}$$

By (5.27), (5.32) and (5.33), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\int_0^t\left|\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\right|dsdx &\le\frac{\lambda}{(T_1+\bar{\varepsilon})^2}e^{\frac{-\lambda}{\bar{\varepsilon}}}\int_{-\infty}^{\infty}\int_0^t|\phi^2(x,s)|dsdx\\ &<\infty.\end{aligned}$$

By Theorem 2, we have

$$\int_{-\infty}^{\infty}\int_0^t\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx=\int_0^t\int_{-\infty}^{\infty}\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds \tag{5.34}$$

From (5.34) and (5.31), we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_0^t(\frac{1}{2}\phi^2(x,s))_s e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dsdx\\ &\quad=\frac{1}{2}\int_{-\infty}^{\infty}e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t)dx-\frac{1}{2}\int_0^t\int_{-\infty}^{\infty}\phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds\end{aligned} \tag{5.35}$$

By (5.25), (5.28), (5.29) and (5.35), then we obtain

$$\int_0^t \int_{-\infty}^{\infty} u_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t)dx - \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds \tag{5.36}$$

Step 2 Consider $\int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$.

We have

$$\int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds = \int_0^t e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\left[\int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)dx\right]ds. \tag{5.37}$$

Now, we consider $\int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)dx$. Integrating by parts, we get that

$$\int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)dx = p(v(x,s))_x\phi(x,s)|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx.$$

Since $u(x,s)=0$ at $x=\pm\infty$,

$$\phi(x,s)=0 \quad at \quad x=\pm\infty.$$

Thus $p(v(x,s))_x\phi(x,s)|_{x=-\infty}^{x=\infty}=0$. Hence

$$\int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)dx = -\int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx. \tag{5.38}$$

From (5.37) and (5.38), we have

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds &= \int_0^t e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\left[-\int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx\right]ds\\ &= -\int_0^t e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\left[\int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx\right]ds.\end{aligned} \tag{5.39}$$

Consider $\int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx$. By Theorem 1, we have

$$\int_{-\infty}^{\infty} p(v(x,s))\phi_x(x,s)dx \le \frac{1}{4\varepsilon}\int_{-\infty}^{\infty} p(v(x,s))^2dx + \varepsilon\int_{-\infty}^{\infty} \phi_x^2(x,s)dx. \tag{5.40}$$

By (5.39) and (5.40), we obtain

$$-\int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x \phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$$

$$\leq \frac{1}{4\varepsilon} \int_0^t \int_{-\infty}^{\infty} p(v(x,s))^2 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds + \varepsilon \int_0^t \int_{-\infty}^{\infty} \phi_x^2(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds. \tag{5.41}$$

Step 3 Consider $\varepsilon \int_0^t \int_{-\infty}^{\infty} u_{xx}(x,s)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$. Then

$$\begin{aligned} \varepsilon \int_0^t \int_{-\infty}^{\infty} u_{xx}(x,s)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds &= \varepsilon \int_0^t \int_{-\infty}^{\infty} \phi_{xx}(x,s)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \\ &= \varepsilon \int_0^t \int_{-\infty}^{\infty} (\phi_x)_x(x,s)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \\ &= \varepsilon \int_0^t e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \int_{-\infty}^{\infty} (\phi_x)_x(x,s)\phi(x,s) dxds. \end{aligned} \tag{5.42}$$

Now, we consider $\int_{-\infty}^{\infty} (\phi_x)_x(x,s)\phi(x,s)dx$. Integrating by parts, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} (\phi_x)_x(x,s)\phi(x,s)dx &= \phi_x(x,s)\phi(x,s)|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \phi_x(x,s)\phi_x(x,s)dx \\ &= \phi_x(x,s)\phi(x,s)|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} \phi_x^2(x,s)dx. \end{aligned}$$

Since $u(x,s) = 0$ at $x = \pm\infty$ and $\phi(x,s) = \min\{0, u(x,s)\}$, $\phi(x,s) = 0$ at $x = \pm\infty$. Hence $\phi_x(x,s)\phi(x,s)|_{x=-\infty}^{x=\infty} = 0$. Therefore,

$$\int_{-\infty}^{\infty} (\phi_x)_x(x,s)\phi(x,s)dx = -\int_{-\infty}^{\infty} \phi_x^2(x,s)dx. \tag{5.43}$$

From (5.42) and (5.43), we have

$$\varepsilon \int_0^t \int_{-\infty}^{\infty} u_{xx}(x,s)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds = -\varepsilon \int_0^t \int_{-\infty}^{\infty} e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \phi_x^2(x,s) dxds. \tag{5.44}$$

Step 4 Consider $\int_0^t \int_{-\infty}^{\infty} f(v,u)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$. We have

$$\int_0^t \int_{-\infty}^{\infty} f(v,u)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$$

$$= \int_0^t \int_{-\infty}^{\infty} (g'S - C_f u^2(x,s)v(x,s))\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$$

$$= \int_0^t \int_{-\infty}^{\infty} \left[g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} - C_f u^2(x,s)v(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \right] dxds$$

$$= \int_0^t \int_{-\infty}^{\infty} \left[g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} - C_f \phi^2(x,s)v(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \right] dxds$$

$$= \int_0^t \int_{-\infty}^{\infty} \left[g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} - C_f v(x,s)\phi^3(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \right] dxds$$

$$= \int_0^t \int_{-\infty}^{\infty} g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds - \int_0^t \int_{-\infty}^{\infty} C_f v(x,s)\phi^3(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds. \tag{5.45}$$

Consider $\int_0^t \int_{-\infty}^{\infty} g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$. Since $\phi(x,s) = \min\{0, u(x,s)\}$ for all $(x,s) \in \mathbb{R} \times (0,T]$,

$$\phi(x,s) \leq 0.$$

Since $g' > 0$, $S > 0$ and $e^{\frac{-\lambda}{s+\bar{\varepsilon}}}$ is nonnegative function for all $0 < s < T_1$,

$$g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \leq 0.$$

Hence

$$\int_0^t \int_{-\infty}^{\infty} g'S\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \leq 0. \tag{5.46}$$

We consider $-\int_0^t \int_{-\infty}^{\infty} C_f v(x,s)\phi^3(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds$. By Theorem 6, we have

$$|v| \ \leq \ ||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}}.$$

Since $\phi(x,s) \leq 0$ for all $(x,s) \in \mathbb{R} \times (0,T_1]$, $\phi^3(x,s) \leq 0$. Thus

$$v\phi^3 \geq \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}} \right) \phi^3.$$

Since $e^{\frac{-\lambda}{s+\bar{\varepsilon}}}$ is nonnegative for all $0 < s \leq T_1$,

$$v\phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \geq \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}} \right) \phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}}.$$

Since $C_f > 0$,

$$C_f v\phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \geq C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}}.$$

That is

$$\begin{aligned} -C_f v\phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} &\leq -C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \\ &= C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 (-\phi) e^{\frac{-\lambda}{s+\bar{\varepsilon}}}. \end{aligned}$$

Since $\phi(x,s) = \min\{0, u(x,s)\}$,

$$0 \leq -\phi \leq ||u||_\infty.$$

Hence

$$-C_f v\phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} \leq C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 ||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}}.$$

Thus

$$\begin{aligned} &-\int_0^t \int_{-\infty}^\infty C_f v\phi^3 e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \\ &\qquad \leq \int_0^t \int_{-\infty}^\infty C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 ||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds. \qquad (5.47) \end{aligned}$$

By (5.46) and (5.47), we obtain

$$\begin{aligned} &\int_0^t \int_{-\infty}^\infty g' S\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds - \int_0^t \int_{-\infty}^\infty C_f v(x,s)\phi^3(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \\ &\qquad \leq \int_0^t \int_{-\infty}^\infty C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 ||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds. \qquad (5.48) \end{aligned}$$

By (5.45) and (5.48), we have

$$\begin{aligned} &\int_0^t \int_{-\infty}^\infty f(v,u)\phi(x,s) e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds \\ &\qquad \leq \int_0^t \int_{-\infty}^\infty C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 ||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}} dxds. \qquad (5.49) \end{aligned}$$

Step 5 From (5.24), we have

$$\int_0^t \int_{-\infty}^{\infty} u_s(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds = -\int_0^t \int_{-\infty}^{\infty} p(v(x,s))_x\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds$$

$$\varepsilon\int_0^t \int_{-\infty}^{\infty} u_{xx}(x,s)\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds + \int_0^t \int_{-\infty}^{\infty} f(v(x,s),u(x,s))\phi(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds.$$

By (5.36), (5.41), (5.44) and (5.49), we have

$$\begin{aligned}
&\left[\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t)dx - \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds\right] \\
\leq \quad & \left[\frac{1}{4\varepsilon}\int_0^t \int_{-\infty}^{\infty} p(v(x,s))^2 e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds + \varepsilon\int_0^t \int_{-\infty}^{\infty} \phi_x^2(x,s)e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds\right] \\
& - \left[\varepsilon\int_0^t \int_{-\infty}^{\infty} e^{\frac{-\lambda}{s+\bar{\varepsilon}}}\phi_x^2(x,s)dxds\right] \\
& + \left[\int_0^t \int_{-\infty}^{\infty} C_f\left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}}\right)\phi^2||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds\right].
\end{aligned}$$

That is

$$\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t+\bar{\varepsilon}}}\phi^2(x,t)dx \quad \leq \quad & \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \phi^2(x,s)\cdot\frac{\lambda}{(s+\bar{\varepsilon})^2}\cdot e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds \\
& + \frac{1}{4\varepsilon}\int_0^t \int_{-\infty}^{\infty} p(v(x,s))^2 e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds \\
& + \int_0^t \int_{-\infty}^{\infty} C_f\left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}}\right)\phi^2||u||_\infty e^{\frac{-\lambda}{s+\bar{\varepsilon}}}dxds. \qquad (5.50)
\end{aligned}$$

Taking limit $\bar{\varepsilon} \to 0$ in (5.50), we have

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t}}\phi^2(x,t)dx \\
&\leq \frac{\lambda}{2}\int_0^t \int_{-\infty}^{\infty} \phi^2(x,s)\cdot\frac{e^{\frac{-\lambda}{s}}}{s^2}dxds + \frac{1}{4\varepsilon}\int_0^t \int_{-\infty}^{\infty} p(v(x,s))^2 e^{\frac{-\lambda}{s}}dxds \\
&\qquad + \int_0^t \int_{-\infty}^{\infty} C_f\left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}}\right)\phi^2||u||_\infty e^{\frac{-\lambda}{s}}dxds. \qquad (5.51)
\end{aligned}$$

Recall that $\lambda < \dfrac{-2K}{\varepsilon\Phi''(0)}$. Then

$$-2\lambda\varepsilon\frac{\Phi''(0)}{4} = -\frac{1}{2}\lambda\varepsilon\Phi''(0) > K.$$

Since $s > 0$ and $e^{\frac{-\lambda}{s}} > 0$,

$$-2s^2e^{\frac{-\lambda}{s}}\lambda\varepsilon\frac{\Phi''(0)}{4} > s^2e^{\frac{-\lambda}{s}}K. \tag{5.52}$$

If $T_1 > 0$ is sufficiently small, then $0 < s < T_1$ is also small. Since $\lim_{s\to 0} h_2(s) = 0$ and $0 < s < T_1$; by definition of limit,

$$|h_2(s)| < \frac{\Phi''(0)}{4}.$$

That is

$$-\frac{\Phi''(0)}{4} < h_2(s) < \frac{\Phi''(0)}{4}. \tag{5.53}$$

Adding $\dfrac{\Phi''(0)}{2}$ to the equation (5.53), then

$$\frac{\Phi''(0)}{2} - \frac{\Phi''(0)}{4} < \frac{\Phi''(0)}{2} + h_2(s) < \frac{\Phi''(0)}{2} + \frac{\Phi''(0)}{4}.$$

That is

$$\frac{\Phi''(0)}{4} < \frac{\Phi''(0)}{2} + h_2(s) < \frac{3\Phi''(0)}{4}.$$

Since $s > 0$, $e^{\frac{-\lambda}{s}}$, $-\lambda > 0$ and $\varepsilon > 0$, we have

$$-2s^2e^{\frac{-\lambda}{s}}\lambda\varepsilon\left[\frac{\Phi''(0)}{2} + h_2(s)\right] > -2s^2e^{\frac{-\lambda}{s}}\lambda\varepsilon\frac{\Phi''(0)}{4}. \tag{5.54}$$

By (5.52) and (5.54), we have

$$-2s^2e^{\frac{-\lambda}{s}}\lambda\varepsilon\left[\frac{\Phi''(0)}{2} + h_2(s)\right] > s^2e^{\frac{-\lambda}{s}}K. \tag{5.55}$$

Since $K = \max_{s\in[0,T]} \int_{-\infty}^{\infty} p(v)^2 dx < \infty$,

$$s^2e^{\frac{-\lambda}{s}}K \geq s^2e^{\frac{-\lambda}{s}}\int_{-\infty}^{\infty} p(v)^2 dx \tag{5.56}$$

for all $0 < s \leq T_1$. From (5.55) and (5.56), we have

$$-2s^2 e^{\frac{-\lambda}{s}} \lambda\varepsilon \left[\frac{\Phi''(0)}{2} + h_2(s)\right] > s^2 e^{\frac{-\lambda}{s}} \int_{-\infty}^{\infty} p(v)^2 dx.$$

That is

$$-2\varepsilon\lambda e^{\frac{-\lambda}{s}} \left[\frac{\Phi''(0)s^2}{2!} + R_2(s)\right] \geq s^2 e^{\frac{-\lambda}{s}} \int_{-\infty}^{\infty} p(v)^2 dx. \tag{5.57}$$

By (5.58) and (5.23), we obtain

$$\frac{-4\varepsilon\lambda}{2} e^{\frac{-\lambda}{s}} \Phi(s) = -2\varepsilon\lambda e^{\frac{-\lambda}{s}} \Phi(s) \geq s^2 e^{\frac{-\lambda}{s}} \int_{-\infty}^{\infty} p(v)^2 dx.$$

That is

$$\frac{-4\varepsilon\lambda e^{\frac{-\lambda}{s}}}{2} \Phi(s) - s^2 e^{\frac{-\lambda}{s}} \int_{-\infty}^{\infty} p(v)^2 dx \geq 0.$$

Then

$$\frac{4\varepsilon\lambda e^{\frac{-\lambda}{s}}}{2} \Phi(s) + s^2 e^{\frac{-\lambda}{s}} \int_{-\infty}^{\infty} p(v)^2 dx \leq 0.$$

Since $4\varepsilon > 0$ and $s^2 > 0$,

$$\frac{\lambda e^{\frac{-\lambda}{s}} \Phi(s)}{2s^2} + \frac{e^{\frac{-\lambda}{s}}}{4\varepsilon} \int_{-\infty}^{\infty} p(v)^2 dx \leq 0.$$

Since $\Phi(s) = \int_{-\infty}^{\infty} \phi^2(x, s) dx$,

$$\frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{\phi^2(x, s) e^{\frac{-\lambda}{s}}}{s^2} dx + \frac{1}{4\varepsilon} \int_{-\infty}^{\infty} p(v)^2 e^{\frac{-\lambda}{s}} dx \leq 0. \tag{5.58}$$

From (5.58), (5.51) becomes

$$\frac{1}{2} \int_{-\infty}^{\infty} e^{\frac{-\lambda}{t}} \phi^2(x, t) dx \leq \int_0^t \int_{-\infty}^{\infty} C_f \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \phi^2 ||u||_\infty e^{\frac{-\lambda}{s}} dx ds.$$

That is

$$\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t}} \phi^2(x, t) dx \leq 2C_f ||u||_\infty \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}} ||u_x||_{L^2_{x,s}} \right) \int_0^t \int_{-\infty}^{\infty} \phi^2 e^{\frac{-\lambda}{s}} dx ds. \tag{5.59}$$

Define

$$\xi(s) = \int_{-\infty}^{\infty} e^{\frac{-\lambda}{s}} \phi^2(x, s) dx ds. \tag{5.60}$$

By (5.59) and (5.60), we have

$$\xi(t) \leq C \int_0^t \xi(s)ds.$$

where $C = 2C_f||u||_\infty \left(||v_0||_\infty + \sqrt[4]{\frac{T}{2\pi\varepsilon}}||u_x||_{L^2_{x,s}} \right)$. By Theorem 2,

$$\xi(t) = 0. \tag{5.61}$$

By (5.60) and (5.61), we obtain

$$\int_{-\infty}^{\infty} e^{\frac{-\lambda}{t}} \phi^2(x,t)dx = 0$$

which implies that

$$\phi(x,t) = 0$$

for all $0 < t < T$. This is a contradiction. Therefore, we conclude that $\phi(x,t) \equiv 0$ for all $(x,t) \in \mathbb{R} \times (0,T]$. Hence $u(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0,T_0]$ for some $T_0 > 0$.

Case 2 : Either $u(x_0,0) > 0$ or $u_s(x_0,0) \geq 0$ for all $x_0 \in \mathbb{R}$.

In this case, for all $x_0 \in \mathbb{R}$, either $u(x_0,0) > 0$ or $u(x_0,0) = 0$ and $u_s(x_0,0) \geq 0$. Fix $x_0 \in \mathbb{R}$. First, we suppose that $u(x_0,0) = 0$ and $u_s(x_0,0) \geq 0$. Note that $x_0 \notin \text{supp} u(x,0)$. Consider

$$u_s(x_0,0) + p(v(x_0,0))_x = \varepsilon u_{xx}(x_0,0) + f(v(x_0,0),u(x_0,0)).$$

Since $f(v,u) = g'S - C_f u^2 v$,

$$u_s(x_0,0) = -p(v(x_0,0))_x + \varepsilon u_{xx}(x_0,0) + g'S - C_f u^2(x_0,0)v(x_0,0).$$

Since $u(x_0,0) = 0$, $u_{xx}(x_0,0) = 0$. Hence

$$u_s(x_0,0) = -p(v(x_0,0))_x + g'S.$$

By assumption $u_s(x_0,0) \geq 0$, we obtain

$$p(v(x_0,0))_x \leq g'S.$$

Integrating over (a,x) for $(a,x) \in \mathbb{R} \setminus \text{supp}u(x,0)$ yields

$$p(v(x_0,0))|_a^x \leq \int_a^x g'Sdx.$$

Then

$$p(v(x,0)) - p(v(a,0)) \leq g'Sx. \tag{5.62}$$

Since $p(v) = \dfrac{1}{2}g'v^{-2}$, (5.62) becomes

$$\frac{1}{2}g'v^{-2}(x,0) - \frac{1}{2}g'v^{-2}(a,0) \leq g'Sx.$$

Then

$$\frac{1}{v^2(x,0)} \leq 2Sx + C_1. \tag{5.63}$$

where $C_1 = \dfrac{1}{v^2(a,0)}$. If $x \leq \dfrac{-C_1}{2S}$, we have $2Sx + C_1 \leq 0$. By (5.63), $v^2(x,0) \leq 0$ which is undefined. This is a contradiction. Hence $u(x_0,0) > 0$ for all $x_0 \in \mathbb{R}$ which is contradiction. □

5.3 Expanding invariant region

The expanding invariant region is introduced in Chapter 2. In this section, we establish an expanding invariant region and prove that the trajectory of the solution to the system (4.6) is inside this expanding invariant region.

Theorem 8. *Let $U(x,t)$ be the smooth solution to (4.6), $K = \max_{t\in[0,T]} \int_{-\infty}^{\infty} p(v)^2 dx$ is finite and $S < 4C_f$. Suppose $u(x,0) \geq 0$, $u(x,t) = 0$ at $x = \pm\infty$ and $u(x,0)$ has compact support. Define the expanding invariant region*

$$\Sigma(t) = \bigcap_{j=1}^{2} \{U \in \mathbb{R}^2 : G_j(U) \leq Ae^{\omega t}\} \tag{5.64}$$

where $\omega = 4C_f g'$, $A = e$, $G_1(U) = u + 2\sqrt{g'}v^{-1/2}$ and $G_2(U) = -u + 2\sqrt{g'}v^{-1/2}$ are Riemann invariants. Then the trajectory $U(x,t)$ is inside $\Sigma(t)$ for all time $t > 0$; that is, $G_j(U(x,t)) \leq Ae^{\omega t}$ for all $j = 1,2$.

Proof. We will apply Theorem 4 to prove this theorem. Suppose that $U(x,t_0) \in \partial\Sigma(t_0)$. That is $G_j(U(x,t_0)) = Ae^{\omega t_0}$ for some $j = 1,2$. From Proposition 6, we have two Riemann invariants $G_1(U) = u+2\sqrt{g'}v^{-1/2}$ and $G_2(U) = -u+2\sqrt{g'}v^{-1/2}$. Then

$$\nabla G_1^T = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & 1 \end{pmatrix} \tag{5.65}$$

and

$$\nabla G_2^T = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & -1 \end{pmatrix}. \tag{5.66}$$

Step 1 We will show that ∇G_j at $U(x,t_0)$ is a left eigenvector of M for all $j = 1,2$.

By (5.65) and (4.7), we have

$$\begin{aligned}
\nabla G_1^T \cdot M &= \begin{pmatrix} -\sqrt{g'}v_0^{-3/2} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ g'v_0^{-3} & 0 \end{pmatrix} \\
&= \begin{pmatrix} g'v^{-3} & -\sqrt{g'}v^{-3/2} \end{pmatrix} \\
&= -\sqrt{g'}v^{-3/2} \begin{pmatrix} -\sqrt{g'}v^{-3/2} & 1 \end{pmatrix} \\
&= -\sqrt{g'}v^{-3/2} \nabla G_1^T \\
&= \lambda_1 \nabla G_1^T.
\end{aligned}$$

Hence ∇G_1^T at $U(x,t_0)$ is a left eigenvector of M.

By (5.66) and (4.7), we have

$$\begin{aligned}
\nabla G_2^T \cdot M &= \begin{pmatrix} -\sqrt{g'}^{-3/2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ g'v^{-3} & 0 \end{pmatrix} \\
&= \begin{pmatrix} -g'v^{-3} & -\sqrt{g'}v^{-3/2} \end{pmatrix} \\
&= \sqrt{g'}v^{-3/2} \begin{pmatrix} -\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} \\
&= \sqrt{g'}^{-3/2} \nabla G_2^T \\
&= \lambda_2 \nabla G_2^T.
\end{aligned}$$

Hence ∇G_2^T at $U(x,t_0)$ is a left eigenvector of M.

Step 2 We will show that G_j is quasi-convex at $U(x,t_0)$ i.e. at $U(x,t_0)$ whenever $\nabla G_j^T \cdot r = 0$, $\nabla^2 G_j(r,r) \geq 0$ for all $j = 1,2$.

By (5.65) and (3.6), we have

$$\nabla G_1^T \cdot r_1 = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} = 0.$$

Consider

$$\begin{aligned}
\nabla^2 G_1(r_1,r_1) &= r_1^T \begin{pmatrix} \frac{\partial^2 G_1}{\partial v^2} & \frac{\partial^2 G_1}{\partial v \partial u} \\ \frac{\partial^2 G_1}{\partial u \partial v} & \frac{\partial^2 G_1}{\partial u^2} \end{pmatrix} r_1 \\
&= \begin{pmatrix} 1 & \sqrt{g'}v^{-3/2} \end{pmatrix} \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{g'}v^{-3/2} \end{pmatrix} \\
&= \frac{3}{2}\sqrt{g'}v^{-5/2}.
\end{aligned}$$

By Proposition 2, we have $0 < v < \infty$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$. Since $g' > 0$ and $v > 0$, $\nabla^2 G_1(r_1,r_1)(v,u) = \frac{3}{2}\sqrt{g'}v^{-5/2} > 0$. So G_1 is quasi-convex at $U(x,t_0)$.

By (5.66) and (3.7), we have

$$\nabla G_2 \cdot r_2 = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix} = 0.$$

Consider

$$\nabla^2 G_2(r_2, r_2) = r_2^T \begin{pmatrix} \frac{\partial^2 G_2}{\partial v^2} & \frac{\partial^2 G_2}{\partial v \partial u} \\ \frac{\partial^2 G_2}{\partial u \partial v} & \frac{\partial^2 G_2}{\partial u^2} \end{pmatrix} r_2.$$

$$= \begin{pmatrix} 1 & -\sqrt{g'}v^{-3/2} \end{pmatrix} \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}\sqrt{g'}v^{-5/2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{g'}v^{-3/2} \end{pmatrix}$$

$$= \frac{3}{2}\sqrt{g'}v^{-5/2}$$

Since $g' > 0$ and $v > 0$, $\nabla^2 G_2(r_2, r_2)(v, u) = \frac{3}{2}\sqrt{g'}v^{-5/2} > 0$. So G_2 is quasi-convex at $U(x, t_0)$.

Step 3 We will show that $\nabla G_j \cdot H < \omega A e^{\omega t_0}$ for all $x \in \mathbb{R}$ and at $(t, U) = (t_0, U(x, t_0))$.

Step 3.1: $G_1 = u + 2\sqrt{g'}v^{-1/2} = Ae^{wt_0} \equiv \tilde{A}$. We want to show that $\nabla G_1 \cdot H < \omega A e^{\omega t_0}$. Consider

$$\nabla G_1 \cdot H = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ g'S - C_f u^2 v \end{pmatrix}$$

$$= g'S - C_f u^2 v.$$

Since $C_f > 0$, $v > 0$ and $u^2 \geq 0$, $C_f u^2 v \geq 0$. Thus $-C_f u^2 v \leq 0$ and hence

$$\nabla G_1 \cdot H \leq g'S.$$

Since $S < 4C_f$ and $1 < e \leq ee^{4C_f g' t_0}$,

$$\nabla G_1 \cdot H < 4C_f g' < (4C_f g')ee^{(4C_f g')t_0} = \omega A e^{\omega t_0} \equiv \omega \tilde{A}.$$

Step 3.2: $G_2 = -u + 2\sqrt{g'}v^{-1/2} = Ae^{wt_0} \equiv \tilde{A}$. We want to show that $\nabla G_2 \cdot H <$

$\omega A e^{\omega t_0}$. Consider

$$\nabla G_2 \cdot H = \begin{pmatrix} -\sqrt{g'}v^{-3/2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ g'S - C_f u^2 v \end{pmatrix}$$
$$= -g'S + C_f u^2 v.$$

Since $g' > 0$ and $S > 0$, $-g'S < 0$. Hence

$$\nabla G_2 \cdot H < C_f u^2 v.$$

Since $-u + 2\sqrt{g'}v^{-1/2} = \tilde{A}$,

$$v = \frac{4g'}{(\tilde{A} + u)^2}.$$

Thus

$$\nabla G_2 \cdot H < C_f u^2 \frac{4g'}{(\tilde{A} + u)^2} \tag{5.67}$$

Since $\tilde{A} > 0$, $\tilde{A} + u > u$. By Theorem 7, we have $u \geq 0$. So $(\tilde{A} + u)^2 > u^2$ and hence

$$\frac{1}{(\tilde{A} + u)^2} < \frac{1}{u^2}. \tag{5.68}$$

By (5.67) and (5.68), we have

$$\nabla G_2 \cdot H < C_f u^2 \frac{4g'}{u^2} = 4C_f g' \tag{5.69}$$

From (5.69) and $1 < e \leq e e^{4C_f g' t_0}$, we have

$$\nabla G_2 \cdot H < (4C_f g') e e^{(4C_f g')t_0} = \omega A e^{\omega t_0} \equiv \omega \tilde{A}.$$

Hence $\nabla G_j \cdot H < \omega A e^{\omega t_0}$ for all $x \in \mathbb{R}$ and at $(t, U) = (t_0, U(x, t_0))$. From Step 1, Step 2 and Step 3, by Theorem 4, the trajectory $U(x, t)$ is inside $\Sigma(t)$ for all time $t > 0$; that is, $G_j(U(x, t)) \leq A e^{\omega t}$ for all $j = 1, 2$. □

5.4 A priori bound of solution (v, u)

In this section, we give the upper bound of the solution u and the lower bound of the solution v of the system (4.6).

Theorem 9. *Let (v, u) be a solution to the system (4.6) that satisfies all of condition to the Theorem 8. Then*

$$0 \leq u(x,t) \leq Ae^{\omega t} - 2\sqrt{\frac{g'}{v}} \quad \text{and} \quad v(x,t) \geq \frac{4g'}{(Ae^{\omega t})^2}$$

for all $(x,t) \in \mathbb{R} \times (0,T]$ where $\omega = 4C_f g'$ and $A = e$.

Proof. By Theorem 8, we have

$$G_1(v,u) = u + 2\sqrt{\frac{g'}{v}} \leq Ae^{\omega t} \tag{5.70}$$

and

$$G_2(v,u) = -u + 2\sqrt{\frac{g'}{v}} \leq Ae^{\omega t}. \tag{5.71}$$

From (5.70) and (5.71), we obtain

$$u \leq Ae^{\omega t} - 2\sqrt{\frac{g'}{v}}.$$

By Theorem 7, we have

$$0 \leq u(x,t) \leq Ae^{\omega t} - 2\sqrt{\frac{g'}{v}}.$$

Adding (5.70) and (5.71), we have

$$4\sqrt{\frac{g'}{v}} \leq 2Ae^{\omega t}.$$

Then

$$\sqrt{\frac{g'}{v}} \leq \frac{Ae^{\omega t}}{2}$$

and hence

$$v \geq \frac{4g'}{(Ae^{\omega t})^2}.$$

□

CHAPTER VI

CONCLUSION

The initial aim of this thesis is to prove existence and uniqueness of weak solutions to the system of hyperbolic conservation laws (3.1) for a motion of flood wave with initial condition (5.1) by the vanishing viscosity method. The idea of the vanishing viscosity method is to establish solutions for the viscosity parabolic system (4.6) and then apply Theorem 5. In this work, we can prove a priori bounds for solutions of (4.6) given in Theorem 9, that is

$$0 \leq u(x,t) \leq Ae^{\omega t} - 2\sqrt{\frac{g'}{v}} \quad \text{and} \quad v(x,t) \geq \frac{4g'}{(Ae^{\omega t})^2}$$

for all $(x,t) \in \mathbb{R} \times (0,T]$ where $\omega = 4C_f g'$ and $A = e$. Using these estimates, the author believes that we can prove an a priori bound of the form (4.3) for the solution v^ε as well. Combining the obtained estimates with a standard (local) existence of smooth solutions for parabolic systems, we obtain by Theorem 5 the existence and uniqueness of weak solution to the flood wave equations (3.1).

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VITA

Name	Ms. Piraya Thipakorn
Date of Birth	26 April 1988
Place of Birth	Satun, Thailand
Education	B.Sc. (Mathematics) (Second Class Honours), Prince of Songkla University, 2009
Scholarship	Development and Promotion of Science and Technology Talents Project (DPST)