จุดตรึงในปริภูมิเอกรูป

นายสิทธิโชค ทรงสอาด

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

FIXED POINTS IN UNIFORM SPACES

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2014

Thesis Title	FIXED POINTS IN UNIFORM SPACES
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Field of Study	Mathematics
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สิทธิโชค ทรงสอาด : จุดตรึงในปริภูมิเอกรูป. (FIXED POINTS IN UNIFORM SPACES)
อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร.พิเชฐ ชาวหา, 53 หน้า.

เราได้แนะนำสัญลักษณ์ของการส่งแบบ J-contraction และ การส่งแบบ J-nonexpansive บนปริภูมิเอกรูปที่ก่อกำเนิดโดยกลุ่มเมตริกเทียมซึ่งเป็นการวางนัยทั่วไปของการส่งแบบ Φ-contraction และการส่งแบบ j-nonexpansive ตามลำดับ ในงานนี้ได้นำเสนอทฤษฎีจุดตรึงสำหรับการส่งแบบ J-contraction และ การส่งแบบ J-nonexpansive ภายใต้เงื่อนไขที่แตกต่างซึ่งคลุมผลงานของ Angelov และยังได้ให้เกณฑ์สำหรับการส่งที่เป็นการส่งแบบ J-contraction และ การส่งแบบ J-nonexpansive นอกจากนี้เรายังได้สืบค้นเซตของจุดตรึงของการส่งเหล่านั้นโดยใช้แนวคิดของเสถียรภาพเสมือน และยังได้ให้ตัวอย่างที่น่าสนใจไว้อีกด้วย

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..............................

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..............................

ปีการศึกษา 2557

##5373890923 : MAJOR MATHEMATICS

KEYWORDS : FIXED POINTS / Φ-CONTRACTION / UNIFORM SPACES

SITTICHOKE SONGSA-ARD : FIXED POINTS IN UNIFORM SPACES. ADVISOR : ASSOC. PROF. PICHET CHAOHA, Ph.D., 53 pp.

We introduce notions of J-contractions and J-nonexpansive maps on a uniform space generated by a collection of pseudometrics which generalize Φ-contractions and j-nonexpansive maps, respectively. Fixed point theorems for J-contractions and J-nonexpansive maps under various conditions which cover Angelov's results are presented, and some criteria for maps to be J-contractions and J-nonexpansive maps are also given. Furthermore, we investigate their fixed point sets using the concept of virtual stability and give some interesting examples.

Department : Mathematics and Computer Science .... Student's Signature ....

Field of Study : Mathematics .... Advisor's Signature ....

Academic Year : 2014 ....

ACKNOWLEDGEMENTS

The most rewarding achievement in my life is the completion of my doctoral dissertation. I would like to take this opportunity to express my immense gratitude to all those persons who have given their invaluable support and assistance. In particular, I am profoundly indebted to my advisor, Associate Professor Dr. Phichet Chaoha, who was very generous with his time and knowledge and assisted me to complete the dissertation. I also sincerely grateful to my thesis committee, Associate Professor Dr. Imchit Termwuttipong, Associate Professor Dr. Wicharn Lewkeeratiyutkul, Assistant Professor Dr. Songkiat Sumetkijakan and Dr. Annop Kaewkhao, who gave valuable suggestions on this thesis.

I would like to acknowledge The Development and Promotion of Science and Technology Talents Project (DPST) for financial support throughout my graduate study.

Notwithstanding all the above support for this project, any error and/or omission are solely my own.

Finally, this thesis is dedicated to my parent.

CONTENTS

CHAPTER I

INTRODUCTION

Considering a selfmap f on a set X, there is a simple question whether the map has a fixed point in X, or in other words, whether an equation $f(x) = x$ has a solution. However, solving such equations is not easy in general, and sometimes there is no such solution (for example $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$). Therefore, some mathematicians who would like to discover some conditions that guarantee the existence of a fixed point develop fixed point theorems. In general fixed point theory, we try to identify conditions imposed on a set X and a selfmap f on X for the existence of a fixed point.

Fixed point theory have many applications in various branches of mathematics, and also in sciences such as chemistry, physics, engineerings, biology, or medicine. It is an important tool for solving equations or systems of equations. For every equation, we can find a corresponding map whose fixed point is a solution of the original equation. Sometimes scientists need to use a complicated equation that represents a real world problem, so it is not easy to find a solution for the equation. However, if we have a fixed point theorem for a map corresponding to the equation, we immediately know when the solution exists. Moreover, some fixed point theorems also provide an approximation of fixed points.

Many fixed point theorems are developed in the context of metric spaces, for example, the celebrated Banach's Contraction Theorem. Banach's contraction theorem is simple but powerful because it requires only a contraction selfmap and completeness of its domain to guarantee the existence and uniqueness of fixed points. Additionally, there is another well-known fixed point theorem in a Banach space called Schauder's fixed point theorem. Schauder's fixed point theorem requires convexity and compactness of the ambient space and continuity of a map; however, Schauder's result does not guarantee the uniqueness of fixed points. For solving a differential equation, we usually need the uniqueness of solutions. With this fact in mind, we are focusing on Banach's contraction theorem.

In 1971 [5], Cain and Nashed introduced a notion of γ-contraction selfmaps on a locally convex space and proved a fixed point theorem for a γ-contraction as follows:

Theorem 1.1. *Let $(E, \mathcal{A})$ be a locally convex space generated by a collection $\mathcal{A}$ of seminorms, $X \subseteq E$ and $T : X \to X$. If X is sequentially complete and T is a γ-contraction for any $\gamma \in \mathcal{A}$, then T has a unique fixed point in X and for each $x \in X$, the iterative sequence of x converges to the fixed point of T.*

Furthermore, there is an application of a fixed point theorem for a γ-contraction that solves the differential equation in a locally convex space by Chonwerayuth, Termwuttipong and Chaoha in 2011 [7].

In 1987 [1], Angelov gave a notion of Φ-contraction selfmaps on a uniform space whose uniformity is generated by a saturated collection of pseudometrics. It is vivid that a Φ-contraction simultaneously generalizes Banach's contraction theorem on a metric space as well as the fixed point theorem for γ-contractions mentioned above on a locally convex space. Moreover, Angelov presented fixed point theorems for a Φ-contraction and a special kind of a Φ-contraction under various conditions, and applied those theorems to find a solution of differential equations.

Later in 1991 [2], Angelov extended the notion of a Φ-contraction to a j-nonexpansive selfmap on a uniform space whose uniformity is generated by a saturated collection of pseudometrics, and gave some conditions for a j nonexpansive selfmap and its domain to guarantee the existence of a fixed point. Moreover, his result could be applied to some complicated differential equations for solving a solution. However, we found that there was a minor flaw in the proof of Theorem 1 in [2] where the surjectivity of the map j was implicitly used without any prior assumption.

In 2009 [12], the author introduced the notions of a functionally lipschitzian selfmap and a functionally uniformly lipschitzian selfmap on a normed space and gave criteria for a map to be functionally lipschitzian and functionally uniformly lipschitzian in a Banach space with a Schauder basis. As results, being functionally lipschitzian and being functionally uniformly lipschitzian are sufficient conditions for weak continuity and weak virtual stability (Definition 2.67), respectively, and being functionally (uniformly) lipschitzian and being (uniformly) lipschitzian are equivalent in a finite-dimensional normed space. However, there is a functionally uniformly lipschitzian map whose fixed point set free, so we investigate some fixed point theorems for a functionally uniformly lipschitzian selfmap and improve definitions of a functionally lipschitzian selfmap and a functionally uniformly lipschitzian selfmap in more general spaces.

According to the above observation, if we have finitely many Φ-contractions and j-nonexpansive maps on a locally convex space, then the sum of those maps is similar to a functionally lipschitzian map, so a concept of a map j can be naturally replaced by a multi-valued map J to obtain more general, yet interesting, notions of J-contractions and J-nonexpansive maps (Definition 3.8 and Definition 3.1, respectively). Therefore, in this dissertation, we present main concepts of J-contractions and J-nonexpansive maps on a uniform space generated by a collection of pseudometrics which generalize Φ-contractions and j-nonexpansive maps, respectively, aim to correct and simplify the proof of Theorem 1 in [2], and investigate the existence of a fixed point of J-contractions and J-nonexpansive maps. Furthermore, a J-contraction which is a special kind of a J-nonexpansive map plays a similar role as a contraction in yielding the uniqueness of fixed points and we are able to recover results of a Φ-contraction in [1].

In 1973 [4], Bruck's result showed that if X is a weakly compact convex subset of a Banach space and $T : X \to X$ is a nonexpansive selfmap satisfying the conditional fixed point property, then the fixed point set of T is a nonexpansive retract of X. Later in 2001 [3], Benavides and Ramirez improved Bruck's result to include a larger class of asymptotically nonexpansive and weakly asymptotically nonexpansive selfmaps. A benefit of these Bruck-type results is a connection between the fixed point set and the domain of the map that results in some topological structures such as connectedness and contractibility of the fixed point set.

In 2009 [6], Chaoha and Atiponrat gave a notion of virtually stable selfmaps on a Hausdorff space generalizing a nonexpansive-type selfmap. They also gave the connection between the fixed point set and the convergence set of a virtually stable selfmap via a retraction.

Although the concept of being J-nonexpansive is developed by using the concept of being functionally uniformly lipschitzian which is a sufficient condition for weak virtual stability, a J-nonexpansive selfmap may not be virtually stable, so we would like to investigate some sufficient conditions for a J-nonexpansive selfmap to be virtually stable.

In this dissertation, we will recall some backgrounds in topology, functional analysis and fixed point theory in the second chapter. Then, in Chapter 3, we introduce notions of J-nonexpansive selfmaps and J-contraction selfmaps on a uniform space whose uniformity is generated by a saturated collection of pseudometrics, and present interesting results of fixed point theorems under various conditions, and hence, our results cover Angelov's results in [1] and [2]. Furthermore, a notion of a functionally uniformly lipschitzian selfmap with respect to the sequence on a normed space, which generalizes a functionally uniformly lipschitzian selfmap, is introduced and fixed point theorems of a functionally uniformly lipschitzian selfmap with respect to the sequence are given. Then we also show that every functionally uniformly lipschitzian selfmap with respect to the sequence is a J-contraction, so it is J-nonexpansive. In Chapter 4, criteria for a selfmap to be J-nonexpansive on a Banach space having a normalized Schauder basis are given. Finally, in Chapter 5, we give some sufficient conditions for J-nonexpansive selfmaps to be virtually stable. Since a Hausdorff uniform spaces is completely regular, by Theorem 2.6 in [6], we immediately obtain that the fixed point set of a J-nonexpansive selfmap is a retract of its convergence set. As the result, we obtain some explicit examples of functionally uniformly lipschitzian selfmaps with respect to the sequence including the one in Example 5.5 that is not nonexpansive and hence falls outside the framework of Bruck in [4].

CHAPTER II

PRELIMINARIES

In this chapter, we recall and collect some definitions, propositions and theorems used in this dissertation.

2.1 Topological Spaces

Definition 2.1. A **topology** on a set X is a collection $\mathcal{T}$ of subset of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$,

2. the union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$,

3. the intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X, but we often omit specific mention of $\mathcal{T}$ if no confusion will arise.

Definition 2.2. Let Y be a subset of a topological space $(X, \mathcal{T}_X)$. We define the topology $\mathcal{T}_Y$ on Y by $\mathcal{T}_Y = \{A \cap Y \subseteq Y : A \in \mathcal{T}_X\}$ and call it the **subspace topology of** Y.

Definition 2.3. Let $(X, \mathcal{T})$ be a topological space and O a subset of X. We say that

1. O is **open** if O belongs to the collection $\mathcal{T}$.

2. O is **closed** if $X - O$ is open.

3. For $x \in X$, O is a **neighborhood of** x if O is an open set containing x.

Definition 2.4. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **closure** of A is defined as the intersection of all closed sets (in X) containing A. The closure of A is denoted by $\overline{A}$.

Note that : $\overline{A}$ is the smallest closed set containing A.

Definition 2.5. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

1. for each $x \in X$, there is at least one basis element B containing x.

2. if x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset O of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in O$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq O$. Equivalently, $\mathcal{T}$ is the collection of all unions of basis elements.

Definition 2.6. A **subbasis $\mathcal{S}$** for a topology on X is a collection of subset of X whose union equals X. The **topology generated by the subbasis $\mathcal{S}$** is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$

Note that the collection of all finite intersection of elements of $\mathcal{S}$ is a basis for a topology $\mathcal{T}$.

Definition 2.7. A **metric** on a nonempty set X is a mapping

$$d : X \times X \to \mathbb{R}$$

having the following properties :

1. $d(x, y) \geq 0$ for all $x, y \in X$; the equality holds if and only if $x = y$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Given $x \in X$ and $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

It is called the **ϵ-ball centered at x.** Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Example 2.8. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x.y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$. Now we say that d is a **euclidean metric** on $\mathbb{R}$, denoted by d_E.

Definition 2.9. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

Definition 2.10. If X is a topological space, X is said to be **metrizable** if there exists a metric d on X that induces the topology of X. A **metric space** (X, d) is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 2.11. A **filter** on a set X is a nonempty collection $\mathcal{F}$ of nonempty subsets of X such that the followings are true:

1. if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$,

2. if $F \in \mathcal{F}$ and $F \subseteq G$, then $G \in \mathcal{F}$.

Definition 2.12. A **filter base** on a set X is a nonempty collection $\mathcal{C}$ of nonempty subsets of X such that if $U_1, U_2 \in \mathcal{C}$, then there exists $U_3 \in \mathcal{C}$ such that $U_3 \subseteq U_1 \cup U_2$. Then $\mathcal{F} = \{F \subseteq X : \exists U \in \mathcal{C} \text{ with } U \subseteq F\}$ is a filter, and $\mathcal{F}$ is called **the filter generated by the filter base** $\mathcal{C}$.

Definition 2.13. Suppose $\mathcal{F}$ is a filter on a topological space X and $x \in X$. We say $\mathcal{F}$ **converges** to x if $\mathcal{N}_x \subseteq \mathcal{F}$ where $\mathcal{N}_x$ is the family of all neighborhoods of x and we say x is a **cluster point** of $\mathcal{F}$ if $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$.

Definition 2.14. A relation $\leq$ on a set A is called a **partial order** relation if the following conditions hold for all $\alpha, \beta, \gamma \in A$

1. $\alpha \leq \alpha$.
2. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$.
3. If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.

A **directed set** J is a set with a partial order $\leq$ such that for each pair α, β of elements of J, there exists an element γ of J having the property that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Example 2.15. $\mathbb{N}$ with a partial order $\leq$ is a directed set.

Definition 2.16. Let X be a topological space. A **net** in X is a function f from a directed set Λ to X. If $\alpha \in \Lambda$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by symbol $(x_\alpha)_{\alpha\in\Lambda}$, or merely by (x_α) if the index set is understood. Moreover if $\Lambda = \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is called a **sequence**.

The net (x_α) is said to **converge** to the point x of X(written $x_\alpha \to x$) if for each neighborhood U of x, there exists $\beta \in \Lambda$ such that for all $\gamma \geq \beta$, then $x_\gamma \in U$.

Definition 2.17. Suppose that $\mathcal{F}$ is a filter on a set X. We define

$$\Lambda_{\mathcal{F}} := \{(x, F) : x \in F \in \mathcal{F}\}$$

and $(x, F) \leq (y, G)$ if and only if $G \subseteq F$ for any $(x, F), (y, G) \in \Lambda_{\mathcal{F}}$. Since $\mathcal{F}$ is a filter, $(\Lambda_{\mathcal{F}}, \leq)$ is a directed set. We defined $(x, F) \longmapsto x$ is **the net generated by the filter** $\mathcal{F}$.

Definition 2.18. Suppose that $(x_\alpha)_{\alpha\in\Lambda}$ is a net in a set X. We define

$$T_\mu := \{x_\lambda : \lambda \geq \mu\}.$$

Since Λ is a directed set, $\{T_\alpha : \alpha \in \Lambda\}$ is a filter base. The filter generated by this filter base is called **the filter generated by the net** $(x_\alpha)_{\alpha\in\Lambda}$

Theorem 2.19. *Let $\mathcal{F}$ be a filter on a topological space X, $(x_\alpha)_{\alpha\in\Lambda}$ a net in X, and $x \in X$.*

1. *$\mathcal{F}$ converges to x if and only if the net generated by the filter $\mathcal{F}$ converges to x.*

2. *$(x_\alpha)_{\alpha\in\Lambda}$ converges to x if and only if the filter generated by the net $(x_\alpha)_{\alpha\in\Lambda}$ converges to x.*

Definition 2.20. Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a **Cauchy sequence** in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \text{ whenever } n, m \geq N.$$

The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to some point in X.

Example 2.21. Let X be a metric space.

Every convergent sequence in X is necessarily a Cauchy sequence.

Definition 2.22. A topological space X is said to be **Hausdorff** if each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.

Definition 2.23. Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

Definition 2.24. Suppose that one-point sets are closed in X. Then X is said to be **completely regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exists a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in B$.

Remark 2.25. Every regular space is Hausdorff, and a completely regular space is regular.

Example 2.26. Every metric space is completely regular.

Definition 2.27. Let X and Y be topological spaces.

A mapping $T : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $T^{-1}(V) = \{x \in X : T(x) \in V\}$ is an open subset of X.

Theorem 2.28. [10, pp. 104] *Let X and Y be topological spaces; let $T : X \to Y$. Then the followings are equivalent:*

1. *T is continuous;*

2. *For every open set C of Y, the set $T^{-1}(C)$ is open in X;*

3. *For every subset A of X, one has $T(\overline{A}) \subseteq \overline{T(A)}$;*

4. *For every closed set B of Y, the set $T^{-1}(B)$ is closed in X;*

5. *For each $x \in X$ and each neighborhood V of $T(x)$, there is a neighborhood U of x such that $T(U) \subseteq V$.*

If the condition 5 holds for the point x of X, we say that T is ***continuous at the point*** *x.*

Definition 2.29. Let X be a Hausdorff space and $T : X \to X$ a continuous mapping. We say that

1. $F(T) = \{x \in X : Tx = x\}$ is the **fixed point set of** T.

2. $C(T) = \{x \in X : \text{the sequence } (T^n x) \text{ converges}\}$ is the **convergence set of** T.

When $F(T) \neq \varnothing$, let $T^{\infty} : C(T) \to F(T)$ be defined by $T^{\infty}x = lim_{n\to\infty} T^n x$ for all $x \in C(T)$.

Definition 2.30. [6] Let X be a nonempty Hausdorff space and $T : X \to X$ a continuous mapping. A fixed point x of T is said to be **virtually T-stable** if for each neighborhood U of x, there exist a neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call T **virtually stable** if every fixed point of T is virtually T-stable.

Definition 2.31. Let X be a topological space and $A \subset B \subseteq X$. A continuous map $r : B \to A$ is called a **retraction** if for each $a \in A$, $r(a) = a$ and A is called a **retract** of B if there exists a retraction from B onto A.

Theorem 2.32. [6] *Suppose that X is a regular space and $T : X \to X$ a selfmap with $F(T) \neq \varnothing$. If T is a virtually stable, then T^{∞} is continuous, so the fixed point set of T is a retract of the convergence set of T.*

2.2 Uniform Spaces

Definition 2.33. [11] Let X be a set and $\mathcal{W}$ a filter on $X \times X$. We say that the filter $\mathcal{W}$ defines a **uniformity (or uniform structure)** on X if $\mathcal{W}$ satisfies these axioms :

- Each $W \in \mathcal{W}$ contains the diagonal Δ where $\Delta = \{(x,x) : x \in X\}$.
- $W \in \mathcal{W}$ implies $W^{-1} \in \mathcal{W}$ where $W^{-1} = \{(y,x) : (x,y) \in W\}$.
- For each $W \in \mathcal{W}$, there exists $V \in \mathcal{W}$ such that $V \circ V \subseteq W$ where $V \circ V = \{(x,z) : \exists y \in X, (x,y) \in V \text{ and } (y,z) \in V\}$.

Each $W \in \mathcal{W}$ being called a **vicinity** of the uniformity.

Let $x \in X, W \in \mathcal{W}$ and $B_x(W) = \{y \in X : (x,y) \in W\}$. Then $\mathcal{T}_{\mathcal{W}}$ is a topology generated by a neighborhood base $\mathcal{B} = \{B_x(W) : W \in \mathcal{W}\}$ at x. The space $(X, \mathcal{W})$, endowed with the topology $\mathcal{T}_{\mathcal{W}}$, is called a **uniform space**. A topological space X is **uniformisable** if its topology can be derived from a uniformity on X. In general, a uniformity is not unique.

Theorem 2.34. [11] *A Hausdorff topological space is uniformisable if and only if it is completely regular.*

Definition 2.35. Let X and Y be uniform spaces. A mapping $T : X \to Y$ is **uniformly continuous** if for each vicinity V of Y, there exists a vicinity U of X such that $(x,y) \in U$ implies $(Tx, Ty) \in V$.

A filter $\mathcal{F}$ on a subset X of a uniform space E is a **Cauchy filter** if, for each vicinity V, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq V$. A net $(x_\alpha)_{\alpha \in \Lambda}$ in X is a **Cauchy net** if, for each vicinity V, there exists $\gamma \in \Lambda$ such that for any $\delta, \beta \geq \gamma$, $(x_\delta, x_\beta) \in V$. If each Cauchy filter (or Cauchy net) converges to an element of X then X is called a **complete** subset of E.

If each Cauchy sequence in X converges to an element of X, then X is called a **sequentially complete** subset of E.

Definition 2.36. A **pseudometric** on a nonempty set X is a mapping

$$d : X \times X \to \mathbb{R}$$

having the following properties :

1. $d(x, y) \geq 0$ for all $x, y \in X$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Definition 2.37. Suppose A is an index set and $\mathcal{A} = \{d_\alpha : \alpha \in A$ and d_α is a pseudometric on $X\}$. Let $\epsilon > 0, \alpha \in A$, and $W_\alpha(\epsilon) = \{(x, y) \in X \times X : d_\alpha(x, y) < \epsilon\}$. Then we have $\mathcal{W}_\mathcal{A} = \{W_\alpha(\epsilon) : \alpha \in A$ and $\epsilon > 0\}$ is a uniformity on X and $\mathcal{T}_{\mathcal{W}_\mathcal{A}}$ is a topology on X. The space $(X, \mathcal{A})$, endowed with the topology $\mathcal{T}_{\mathcal{W}_\mathcal{A}}$, is called a **uniform space whose uniformity is generated by a collection $\mathcal{A}$ of pseudometrics**.

A collection $\mathcal{A}$ is **saturated** if $\forall \alpha \in A, d_\alpha(x, y) = 0$ implies $x = y$ for any $x, y \in X$.

Proposition 2.38. *Let $(X, \mathcal{A})$ be a uniform space whose uniformity is generated by a collection $\mathcal{A} = \{d_\alpha : \alpha \in A\}$ of pseudometrics indexed by A. Then (x_n) is a Cauchy sequence if and only if for all $\epsilon > 0$, $\alpha \in A$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $d_\alpha(x_m, x_n) < \epsilon$.*

2.3 Vector Spaces

Definition 2.39. A set X is called a **vector space** (or a **linear space**) over $\mathbb{R}$ if we have a mapping $+$ from $X \times X$ to X and a mapping $\cdot$ from $\mathbb{R} \times X$ to X that satisfy the following conditions :

1. $x + y = y + x$ for all $x, y \in X$.
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$.
3. There is a vector $0 \in X$ such that $x + 0 = x$ for all $x \in X$.
4. $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.
5. $(\lambda + \mu)x = \lambda x + \mu x$ for all $\lambda, \mu \in \mathbb{R}$ and $x \in X$.
6. $\lambda(\mu x) = (\lambda\mu)x$ for all $\lambda, \mu \in \mathbb{R}$ and $x \in X$.
7. $0 \cdot x = 0$ and $1 \cdot x = x$ for all $x \in X$.

We call $+$ addition and $\cdot$ multiplication by scalars. Suppose that Y is a nonempty subset of X. We say that Y is a **subspace of** X if for any $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y$ belongs to Y.

Definition 2.40. A mapping $\|\cdot\|$ from a vector space X to $\mathbb{R}$ is called a **norm** on X if

1. $\|x\| \geq 0$ for all $x \in X$; the equality holds if and only if $x = 0$.
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
3. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$.

We call X with a norm $\|\cdot\|$, a **normed space** (or **normed vector space**), denoted by $(X, \|\cdot\|)$.

Proposition 2.41. *Let $(X, \|\cdot\|)$ be a normed spaces and $d_{\|\cdot\|} : X \times X \to \mathbb{R}$ be defined by $d_{\|\cdot\|}(x, y) = \|x - y\|$. Then $d_{\|\cdot\|}$ is a metric on X, so $(X, d_{\|\cdot\|})$ is a metric space. Therefore every normed space is a metric space.*

Example 2.42. For $0 < p < \infty$, $\ell_p = \{(x_n) \subseteq \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ and $\|\cdot\|_p : \ell_p \to \mathbb{R}$ be defined by $\|(x_n)\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Thus ℓ_p is a vector space (over $\mathbb{R}$) and $\|\cdot\|_p$ is a norm on ℓ_p. So, ℓ_p with this norm is a normed space.

Definition 2.43. A normed space $(X, \|\cdot\|)$ is a **Banach space** if $(X, d_{\|\cdot\|})$ is a complete metric space. If (x_n) is a sequence in X, the series $\sum_{i=1}^{\infty} x_i$ (or $\sum x_n$) is said to be **summable** if a sequence of partial sums $(\sum_{i=1}^{n} x_i)$ converges to some point in X, and it is called **absolutely summable** if $\sum \|x_n\| < \infty$.

Theorem 2.44. [8, pp. 152] *A normed space X is complete if and only if every absolutely summable series in X is summable.*

Example 2.45. For any $1 \leq p < \infty$, ℓ_p is complete and hence ℓ_p is a Banach space.

Definition 2.46. Let X and Y be normed vector spaces and $T : X \to Y$. We say that T is a **linear mapping or linear operator** if for each $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. In particular, if $Y = \mathbb{R}$, we call T a **linear functional**.

A linear mapping T is called **bounded** if there exists $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C \|x\|$.

Proposition 2.47. [8, pp. 153] *If X and Y are normed vector spaces and $T : X \to Y$ a linear mapping, the followings are equivalent :*

1. *T is continuous;*

2. *T is continuous at 0;*

3. *T is bounded.*

Definition 2.48. If X and Y are normed vector spaces, we denote the space of all bounded linear mappings from X to Y by $L(X,Y)$. Thus $L(X,Y)$ is a vector space. Let $\|\cdot\| : L(X,Y) \to \mathbb{R}$ be defined by

$$\begin{aligned}\|T\| &= \sup\{\|T(x)\| : x \in X \text{ and } \|x\| = 1\} \\ &= \sup\{\frac{\|T(x)\|}{\|x\|} : x \in X \text{ and } x \neq 0\},\end{aligned}$$

for all $T \in L(X,Y)$. Then $\|\cdot\|$ is a norm on $L(X,Y)$ and called the **operator norm**. Hence, $L(X,Y)$ with the operator norm is a normed vector space. In particular, The space $L(X,\mathbb{R})$ of bounded linear functionals on X is called the **dual space** of X and denoted by X^*.

Remark 2.49. Every dual space of a normed vector space with the operator norm is a Banach space.

Example 2.50. Let $1 \leq p < \infty$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then the dual space of ℓ_p is isometrically isomorphic to ℓ_q; i.e., for each $f \in (\ell_p)^*$ there exists $(x_n) \in \ell_q$ such that $f(y_n) = \sum x_n y_n$ for all $(y_n) \in \ell_p$ and $\|f\| = \|(x_n)\|_q$.

Definition 2.51. Let A be a subset of a vector space. A is **convex** if for all $x, y \in A$ and $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y$ belongs to A.

Definition 2.52. A mapping $p(\cdot)$ from a vector space X to $\mathbb{R}$ is called a **semi-norm** if

1. $p(x) \geq 0$ for all $x \in X$.

2. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

3. $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$.

Proposition 2.53. *Let* $d_p : X \times X \to \mathbb{R}$ *be defined by* $d_p(x, y) = p(x - y)$*. Then* d_p *is a pseudometric on* X *induced by the seminorm* p*.*

Definition 2.54. Let (X, d) and (Y, ρ) be metric spaces and $T : X \to Y$.
Then T is said to be

1. **nonexpansive** if $\rho(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$;
2. **quasi-nonexpansive** if $T(X) \subseteq X \subseteq Y$, and $\rho(Tx, p) \leq d(x, p)$ for any $x \in X$ and $p \in F(T)$;
3. **lipschitzian** if there is $k \geq 0$ such that $\rho(Tx, Ty) \leq kd(x, y)$ for any $x, y \in X$;
4. **uniformly lipschitzian** if $T(X) \subseteq X \subseteq Y$ and there is $k \geq 0$ such that $\rho(T^n x, T^n y) \leq kd(x, y)$ for any $x, y \in X$ and $n \in \mathbb{N}$;
5. **contraction** if if there is $k \in (0, 1)$ such that $\rho(Tx, Ty) \leq kd(x, y)$ for any $x, y \in X$.

2.4 Locally Convex Spaces and The Weak Topology

Proposition 2.55. [9, pp. 203] *Let* X *be a set and let* $\mathcal{F}$ *be a family of maps and* $\{(Y_f, \mathcal{T}_f) : f \in \mathcal{F}\}$ *a family of topological spaces such that each* $f \in \mathcal{F}$ *maps* X *into the corresponding* Y_f*. Then there is the smallest topology for* X *with respect to which each member of* $\mathcal{F}$ *is continuous. That is, there is a unique topology* $\mathcal{T}_{\mathcal{F}}$ *for* X *such that the followings hold :*

1. *For each* $f \in \mathcal{F}$*,* f *is a continuous mapping from* $(X, \mathcal{T}_{\mathcal{F}})$ *into* $(Y_f, \mathcal{T}_f)$*.*
2. *If* $\mathcal{T}$ *is any topology for* X *such that for each* $f \in \mathcal{F}$*,* f *is a continuous mapping from* $(X, \mathcal{T})$ *into* $(Y_f, \mathcal{T}_f)$*, then* $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$*.*

The topology $\mathcal{T}_{\mathcal{F}}$ has $\{f^{-1}(U) : f \in \mathcal{F}, U \in \mathcal{T}_f\}$ as a subbasis, and it is called ***the weak topology on*** X ***induced by*** $\mathcal{F}$.

Definition 2.56. Let $\mathcal{A}$ be a collection of seminorms on a vector space X. If the collection of pseudometrices induced by all seminorms in $\mathcal{A}$ is saturated, then X equipped with the weak topology on X induced by $\mathcal{A}$ (or $(X, \mathcal{A})$) is a **locally convex space generated by** $\mathcal{A}$.

Proposition 2.57. *Every locally convex space generated by a collection $\mathcal{A}$ of seminorms is a Hausdorff uniform space generated by the collection of pseudometrics induced by all seminorms in $\mathcal{A}$.*

Definition 2.58. Let X be a normed space. Then **the weak topology** on X is the weak topology induced by the dual space of X and **the strong topology** on X is the topology induced by its norm.

Example 2.59. Let X be an infinite-dimensional normed space. Then X equipped with the weak topology (or (X, X^*)) is a non-metrizable locally convex space.

Example 2.60. Let X be a normed space, $f \in X^*$, and a function $\|\cdot\|_f : X \to [0, \infty)$ defined by

$$\|x\|_f = |f(x)|, \quad \text{for any } x \in X.$$

Then $\|\cdot\|_f$ is a seminorm on X and a function $d_f(\cdot,\cdot) : X \times X \to [0, \infty)$ defined by $d_f(x, y) = \|x - y\|_f$ for any $x, y \in X$ is a pseudometric on X. Hence (X, X^*) is a uniform space generated by a collection of pseudometrics.

Definition 2.61. Let Y be a subset of a normed space X. **The weak topology on** Y is a subspace topology of the weak topology on X

Theorem 2.62. [9, pp. 215] *Let X be a normed space. If X has finite dimension, the weak topology and strong topology are the same.*

Remark 2.63. [9, pp. 212] The weak topology on a normed space is completely regular.

Definition 2.64. Let X be a normed space and O a subset of X. We say that

1. O is **weakly open** if O belongs to the weak topology.

2. O is **weakly closed** if $X - O$ is weakly open.

3. For $x \in X$, O is a **weak neighborhood of** x if O is a weak open set containing x.

Definition 2.65. Let X and Y be normed spaces. A mapping $T : X \to Y$ is said to be **weakly continuous** (or **weak-to-weak continuous**) if for each weakly open subset V of Y, the set $T^{-1}(V)$ is a weakly open subset of X.

Theorem 2.66. *Let X and Y be normed spaces. A mapping $T : X \to Y$ is weakly continuous if and only if for any $f \in Y^*$, $f \circ T$ is a weakly continuous functional.*

Definition 2.67. Let X be a normed space and $T : X \to X$ a weakly continuous mapping. A fixed point x of T is said to be **weakly virtually T-stable** if for each weak neighborhood U of x, there exist a weak neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call T **weakly virtually stable** if every fixed point of T is weakly virtually T-stable.

Proposition 2.68. [9, pp. 212] *A linear functional on a normed space is continuous with respect to the weak topology if and only if it is continuous with respect to the metric induced by its norm.*

Definition 2.69. Let X be a subset of a normed space E and $T : X \to X$. We say that

1. T is **functionally lipschitzian** if for each $f \in E^*$ there exist $N \in \mathbb{N}$ and $g_1, g_2, \ldots, g_N \in E^*$ such that

$$\|Tx - Ty\|_f \leq \sum_{i=1}^{N} \|x - y\|_{g_i},$$

for any $x, y \in X$.

2. T is **functionally uniformly lipschitzian** if for each $f \in E^*$ there exist $N \in \mathbb{N}$ and $g_1, g_2, \ldots, g_N \in E^*$ such that

$$\|T^k x - T^k y\|_f \leq \sum_{i=1}^{N} \|x - y\|_{g_i},$$

for any $x, y \in X$ and $k \in \mathbb{N}$.

Proposition 2.70. [12] *Every functionally lipschitzian selfmap is weakly continuous and every functionally uniformly lipschitzian selfmap is weakly virtually stable with respect to the sequence of all natural numbers.*

Definition 2.71. [9] A sequence (x_n) in an infinite-dimensional Banach space X is a **Schauder basis** for X if for each x in X there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n x_n$.

Example 2.72. Let $i \in \mathbb{N}$ and $e_i = (0, 0, \ldots, 0, \overbrace{1}^{i^{th}}, 0, \ldots)$. Then $e_i \in \ell_p$ for all $1 < p < \infty$ and a sequence (e_n) in ℓ_p is a Schauder basis for ℓ_p for all $1 < p < \infty$.

Theorem 2.73. [9, pp. 351] *If (x_n) is a Schauder basis for an infinite-dimensional Banach space, then $\|x_n\|^{-1} x_n$ is a Schauder basis for the space, so there is a Schauder basis (e_n) such that $\|e_i\| = 1$ for any $i \in \mathbb{N}$.*

Definition 2.74. Let X be a Banach space. We say that X has a **normalized Schauder basis** if there is a Schauder basis (e_n) for X such that $\|e_i\| = 1$ for any $i \in \mathbb{N}$.

Definition 2.75. Let X be an infinite-dimensional Banach space with a Schauder basis (x_n). For each positive integer m, the m^{th} **coordinate functional** x_m^* for (x_n) is the mapping $\sum_n \alpha_n x_n \longmapsto \alpha_m$ from X into $\mathbb{R}$.

Theorem 2.76. [9] *Each coordinate functional associated with a basis for a Banach space is a continuous linear functional.*

CHAPTER III

FIXED POINT THEOREMS

In this chapter, new kinds of selfmaps on a uniform space and fixed point theorems for those maps are presented. For any set S, we will use $\mathcal{P}^f(S)$ and $|S|$ to denote the set of all nonempty finite subsets of S and the cardinality of S, respectively. Let $(E, \mathcal{A})$ be a Hausdorff uniform space whose uniformity is generated by a saturated collection of pseudometrics $\mathcal{A} = \{d_\alpha : \alpha \in A\}$ indexed by A, $\varnothing \neq X \subseteq E$, and $J : A \to \mathcal{P}^f(A)$. The definition of a J-nonexpansive map is given as follows :

Definition 3.1. A selfmap $T : X \to X$ is said to be ***J*-nonexpansive** if for each $\alpha \in A$,

$$d_\alpha(Tx, Ty) \leq \sum_{\beta \in J(\alpha)} d_\beta(x, y),$$

for any $x, y \in X$.

Proposition 3.2. *Every functionally lipschitzian selfmap on a subset X of a normed space Y is J-nonexpansive on X equipped with the weak topology.*

Proof. Since T is functionally lipschitzian, for each $f \in Y^*$, there are $n \in \mathbb{N}$ and $g_1, g_2, \ldots, g_n \in Y^*$ such that

$$|f(Tx - Ty)| = \|Tx - Ty\|_f \leq \sum_{i=1}^{n} \|x - y\|_{g_i} = \sum_{i=1}^{n} |g_i(x - y)|,$$

for any $x, y \in X$. Then Y is a uniform space generated by a collection $\mathcal{A} = \{|f| : f \in Y^*\}$ where $|f|(x, y) = |f(x-y)|$ for any $x, y \in Y$. By letting $J : Y^* \to \mathcal{P}^f(Y^*)$ be defined by $J(f) = \{g_1, g_2, \ldots, g_n\}$ for each $f \in Y^*$, it follows that T is J-nonexpansive. $\square$

Following the above proposition, we obtain many examples of J-nonexpansive maps.

Example 3.3. Let $1 < p < \infty$, $E = \ell_p$ equipped with the weak topology, and $T : \ell_p \to \ell_p$ be defined by

$$T(x_1, x_2, \dots) = \left(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, \dots \right),$$

for any $(x_1, x_2, \dots) \in \ell_p$. Then $\mathcal{A} = \{|f| : f \in \ell_p^*\}$.
By example 2.8 in [12], T is functionally lipshitzian, so it is J-nonexpansive. However, we directly show that

$$|f(Tx - Ty)| \leq \left| \frac{\|f\|}{3}(x_1 - y_1 + x_3 - y_3) \right| + \left| \frac{\|f\|}{3}(x_2 - y_2 + x_4 - y_4) \right| \\ + |\|f\|(x_1 - y_1)| + |\|f\|(x_2 - y_2)| + |f(x - y)|,$$

for each $f \in \ell_p^*$, $x = (x_1, x_2, \dots) \in \ell_p$ and $y = (y_1, y_2, \dots) \in \ell_p$.

By letting $J : \ell_p^* \to \mathcal{P}^f(\ell_p^*)$ be defined by $J(f) = \{f, g_1, g_2, g_3, g_4\}$, for each $f \in \ell_p^*$, where

$$g_1(x) = \frac{\|f\|}{3}(x_1 + x_3), g_2(x) = \frac{\|f\|}{3}(x_2 + x_4), g_3(x) = \|f\|x_1, g_4(x) = \|f\|x_2,$$

for each $x = (x_1, x_2, \dots) \in \ell_p$, it follows that T is J-nonexpansive.

The above definition of a J-nonexpansive map clearly extends the definition of a j-nonexpansive map in [2]. Before giving general existence criteria of fixed points for a J-nonexpansive selfmap, we need the following notations. For each $\alpha \in A$ and $n \in \mathbb{N}$, we let

$$A_n(\alpha) = \{(\alpha_1, \dots, \alpha_n) : \alpha_1 \in J(\alpha) \text{ and } \alpha_k \in J(\alpha_{k-1}) \text{ for } 1 < k \leq n\}$$

and

$$A(\alpha) = \{(\alpha_1, \alpha_2, \dots) : \alpha_1 \in J(\alpha) \text{ and } \alpha_k \in J(\alpha_{k-1}) \text{ for } k > 1\}.$$

When there is no ambiguity, we will denote an element of both $A_n(\alpha)$ and $A(\alpha)$ simply by (α_k). Notice that for each $\alpha \in A$ and $n \in \mathbb{N}$, the sets $A_n(\alpha)$ and $\pi_n(A(\alpha))$ are finite, where π_n denotes the n-th coordinate projection $(\alpha_k) \mapsto \alpha_n$.

Lemma 3.4. *Every J-nonexpansive map is continuous.*

Proof. Suppose $T : X \to X$ is J-nonexpansive. Let $x \in X$, $(x_\gamma)_{\gamma\in\Lambda}$ be a net in X converging to x, $\epsilon > 0$, and $\alpha \in A$. Since (x_γ) converges to x, there exists $\eta \in \Lambda$ such that for all $\gamma \geq \eta$ and $\beta \in J(\alpha)$, $d_\beta(x_\gamma, x) < \frac{\epsilon}{|J(\alpha)|}$. Then we have

$$d_\alpha(Tx_\gamma, Tx) \leq \sum_{\beta\in J(\alpha)} d_\beta(x_\gamma, x) < \epsilon,$$

and hence T is continuous. □

An equivalent definition of a fixed point for a J-nonexpansive selfmap is shown in the next theorem.

Theorem 3.5. *Let $T : X \to X$ be J-nonexpansive whose $A(\alpha)$ is finite for any $\alpha \in A$. Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that*

(i) the sequence $(T^n x_0)$ has a convergent subsequence, and

(ii) for each $\alpha \in A$ and $(\alpha_k) \in A(\alpha)$, $\lim_{n\to\infty} d_{\alpha_n}(x_0, Tx_0) = 0$.

Proof. ($\Rightarrow$): It is obvious by letting x_0 be a fixed point of T.
($\Leftarrow$): Suppose that $(T^{n_i} x_0)$ converges to some $z \in X$. Let $\alpha \in A$ and $(\alpha_k) \in A(\alpha)$. Then $\lim_{i\to\infty} d_\alpha(z, T^{n_i} x_0) = 0$ and $\lim_{n\to\infty} d_{\alpha_n}(x_0, Tx_0) = 0$. We can choose $N \in \mathbb{N}$ sufficiently large so that $d_\alpha(z, T^{n_i} x_0) < \epsilon$ and $d_{\alpha_{n_i}}(x_0, Tx_0) < \epsilon$, for all $i \geq N$. It follows that

$$\begin{aligned} d_\alpha(z, T^{n_i+1} x_0) &\leq d_\alpha(z, T^{n_i} x_0) + d_\alpha(T^{n_i} x_0, T^{n_i}(Tx_0)) \\ &\leq d_\alpha(z, T^{n_i} x_0) + \sum_{(\alpha_k)\in A_{n_i}(\alpha)} d_{\alpha_{n_i}}(x_0, Tx_0) \\ &\leq \Big(1 + |A(\alpha)|\Big)\epsilon. \end{aligned}$$

Since α is arbitrary, $(T^{n_i+1}x_0)$ converges to z. By the continuity of T, we have $T(T^n x_0)$ converges to Tz, so $z = Tz$ and hence z is a fixed point of T. □

As a corollary of the previous theorem, we immediately obtain Theorem 1 in [2], with a corrected and simplified proof, as follows :

Corollary 3.6 (Theorem 1 in [2])**.** *Let $T : X \to X$ be a j-nonexpansive map. If there exists $x_0 \in X$ such that*

(i) the sequence $(T^n x_0)$ has a convergent subsequence, and

(ii) for every $\alpha \in A$, $\lim_{n\to\infty} d_{j^n(\alpha)}(x_0, Tx_0) = 0$,

then T has a fixed point.

Proof. Follows directly from the previous theorem by considering the map $J : \alpha \mapsto \{j(\alpha)\}$. Notice that $A(\alpha) = \{(j^n(\alpha))\}$ which is finite. □

We now consider a special kind of J-nonexpansive maps. Let Φ denote the family of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

(Φ1) ϕ is non-decreasing and continuous from the right, and

(Φ2) $\phi(t) < t$ for any $t > 0$.

Notice that $\phi(0) = 0$, and we will call $\phi \in \Phi$ subadditive if $\phi(t_1+t_2) \le \phi(t_1)+\phi(t_2)$ for all $t_1, t_2 \ge 0$. For a subfamily $\{\phi_\alpha\}_{\alpha\in A}$ of Φ, $\alpha \in A$, $(\alpha_k) \in A_n(\alpha)$ and $i \le n$, we let

$$\phi^i_{(\alpha_k)} = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_i}.$$

Example 3.7. Let $c \in (0,\infty)$ and $\phi : [0,\infty) \to [0,\infty)$ be defined by $\phi(t) = ct$. Then ϕ is non-decreasing, continuous from the right, and subadditive. If $c \in (0,1)$, then $\phi \in \Phi$.

Definition 3.8. A selfmap $T : X \to X$ is said to be a J**-contraction** if for each $\alpha \in A$, there exists $\phi_\alpha \in \Phi$ such that

$$d_\alpha(Tx, Ty) \leq \sum_{\beta \in J(\alpha)} \phi_\alpha(d_\beta(x, y)),$$

for any $x, y \in X$, and ϕ_α is subadditive whenever $|J(\alpha)| > 1$.

Clearly, a Φ-contraction as defined in [1] is a J-contraction and a J-contraction is always J-nonexpansive. A natural example of a J-contraction can be obtained by adding (finitely many) appropriate Φ-contractions as shown in the following example.

Example 3.9. Given two Φ-contractions $T_1 : X \to X$ and $T_2 : X \to X$. Then there exist $j_1, j_2 : A \to A$, and for each $\alpha \in A$, there exist $\phi_{1,\alpha}, \phi_{2,\alpha} \in \Phi$ such that

$$d_\alpha(T_1x, T_1y) \leq \phi_{1,\alpha}(d_{j_1(\alpha)}(x, y)) \text{ and } d_\alpha(T_2x, T_2y) \leq \phi_{2,\alpha}(d_{j_2(\alpha)}(x, y)),$$

for any $\alpha \in A$ and $x, y \in X$. If for each $\alpha \in A$, $j_1(\alpha) \neq j_2(\alpha)$ and there is a subadditive $\phi_{3,\alpha} \in \Phi$ so that $\phi_{1,\alpha}(t) \leq \phi_{3,\alpha}(t)$ and $\phi_{2,\alpha}(t) \leq \phi_{3,\alpha}(t)$ for any $t \geq 0$, then the map $H = T_1 + T_2$ is clearly a J-contraction with respect to $J(\alpha) = \{j_1(\alpha), j_2(\alpha)\}$ and $\phi_{H,\alpha} = \phi_{3,\alpha}$ for any $\alpha \in A$.

Considering the iterative sequence of a J-contraction T, because of the multi-valued map J, the sequence (T^n) is very complicated, so a useful lemma which simplifies T^n by a notion $\phi^{n-1}_{(\alpha_k)}$ is shown in the next lemma.

Lemma 3.10. *If $T : X \to X$ is a J-contraction. then we have*

$$d_\alpha(T^n x, T^n y) \leq \sum_{(\alpha_k) \in A_n(\alpha)} \phi_\alpha \circ \phi^{n-1}_{(\alpha_k)}(d_{\alpha_n}(x, y)),$$

for any $\alpha \in A$, $n \geq 2$ and $x, y \in X$.

Proof. Recall that ϕ_α is assumed to be subadditive whenever $|J(\alpha)| > 1$. Then, for any $\alpha \in A$, $n \geq 2$ and $x, y \in X$, we clearly have

$$\begin{aligned}
d_\alpha(T^n x, T^n y) &\leq \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(d_{\alpha_1}(T^{n-1}x, T^{n-1}y)) \\
&\leq \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(\sum_{\alpha_2 \in J(\alpha_1)} \phi_{\alpha_1}(d_{\alpha_2}(T^{n-2}x, T^{n-2}y))) \\
&\leq \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \phi_\alpha \circ \phi_{\alpha_1}(d_{\alpha_2}(T^{n-2}x, T^{n-2}y)) \\
&\vdots \\
&\leq \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \cdots \sum_{\alpha_n \in J(\alpha_{n-1})} \phi_\alpha \circ \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_{n-1}}(d_{\alpha_n}(x, y)) \\
&= \sum_{(\alpha_k) \in A_n(\alpha)} \phi_\alpha \circ \phi^{n-1}_{(\alpha_k)}(d_{\alpha_n}(x, y)).
\end{aligned}$$

□

A general criterion for the existence of fixed points of a J-contraction self-map is obtained in the next theorem. Moreover, a condition which guarantees a uniqueness of a fixed point is given, so a J-contraction plays a similar role as a contraction in yielding the uniqueness of a fixed point.

Theorem 3.11. *Suppose X is sequentially complete and $T : X \to X$ is a J-contraction whose $A(\alpha)$ is finite for any $\alpha \in A$. If T satisfies the following conditions :*

(i) for each $\alpha \in A$, there exists $c_\alpha \in \Phi$ such that

$$\phi_{\alpha_i}(t) \leq c_\alpha(t),$$

for any $(\alpha_k) \in A(\alpha)$, $i \in \mathbb{N}$, $t \geq 0$, and

(ii) there exists $x_0 \in X$ such that for each $\alpha \in A$, $(\alpha_k) \in A(\alpha)$, $i \in \mathbb{N}$ and $n, m \in \mathbb{N}$, we have

$$d_{\alpha_i}(T^n x_0, T^m x_0) \leq M_\alpha(x_0),$$

for some $M_\alpha(x_0) \in \mathbb{R}$,

then T *has a fixed point. Moreover, if for each* $\alpha \in A$ *and* $x, y \in X$*, there exists* $F_\alpha(x,y) \in \mathbb{R}_0^+$ *such that*

$$d_{\alpha_i}(x,y) \le F_\alpha(x,y),$$

for all $(\alpha_k) \in A(\alpha)$ *and* $i \in \mathbb{N}$*, then the fixed point of* T *is unique.*

Proof. For each $\alpha \in A$ and $n, m, N \in \mathbb{N}$, since ϕ_α is non-decreasing, we have

$$\begin{aligned} d_\alpha(T^n x_0, T^m x_0) &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(d_{\alpha_1}(T^{n-1}x_0, T^{m-1}x_0)) \\ &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(\sup\{d_{\alpha_1}(T^{n-1}x_0, T^{m-1}x_0) : n, m \ge N\}), \end{aligned}$$

since ϕ_α is non-decreasing, and by letting $h_N^\alpha := \sup\{d_\alpha(T^n x_0, T^m x_0) : n, m \ge N\}$, it follows that

$$\begin{aligned} h_N^\alpha &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(\sup\{d_{\alpha_1}(T^{n-1}x_0, T^{m-1}x_0) : n, m \ge N\}) \\ &= \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha(h_{N-1}^{\alpha_1}) \\ &\le \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \phi_\alpha(\phi_{\alpha_1}(h_{N-2}^{\alpha_2})) \\ &\ \vdots \\ &\le \sum_{(\alpha_k) \in A_{N-1}(\alpha)} \phi_\alpha \circ \phi_{(\alpha_k)}^{N-1}(h_1^{\alpha_{N-1}}) \\ &\le \sum_{(\alpha_k) \in A_{N-1}(\alpha)} c_\alpha^N(M_\alpha(x_0)) \\ &\le |A(\alpha)| c_\alpha^N(M_\alpha(x_0)). \quad (*) \end{aligned}$$

Since $0 \le c_\alpha^N(t) = c_\alpha(c_\alpha^{N-1}(t)) < c_\alpha^{N-1}(t)$ for any $t \ge 0$, there exists $r_\alpha \ge 0$, $\lim_{N\to\infty} c_\alpha^N(t) = r_\alpha$. Because c_α is right continuous, we have $\lim_{N\to\infty} c_\alpha(c_\alpha^{N-1}(t)) = c_\alpha(r_\alpha)$, and hence $c_\alpha(r_\alpha) = r_\alpha$. Therefore, $r_\alpha = 0$.

By (*), it follows that $\lim_{N\to\infty} h_N^\alpha = 0$.

Since α is arbitrary, $(T^k x_0)$ is a Cauchy sequence, and by sequential completeness, converges to some $z \in X$. Notice also that z must be a fixed point of T by continuity.

Now suppose that for each $x, y \in X$ and $\alpha \in A$, there exists $F_\alpha(x,y) \in \mathbb{R}_0^+$ such that $d_{\alpha_i}(x,y) \le F_\alpha(x,y)$ for all $(\alpha_k) \in A(\alpha)$ and $i \in \mathbb{N}$. If x, y are fixed points of T, then by Lemma 3.10, we have for each $\alpha \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned} d_\alpha(x,y) &= d_\alpha(T^n x, T^n y) \\ &\le \sum_{(\alpha_k)\in A_n(\alpha)} \phi_\alpha \circ \phi_{(\alpha_k)}^{n-1}(d_{\alpha_n}(x,y)) \\ &\le \sum_{(\alpha_k)\in A_n(\alpha)} c_\alpha^n(d_{\alpha_n}(x,y)) \\ &\le |A(\alpha)| c_\alpha^n(F_\alpha(x,y)). \end{aligned}$$

Since $\lim_{n\to\infty} c_\alpha^n(F_\alpha(x,y)) = 0$, we must have $x = y$. □

As a corollary of the previous theorem, we immediately obtain Theorem 1 in [1] as follows :

Corollary 3.12 (Theorem 1 in [1]). *Suppose X is a bounded and sequentially complete subset of E and $T : X \to X$ is Φ-contraction. If*

(i) for each $\alpha \in A$, there exists $c_\alpha \in \Phi$ such that $\phi_{j^n(\alpha)}(t) \le c_\alpha(t)$ for all $n \in \mathbb{N}$ and $t \ge 0$,

(ii) for each $n \in \mathbb{N}$, $\sup\{d_{j^n(\alpha)}(x,y) : x, y \in X\} \le p(\alpha) := \sup\{d_\alpha(x,y) : x, y \in X\}$,

then there exists a unique fixed point $x \in X$ of T.

Proof. For each $x_0, x, y \in X$, $\alpha \in A$, $(\alpha_k) \in A(\alpha)$ and $i, m, n \in \mathbb{N}$, by letting $J(\alpha) = \{j(\alpha)\}$ and $M_\alpha(x_0) = p(\alpha) = F_\alpha(x, y)$, we have

$$A(\alpha) = \{(\alpha, j(\alpha), j^2(\alpha), \ldots, j^k(\alpha), \ldots)\},$$

$d_{\alpha_i}(T^m x_0, T^n x_0) = d_{j^i(\alpha)}(T^m x_0, T^n x_0) \leq M_\alpha(x_0)$ and $d_{\alpha_i}(x, y) \leq F_\alpha(x, y)$. Hence, by Theorem 3.11, T has a unique fixed point. □

In the next theorem, we give a special kind of a selfmap, and obtain a criterion for the existence of its fixed points. Furthermore, a condition that guarantees a uniqueness of its fixed point is also given.

Theorem 3.13. *Suppose X is sequentially complete and $T : X \to X$ is a continuous selfmap satisfying : for each $\alpha \in A$ and $k \in \mathbb{N}$, there exist a function $\phi_{\alpha,k} : [0, \infty) \to [0, \infty)$, a finite set $D_{\alpha,k}$ and a map $P_{\alpha,k} : D_{\alpha,k} \to A$ such that $\phi_{\alpha,k}$ is non-decreasing, and*

$$d_\alpha(T^k x, T^k y) \leq \sum_{\gamma \in D_{\alpha,k}} \phi_{\alpha,k}(d_{P_{\alpha,k}(\gamma)}(x, y)),$$

for any $x, y \in X$.

1. *If there exists $x_0 \in X$ such that for each $\alpha \in A$ there exists $M_\alpha(x_0) \in \mathbb{R}_0^+$ so that $\sum_{k\in\mathbb{N}} |D_{\alpha,k}|\phi_{\alpha,k}(M_\alpha(x_0)) < \infty$ and*

   $$d_{P_{\alpha,k}(\gamma)}(x_0, Tx_0) \leq M_\alpha(x_0),$$

   for all $k \in \mathbb{N}$ and $\gamma \in D_{\alpha,k}$, then T has a fixed point in X.

2. *If for each $\alpha \in A$ and $x, y \in X$, there exists $F_\alpha(x, y) \in \mathbb{R}_0^+$ such that $\sum_{k\in\mathbb{N}} |D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x, y)) < \infty$ and*

   $$d_{P_{\alpha,k}(\gamma)}(x, y) \leq F_\alpha(x, y),$$

for all $k \in \mathbb{N}$ and $\gamma \in D_{\alpha,k}$, then T has a unique fixed point in X and $C(T) = X$.

Notice that if $\phi_{\alpha,1}$ belongs to Φ, then T is clearly a J-contraction.

Proof. 1. Let $\epsilon > 0$ and $\alpha \in A$. Since $\sum_{k\in\mathbb{N}} |D_{\alpha,k}|\phi_{\alpha,k}(M_\alpha(x_0)) < \infty$, there is $N \in \mathbb{N}$ such that for any $m > n \in \mathbb{N}$, $\sum_{n\leq i<m} |D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0)) < \epsilon$. Then

$$\begin{aligned} d_\alpha(T^n x_0, T^m x_0) &\leq \sum_{n\leq i<m} d_\alpha(T^i x_0, T^{i+1}x_0) \\ &\leq \sum_{n\leq i<m} \sum_{\gamma\in D_{\alpha,i}} \phi_{\alpha,i}(d_{P_{\alpha,i}(\gamma)}(x_0, Tx_0)) \\ &\leq \sum_{n\leq i<m} \sum_{\gamma\in D_{\alpha,i}} \phi_{\alpha,i}(M_\alpha(x_0)) \\ &= \sum_{n\leq i<m} |D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0)) < \epsilon. \end{aligned}$$

Therefore, $(T^k x_0)$ is a Cauchy sequence and converges to a fixed point of T by the sequential completeness of X and the continuity of T.

2. Let $\epsilon > 0$, $x \in X$, and $\alpha \in A$. Since $\sum_{k\in\mathbb{N}} |D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x, Tx)) < \infty$, there is $N \in \mathbb{N}$ such that for any $m > n \in \mathbb{N}$, $\sum_{n\leq i<m} |D_{\alpha,i}|\phi_{\alpha,i}(F_\alpha(x, Tx)) < \epsilon$. Then

$$\begin{aligned} d_\alpha(T^n x, T^m x) &\leq \sum_{n\leq i<m} d_\alpha(T^i x, T^{i+1}x) \\ &\leq \sum_{n\leq i<m} \sum_{\gamma\in D_{\alpha,i}} \phi_{\alpha,i}(d_{P_{\alpha,i}(\gamma)}(x, Tx)) \\ &\leq \sum_{n\leq i<m} \sum_{\gamma\in D_{\alpha,i}} \phi_{\alpha,i}(F_\alpha(x, Tx)) \\ &= \sum_{n\leq i<m} |D_{\alpha,i}|\phi_{\alpha,i}(F_\alpha(x, Tx)) < \epsilon. \end{aligned}$$

Therefore, $(T^k x)$ is a Cauchy sequence and converges to a fixed point of T by the sequential completeness of X and the continuity of T. Notice also that, since x is arbitrary, $C(T) = X$.

Now, let $\epsilon > 0$, $\alpha \in A$, and $x, y \in F(T)$. Since $\lim_{k\to\infty} |D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x,y)) = 0$, there is $K \in \mathbb{N}$ such that for any $k \geq K$, $|D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x,y)) < \epsilon$. Then

$$
\begin{aligned}
d_\alpha(x,y) &= d_\alpha(T^k x, T^k y) \\
&\leq \sum_{\gamma \in D_{\alpha,k}} \phi_{\alpha,k}(d_{P_{\alpha,k}(\gamma)}(x,y)) \\
&\leq \sum_{\gamma \in D_{\alpha,k}} \phi_{\alpha,k}(F_\alpha(x,y)) \\
&= |D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x,y)) < \epsilon.
\end{aligned}
$$

Then we have the uniqueness. □

Also, we immediately obtain Theorem 5 in [1] as follows :

Corollary 3.14 (Theorem 5 in [1]). *Let us suppose that*

(i) for each $\alpha \in A$ and $n > 0$, there exists $\phi_{\alpha,n} \in \Phi$ and $j(\alpha,n) \in A$ such that

$$d_\alpha(T^n x, T^n y) \leq \phi_{\alpha,n}(d_{j(\alpha,n)}(x,y)),$$

for any $x, y \in X$,

(ii) there exists $x_0 \in X$ such that $d_{j(\alpha,n)}(x_0, Tx_0) \leq p(\alpha) < \infty)$ $(n = 1, 2, \dots)$, $\sum_n \phi_{\alpha,n}(p(\alpha)) < \infty$ and $j : A \times \mathbb{N} \to A$.

Then T has at least one fixed point in X.

Proof. By letting $D_{\alpha,k} = \{j(\alpha,k)\}$ for any $\alpha \in A$ and $k \in \mathbb{N}$ and $P_{\alpha,k} = \pi_k|_{D_{\alpha,k}}$. Then for each $i \in \mathbb{N}$, we have $|D_{\alpha,i}| = 1$ and $M_\alpha(x_0) = p(\alpha)$. By Theorem 3.13 (2), T has a fixed point. □

In the next theorem, another fixed point theorem for a J-contraction selfmap by using the previous theorem is presented.

Theorem 3.15. *Suppose X is sequentially complete and $T : X \to X$ is a J-contraction whose $A(\alpha)$ is finite for each $\alpha \in A$. If, for each $\alpha \in A$, there exists $c_\alpha \in \Phi$ satisfying :*

(i) $c_\alpha(t)/t$ is non-decreasing in t,

(ii) $\phi_{\alpha_n}(t) \le c_\alpha(t)$ for any $(\alpha_k) \in A(\alpha)$, $n \in \mathbb{N}$ and $t \in [0, \infty)$, and

(iii) there exists $x_0 \in X$ and $M_\alpha(x_0) \in \mathbb{R}^+$ such that $d_{\alpha_n}(x_0, Tx_0) \le M_\alpha(x_0)$ for any $(\alpha_k) \in A(\alpha)$ and $n \in \mathbb{N}$,

then T has a fixed point in X.

Proof. Let $D_{\alpha,i} = A_i(\alpha)$, $P_{\alpha,i}((\alpha_k)) = \alpha_i$, and $\phi_{\alpha,i}(t) = c^i_\alpha(t)$ for any $i \in \mathbb{N}$, $\alpha \in A$, $(\alpha_k) \in A_i(\alpha)$, and $t \in [0, \infty)$. Then for any $\alpha \in A$ and $x, y \in X$, we have, by Lemma 3.10,

$$\begin{aligned} d_\alpha(T^i x, T^i y) &\le \sum_{(\alpha_k)\in A_i(\alpha)} \phi_\alpha \circ \phi^{i-1}_{(\alpha_k)}(d_{\alpha_i}(x, y)) \\ &\le \sum_{(\alpha_k)\in A_i(\alpha)} c^i_\alpha(d_{\alpha_i}(x, y)) \\ &= \sum_{(\alpha_k)\in D_{\alpha,i}} \phi_{\alpha,i}(d_{P_{\alpha,i}((\alpha_k))}(x, y)). \end{aligned}$$

Since

$$\begin{aligned} \frac{|D_{\alpha,i+1}|\phi_{\alpha,i+1}(M_\alpha(x_0))}{|D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0))} &= \frac{|A_{i+1}(\alpha)|c^{i+1}_\alpha(M_\alpha(x_0))}{|A_i(\alpha)|c^i_\alpha(M_\alpha(x_0))} \\ &\le \frac{c_\alpha(c^i_\alpha(M_\alpha(x_0)))}{c^i_\alpha(M_\alpha(x_0))} \\ &\le \frac{c_\alpha(M_\alpha(x_0))}{M_\alpha(x_0)} < 1, \end{aligned}$$

for any $i \in \mathbb{N}$, we have $\sum_{i\in\mathbb{N}} |D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0)) < \infty$. Then by Theorem 3.13 (1), T has a fixed point. □

Corollary 3.16 (Theorem 2 in [1])**.** *Let us suppose* *(i) the operator* $T : X \to X$ *is a* Φ*-contraction,*

(ii) for each $\alpha \in A$ *there exists a* Φ*-function* c_α *such that* $\phi_{j^n(\alpha)}(t) \leq c_\alpha(t)$ *for all* $n \in \mathbb{N}$ *and* $c_\alpha(t)/t$ *is non-decreasing,*

(iii) there exists an element $x_0 \in X$ *such that* $d_{j^n(\alpha)}(x_0, Tx_0) \leq p(\alpha) < \infty$ $(n = 1, 2, \ldots)$.

Then T *has at least one fixed point in* X*.*

Proof. By letting $J(\alpha) = \{j(\alpha)\}$ for any $\alpha \in A$ and $M_\alpha(x_0) = p(\alpha)$. Then $|A(\alpha)| = 1$, and by Theorem 3.15, T has a fixed point. □

Example 3.17. Given a sequentially complete locally convex space X, and two Φ-contractions $T_1, T_2 : X \to X$; i.e., there exist $j_1, j_2 : A \to A$, and for each $\alpha \in A$, there exist $\phi_{1,\alpha}, \phi_{2,\alpha} \in \Phi$ such that

$$d_\alpha(T_1x, T_1y) \leq \phi_{1,\alpha}(d_{j_1(\alpha)}(x, y)) \text{ and } d_\alpha(T_2x, T_2y) \leq \phi_{2,\alpha}(d_{j_2(\alpha)}(x, y)),$$

for any $\alpha \in A$ and $x, y \in X$. Suppose further that

(i) $j_1^{n+1} = j_1^n \circ j_2$ and $j_2^n \circ j_1 = j_2^{n+1}$ for any $n \in \mathbb{N}$,

(ii) for each $\alpha \in A$, $\phi_{1,\alpha}(t) = c_1(\alpha)t$ and $\phi_{2,\alpha}(t) = c_2(\alpha)t$ for some $c_1(\alpha)+c_2(\alpha) \in (0, 1)$, and

(iii) there exist $x_0 \in X$ such that $d_{j_1^n(\alpha)}(x_0, T_1x_0) \leq p_1(x_0, \alpha) < \infty$ and $d_{j_2^n(\alpha)}(x_0, T_2x_0) \leq p_2(x_0, \alpha) < \infty$ for any $\alpha \in A$ and $n = 1, 2, \ldots$.

Then $H = \frac{T_1+T_2}{2}$ is a J-contraction with $J(\alpha) = \{j_1(\alpha), j_2(\alpha)\}$ and $\phi_{H,\alpha}(t) = \big(c_1(\alpha) + c_2(\alpha)\big)t$. Also, by (i) and (iii), we have $|A(\alpha)| = 2 < \infty$ and

$$d_{\alpha_n}(x_0, Hx_0) \leq \frac{d_{\alpha_n}(x_0, T_1x_0) + d_{\alpha_n}(x_0, T_2x_0)}{2} \leq \frac{p_1(x_0, \alpha) + p_2(x_0, \alpha)}{2}.$$

Hence, H satisfies all conditions in Theorem 3.15, and has a fixed point in X. Notice that H may not be a Φ-contraction, by choosing j_1, j_2 such that $d_{j_1(\alpha)} + d_{j_2(\alpha)} \notin \mathcal{A}$ for some $\alpha \in A$, and hence Theorem 2 in[1] cannot be applied.

We now consider a normed space E equipped with the weak topology. Let E^* be the dual space of E, $\|x\|_f = |f(x)|$ for any $x \in E$ and $f \in E^*$. Then for all $f \in E^*$, $\|\cdot\|_f$ is a seminorm on E, and hence $d_f(x, y) = \|x - y\|_f$ is a pseudometric on E. Then $\mathcal{A} = \{d_f : f \in E^*\}$ is a collection of pseudometrics on E. A definition of functionally uniformly lipschitzian selfmaps on a subset X of E with respect to the sequence (z_k) is given as follows :

Definition 3.18. Let (z_k) be a sequence of positive real numbers. The map $T : X \to X$ is called **functionally uniformly lipschitzian with respect to the sequence** (z_k) if for each $f \in E^*$, there exist $n \in \mathbb{N}$, and $g_1, g_2, \ldots, g_n \in E^*$

$$\|T^k x - T^k y\|_f \le z_k \sum_{i=1}^{n} \|x - y\|_{g_i}$$

for any $x, y \in X$ and $k \in \mathbb{N}$.

Remark 3.19. Every functionally uniformly lipschitzian selfmap is always functionally uniformly lipschitzian with respect to the constant sequence (1), while a functionally uniformly lipschitzian selfmap with respect to the bounded sequence is always functionally uniformly lipschitzian.

Proposition 3.20. *Let (z_k) be a sequence of positive real numbers and $T : X \to X$ be functionally uniformly lipshitzian with respect to the sequence (z_k). Then T^k is a J-contraction for any $k \ge 1$.*

Proof. Since T is functionally uniformly lipshitzian with respect to the sequence

(z_k), for each $f \in E^*$, there exist $n \in \mathbb{N}$, and $g_1, g_2, \ldots, g_n \in E^*$

$$\begin{aligned}\|T^k x - T^k y\|_f &\leq z_k \sum_{i=1}^{n} \|x - y\|_{g_i} \\ &= \frac{z_k}{z_k + 1}(z_k + 1) \sum_{i=1}^{n} \|x - y\|_{g_i} \\ &= \frac{z_k}{z_k + 1} \sum_{i=1}^{n} \|x - y\|_{(z_k+1)g_i}\end{aligned}$$

for any $x, y \in X$. By letting $J(f) = \{(z_k + 1)g_1, \ldots, (z_k + 1)g_n\}$, $\phi(t) = \frac{z_k}{z_k+1}t$, then $\phi \in \Phi$ is subadditive, so T^k is a J-contraction. □

Next, a fixed point theorem for functionally uniformly lipschitzian selfmaps with respect to the sequence (z_k) which does not require sequential completeness is presented.

Theorem 3.21. *Suppose $T : X \to X$ is a functionally uniformly lipschitzian selfmap with respect to a sequence (z_k) converging to 0. If there exists $x_0 \in X$ such that the sequence $(T^k x_0)$ has weakly convergent subsequence, then T has a unique fixed point in X.*

Proof. Since T is functionally uniformly lipschitzian, for each $f \in E^*$, there exist $n \in \mathbb{N}$, and $g_1, g_2, \ldots, g_n \in E^*$

$$\|T^k x - T^k y\|_f \leq z_k \sum_{i=1}^{n} \|x - y\|_{g_i}$$

for any $x, y \in X$ and $k \in \mathbb{N}$. Let $f \in E^*$, $\epsilon > 0$, and $(T^{j_k} x_0)$ be a subsequence of $T^k x_0$ such that $(T^{j_k} x_0)$ converges weakly to $p \in X$. Since (z_k) converges to 0, we can choose $N \in \mathbb{N}$ sufficiently large so that $z_k < \frac{\epsilon}{2\sum_{i=1}^{n} \|Tx_0 - x_0\|_{g_i}}$ and $\|T^{j_k} x_0 - p\|_f < \frac{\epsilon}{2}$ for any $k \geq N$. Then we have

$$\begin{aligned}\|T^{j_k+1} x_0 - p\|_f &\leq \|T^{j_k}(Tx_0) - T^{j_k} x_0\|_f + \|T^{j_k} x_0 - p\|_f \\ &\leq z_{j_k} \sum_{i=1}^{n} \|Tx_0 - x_0\|_{g_i} + \|T^{j_k} x_0 - p\|_f < \epsilon.\end{aligned}$$

Then $(T^{j_k+1}x_0)$ converges weakly to p. Since T is J-contraction, T is weakly continuous and hence the sequence $(T^{j_k+1}x_0)$ converges weakly to Tp. Since the weak topology is Hausdorff, $Tp = p$. For the uniqueness, suppose x and y are fixed points of T. Then for each $f \in E^*$ and $k \in \mathbb{N}$

$$\|x-y\|_f = \|T^k x - T^k y\|_f \leq z_k \sum_{i=1}^{n} \|x-y\|_{g_i}.$$

Since the sequence (z_k) converges to 0, we must have $x = y$. □

Also, we immediately obtain this corollary.

Corollary 3.22. *If X is closed, convex and bounded subset of a reflexive Banach space, and $T : X \to X$ is a functionally uniformly lipschitzian selfmap with respect to a sequence (z_k) converges to 0, then T has a unique fixed point in X*

Proof. By the assumption, X is weakly compact. Then every sequence in X has a weakly convergent subsequence, and hence, T has a unique fixed point in X by the previous theorem. □

Theorem 3.23. *Suppose X is weakly sequentially complete and $T : X \to X$ is a functionally uniformly lipschitzian selfmap with respect to a sequence (z_k). If the sequence (z_k) is summable, then T has a unique fixed point in X and $C(T) = X$.*

Proof. Let $\mathcal{A} = \{d_f : f \in E^*\}$. Since T is functionally uniformly lipschitzian with respect to (z_k), for any $f \in E^*$, there exist $n \in \mathbb{N}$, and $g_1, g_2, \ldots, g_n \in E^*$

$$\|T^k x - T^k y\|_f \leq z_k \sum_{i=1}^{n} \|x-y\|_{g_i}$$

for any $x, y \in X$ and $k \in \mathbb{N}$. By letting $\phi_{f,k}(t) = z_k t$, $D_{f,k} = \{1, 2, \ldots, n\}$, $P_{f,k}(i) = g_i$, and $F_f(x,y) = \sum_{i=1}^{n} d_{g_i}(x,y) = \sum_{i=1}^{n} \|x-y\|_{g_i}$ where $f \in E^*$, $k \in \mathbb{N}$, $t \in [0,\infty)$, then $\phi_{f,k}$ is non-decreasing and continuous from the right, and

$P_{f,k}(i) \in E^*$. Since (z_k) is summable,

$$\sum_{k\in\mathbb{N}} |D_{f,k}|\phi_{f,k}(F_f(x,y)) = \sum_{k\in\mathbb{N}} nz_k(\sum_{i=1}^{n} \|x-y\|_{g_i}) < \infty,$$

and

$$d_{P_{f,k}(i)}(x,y) = \|x-y\|_{g_i} \leq \sum_{i=1}^{n} \|x-y\|_{g_i} = F_f(x,y),$$

for all $k \in \mathbb{N}$ and $i \in D_{f,k}$. By Theorem 3.13(2), then T has a unique fixed point in X and $C(T) = X$. $\square$

Example 3.24. Let $0 \neq c = (c_1, c_2, \dots) \in \ell_2$ and $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \dots) = \Big(\frac{1}{2}\sin(g(x)), \frac{1}{8}\sin^2(g(x)), \dots, \frac{1}{n2^n}\sin^n(g(x)), \dots\Big),$$

where $g(x) = \frac{1}{\|c\|_2+1}\sum_{n=1}^{\infty} c_n x_n$ for all $x = (x_1, x_2, \dots) \in \ell_2$. In Chapter 4, we will show that T is functionally uniformly lipschitzian with respect to a sequence (z_k). Moreover, T^k is a J-contraction for any $k \geq 1$.

CHAPTER IV

CRITERIA FOR J-CONTRACTION MAPS

In this chapter, some criteria for a map to be functionally uniformly lipschitzian with respect to a sequence (z_k) on an infinite dimensional Banach space are given. By Proposition 3.20, we also have criteria for a map to be J-contractions. Suppose E is an infinite dimensional Banach space having a normalized Schauder basis (e_n). Let X be a nonempty subset of E, and $T : X \to X$. Moreover, for a lipschitzian selfmap $h : \mathbb{R} \to \mathbb{R}$, we will use $L(h)$ to denote the Lipschitz constant of h.

First, we will recall the definition of functionally uniformly lipshitzian with respect to a sequence (z_k) as follows: for any sequence (z_k) of positive real numbers, a selfmap T on X is functionally uniformly lipshitzian with respect to the sequence (z_k) if for each $f \in E^*$ there are $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in E^*$ such that

$$\|T^k x - T^k y\|_f \leq z_k \sum_{i=1}^{n} \|x - y\|_{g_i},$$

for any $x, y \in X$ and $k \in \mathbb{N}$.

Proposition 4.1. *If T is functionally lipschitzian, then $e_n^* \circ T$ is a lipschitzian functional for each $n \in \mathbb{N}$.*

Proof. Since T is functionally lipschitzian, for each $n \in \mathbb{N}$, there are $m \in \mathbb{N}$ and

$g_1, \ldots, g_m \in E^*$ such that for any $x, y \in X$,

$$\begin{aligned}|e_n^* \circ T(x) - e_n^* \circ T(y)| &= \|T(x) - T(y)\|_{e_n^*} \\ &\leq \sum_{i=1}^{m} \|(x-y)\|_{g_i} \\ &= \sum_{i=1}^{m} |g_i(x-y)| \\ &\leq \left(\sum_{i=1}^{m} \|g_i\|\right) \|x-y\|,\end{aligned}$$

which implies that $e_n^* \circ T$ is lipschitzian. □

The following lemma is important because the definition of being functionally uniformly lipschitzian with respect to the sequence can be simplified in a Banach space with a normalized Schauder basis.

Lemma 4.2. *Let $g_1, g_2, \ldots, g_N \in E^*$, and for any $k \in \mathbb{N}$, (c_n^k) sequences of nonnegative numbers with $\sum_{n=1}^{\infty} c_n^k < \infty$. If for each $n, k \in \mathbb{N}$ and $x, y \in X$,*

$$|e_n^*(T^k(x) - T^k(y))| \leq c_n^k \sum_{i=1}^{N} |g_i(x-y)|,$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \sum_{n=1}^{\infty} c_n^k$.

Proof. Let $z_k = \sum_{n=1}^{\infty} c_n^k$.

For each $f \in E^*$ and $x, y \in X$, we have

$$
\begin{aligned}
\|T^k(x) - T^k(y)\|_f &= \left| f\left(\sum_{n=1}^{\infty} e_n^*(T^k(x) - T^k(y)) e_n \right) \right| \\
&\le \|f\| \sum_{n=1}^{\infty} |e_n^*(T^k(x) - T^k(y))| \\
&\le \|f\| \sum_{n=1}^{\infty} \left(c_n^k \sum_{i=1}^{N} |g_i(x-y)| \right) \\
&= \|f\| z_k \sum_{i=1}^{N} |g_i(x-y)| \\
&= z_k \sum_{i=1}^{N} \|x-y\|_{\|f\| g_i}.
\end{aligned}
$$

□

Therefore, in a Banach space with a normalized Schauder basis, it is enough to check only coordinate functionals to show the functionally uniform lipschitzianess with respect to the sequence.

Theorem 4.3. *Let $g_1, g_2, \ldots, g_N \in E^*$ and $\{h_{n,i} : n \in \mathbb{N}; i = 1, \ldots, N\}$ a collection on lipschitzian selfmaps on $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \ldots, N\} < \infty$. If for each $n \in \mathbb{N}$*

$$
e_n^* \circ T = \sum_{i=1}^{N} h_{n,i} \circ g_i|_X,
$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where

$$
z_k = \left[\left(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \ldots, N\} \right) \left(\sum_{i=1}^{N} \|g_i\| \right) \right]^{k-1}.
$$

Consequently,

1. *if $\left(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \ldots, N\}\right) \left(\sum_{i=1}^{N} \|g_i\|\right) \le 1$, then T is functionally uniformly lipschitzian.*

2. *if $\left(\sum_{n=1}^{\infty} \max\{L(h_{n,i}) : i = 1, \ldots, N\}\right) \left(\sum_{i=1}^{N} \|g_i\|\right) < 1$, then T has a unique fixed point and $C(T) = X$.*

Proof. Let $B = \sum_{i=1}^{N} \|g_i\|$, $C_n = \max\{L(h_{n,i}) : i = 1, \ldots, N\}$ and $C = \sum_{n=1}^{\infty} C_n$.

First notice that, for any $n \in \mathbb{N}$ and $x, y \in X$, we have

$$\begin{aligned}
|e_n^*(T(x) - T(y))| &= \left| \sum_{i=1}^{N} (h_{n,i} \circ g_i(x) - h_{n,i} \circ g_i(y)) \right| \\
&\leq \sum_{i=1}^{N} |h_{n,i}(g_i(x)) - h_{n,i}(g_i(y))| \\
&\leq \sum_{i=1}^{N} L(h_{n,i}) |g_i(x - y)| \\
&\leq C_n \sum_{i=1}^{N} |g_i(x - y)|.
\end{aligned}$$

Next, we claim that for each $n, k \in \mathbb{N}$ and $x, y \in X$,

$$|e_n^*(T^k(x) - T^k(y))| \leq C_n (BC)^{k-1} \sum_{i=1}^{N} |g_i(x - y)|.$$

When $k = 1$, the statement immediately holds from the previous paragraph. Suppose it is also true for some $k \in \mathbb{N}$, we then have

$$\begin{aligned}
|e_n^*(T^{k+1}(x) - T^{k+1}(y))| &\leq C_n (BC)^{k-1} \sum_{i=1}^{N} |g_i(T(x) - T(y))| \\
&\leq C_n (BC)^{k-1} \left(\sum_{i=1}^{N} \|g_i\| \right) \|T(x) - T(y)\| \\
&\leq C_n (BC)^{k-1} B \sum_{n=1}^{\infty} |e_n^*(T(x) - T(y))| \\
&\leq C_n (BC)^{k-1} B \sum_{n=1}^{\infty} \left(C_n \sum_{i=1}^{N} |g_i(x - y)| \right) \\
&= C_n (BC)^{k-1} B \left(\sum_{n=1}^{\infty} C_n \right) \sum_{i=1}^{N} |g_i(x - y)| \\
&= C_n (BC)^{k-1} BC \sum_{i=1}^{N} |g_i(x - y)| \\
&= C_n (BC)^{k} \sum_{i=1}^{N} |g_i(x - y)|,
\end{aligned}$$

which proves the claim.

By the previous Lemma, we have

$$\|T^k(x) - T^k(y)\|_f \leq (BC)^{k-1} \sum_{i=1}^{N} \|x - y\|_{C\|f\|g_i}$$

and hence T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = (BC)^{k-1}$. Therefore, if $BC \leq 1$, then (z_k) is bounded and T is functionally uniformly lipschitzian by Remark 3.19. Also, if $BC < 1$, the sequence (z_k) is summable, and hence by Theorem 3.23, T has a unique fixed point and $C(T) = X$. □

Example 4.4. Consider the map T given in Example 3.24. Let $0 \neq c = (c_1, c_2, \dots) \in \ell_2$ and $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \dots) = \big(\frac{1}{2}\sin(g(x)), \frac{1}{8}\sin^2(g(x)), \dots, \frac{1}{n2^n}\sin^n(g(x))\big),$$

where $g(x) = \frac{1}{\|c\|_2+1}\sum_{n=1}^{\infty} c_n x_n$ for all $x = (x_1, x_2, \dots) \in \ell_2$. By letting $N = 1$, $h_n = \frac{1}{n2^n}\sin^n$ and Theorem 4.3, then T is functionally uniformly lipschitzian with respect to a sequence (z_k) where

$$z_k = \left[\left(\sum_{n=1}^{\infty} L(h_n)\right)(\|g\|)\right]^{k-1}.$$

Since $\sum_{n=1}^{\infty} L(h_n) \leq 1$ and $\|g\| < 1$ and Proposition 3.20, T^k is a J-contraction for any $k \geq 1$. Furthermore, T has a unique fixed point and $C(T) = X$.

Theorem 4.5. *Let $l \in \mathbb{N}$, $\{g_{n,i} : n = 1, \dots, l; i = 1, \dots, m_n\} \subseteq E^*$ and $a \in \mathbb{R}$. Suppose that $e_n^*(T(x) - T(y)) = ae_n^*(x - y)$ for all $n > l$ and $x, y \in X$. If for each $n \leq l$, $k \in \mathbb{N}$ and $x, y \in X$, there is $c_k \geq 0$*

$$|e_n^*(T^k(x) - T^k(y))| \leq c_k \textstyle\sum_{i=1}^{m_n} |g_{n,i}(x - y)|,$$

then T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \max\{c_k, |a|^k\}$.

Proof. Let $f \in E^*$, $k \in \mathbb{N}$ and $x, y \in X$. Then

$$\begin{aligned}
\|T^k(x) - T^k(y)\|_f &= |f(T^k(x) - T^k(y))| \\
&= \left| f\left(\sum_{n=1}^{\infty} e_n^*(T^k(x) - T^k(y))e_n\right)\right| \\
&= \left| f\left(\sum_{n\le l} e_n^*(T^k(x) - T^k(y))e_n\right) + f\left(\sum_{n>l} e_n^*(T^k(x) - T^k(y))e_n\right)\right| \\
&\le \left| f\left(\sum_{n\le l} e_n^*(T^k(x) - T^k(y))e_n\right)\right| + \left| f\left(\sum_{n>l} e_n^*(T^k(x) - T^k(y))e_n\right)\right| \\
&\le c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + \left| f\left(\sum_{n>l} a^k e_n^*(x-y)e_n\right)\right| \quad (**) \\
&= c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + \left| f\left(a^k(x-y) - \sum_{n\le l} a^k e_n^*(x-y)e_n\right)\right| \\
&\le c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + |a^k f(x-y)| \\
&\quad + \left| f\left(\sum_{n\le l} a^k e_n^*(x-y)e_n\right)\right| \\
&\le c_k\|f\| \sum_{n\le l}\sum_{i=1}^{m_n} |g_{n,i}(x-y)| + |a^k f(x-y)| \\
&\quad + \sum_{n\le l} |a^k f(e_n)e_n^*(x-y)| \\
&\le \max\{c_k, |a|^k\} \left(\sum_{n\le l}\sum_{i=1}^{m_n} \|x-y\|_{\|f\|g_{n,i}}\right. \\
&\quad \left. + \|x-y\|_f + \sum_{n\le l} \|x-y\|_{|f(e_n)|e_n^*}\right).
\end{aligned}$$

Therefore, T is functionally uniformly lipschitzian with respect to the sequence (z_k), where $z_k = \max\{c_k, |a|^k\}$. □

CHAPTER V

FIXED POINT SETS AND VIRTUAL STABILITY

In this chapter, we will show that, under a mild condition, a J-nonexpansive map is always virtually stable. This immediately gives a connection between the fixed point set and the convergence set of a J-nonexpansive selfmap.

As in the previous section, let $(E, \mathcal{A})$ be a Hausdorff uniform space whose uniformity is generated by a saturated collection of pseudometrics $\mathcal{A} = \{d_\alpha : \alpha \in A\}$ indexed by A and $\varnothing \neq X \subseteq E$. The following theorem gives a general criterion for a selfmap on X to be virtually stable.

Theorem 5.1. *Let $T : X \to X$ be a selfmap whose fixed point set $F(T)$ is nonempty, and satisfies the following conditions :*

(i) for each $\alpha \in A$ and $k \in \mathbb{N}$, there exist a finite set $D_{\alpha,k}$ and a map $P_{\alpha,k} : D_{\alpha,k} \to A$ such that

$$d_\alpha(T^k x, T^k y) \leq \sum_{\gamma \in D_{\alpha,k}} d_{P_{\alpha,k}(\gamma)}(x, y),$$

for any $x, y \in X$,

(ii) there exists $N \in \mathbb{N}$ such that $|D_{\alpha,n}| \leq |D_{\alpha,N}|$ and $P_{\alpha,n}(D_{\alpha,n}) \subseteq P_{\alpha,N}(D_{\alpha,N})$ for any $n \geq N$ and $\alpha \in A$.

Then T is uniformly virtually stable with respect to the sequence of all natural numbers. Moreover, the fixed point set of T is a retract of the convergence set of T.

Proof. Let $z \in F(T)$ and let U be a neighborhood of z. We may assume that $U = \bigcap_{i=1}^{m}\{w \in X : d_{\alpha_i}(w,z) < \epsilon\}$ for some $\epsilon > 0$ and $\alpha_1, \dots, \alpha_m \in A$. For each $n \in \mathbb{N}$, let

$$V_n = \bigcap_{i=1}^{m} \bigcap_{\gamma \in D_{\alpha_i,n}} \{w \in X : d_{P_{\alpha_i,n}(\gamma)}(w,z) < \frac{\epsilon}{|D_{\alpha_i,n}|}\}.$$

By (ii), there exists $N \in \mathbb{N}$ such that $|D_{\alpha_i,n}| \leq |D_{\alpha_i,N}|$ and $P_{\alpha_i,n}(D_{\alpha_i,n}) \subseteq P_{\alpha_i,N}(D_{\alpha_i,N})$ for any $n \geq N$ and $i = 1, \dots, m$. Let $V = V_1 \cap V_2 \cap \cdots \cap V_N$ which is clearly a nonempty open subset of X, $y \in V$, $l \in \mathbb{N}$ and $i \in \{1, \dots, m\}$. It follows that

$$d_{\alpha_i}(T^l y, z) = d_{\alpha_i}(T^l y, T^l z) \leq \sum_{\gamma \in D_{\alpha_i,l}} d_{P_{\alpha_i,l}(\gamma)}(y,z).$$

If $l < N$, then

$$d_{\alpha_i}(T^l y, z) < \sum_{\gamma \in D_{\alpha_i,l}} \frac{\epsilon}{|D_{\alpha_i,l}|} = \epsilon.$$

If $l \geq N$, since $P_{\alpha_i,l}(\gamma) \in P_{\alpha_i,l}(D_{\alpha_i,l}) \subseteq P_{\alpha_i,N}(D_{\alpha_i,N})$, we have $d_{P_{\alpha_i,l}(\gamma)}(y,z) < \frac{\epsilon}{|D_{\alpha_i,N}|}$ for each $\gamma \in D_{\alpha_i,l}$, and hence

$$d_{\alpha_i}(T^l y, z) < \sum_{\gamma \in D_{\alpha_i,l}} \frac{\epsilon}{|D_{\alpha_i,N}|} = \frac{\epsilon |D_{\alpha_i,l}|}{|D_{\alpha_i,N}|} \leq \epsilon.$$

Hence, T is uniformly virtually stable with respect to the sequence of all natural numbers. Since $(E, \mathcal{A})$ is regular, by Theorem 2.32, we immediately obtain that the fixed point set of T is a retract of the convergence set of T. □

Corollary 5.2. *Suppose that T is J-nonexpansive with $F(T) \neq \varnothing$. If there exists $N \in \mathbb{N}$ such that $|A_n(\alpha)| \leq |A_N(\alpha)|$ and $\pi_n(A_n(\alpha)) \subseteq \pi_N(A_N(\alpha))$ for any $n \geq N$ and $\alpha \in A$, then T is uniformly virtually stable with respect to the sequence of all natural numbers and the fixed point set of T is a retract of the convergence set of T.*

Proof. By letting $D_{\alpha,n} = A_n(\alpha)$ and $P_{\alpha,n} = \pi_n|_{A_n(\alpha)}$ for any $n \in \mathbb{N}$ and $\alpha \in A$, we have

$$d_\alpha(T^l x, T^l y) \leq \sum_{\gamma \in D_{\alpha,l}} d_{P_{\alpha,l}(\gamma)}(x, y),$$

for any $x, y \in X$. The result then follows from Theorem 5.1. □

Corollary 5.3. *Suppose that T is a functionally uniformly lipschitzian selfmap on a subset X of a normed space E. If $F(T) \neq \varnothing$, then T is uniformly virtually stable with respect to the sequence of all natural numbers and the fixed point set of T is a retract of the weak convergence set of T.*

Proof. Since T is functionally uniformly lipschitzian, for each $f \in E^*$ there exist $N \in \mathbb{N}$ and $g_1, g_2, \dots, g_N \in E^*$ such that

$$\|T^k x - T^k y\|_f \leq \sum_{i=1}^{N} \|x - y\|_{g_i},$$

for any $x, y \in X$ and $k \in \mathbb{N}$. Clearly, T is J-nonexpansive where $J(f) = \{g_1, g_2, \dots, g_N\}$ and $\pi_n(A(f)) = \pi_m(A(f))$ for any $f \in E^*$ and $n, m \in \mathbb{N}$. By the previous corollary, T is uniformly virtually stable with respect to the sequence of all natural numbers and the fixed point set of T is a retract of the weak convergence set of T. □

In the final part of this chapter, examples of J-nonexpansive selfmaps are given. They will help us, for the sake of completeness, construct a simple selfmap, the last example, whose fixed point set is not convex and hence guaranteed be a retract of its weak convergence set only by our results.

Example 5.4. Let $E = \ell_2$ equipped with the weak topology, and $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \dots) = \left(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, \dots \right),$$

for any $(x_1, x_2, \dots) \in \ell_2$. Then $\mathcal{A} = \{|f| : f \in \ell_2\}$, and by example 3.8 in [12], we have T is functionally uniformly lipshitzian, so it is J-nonexpansive. However, we can directly show that

$$\begin{aligned}|f(T^n x - T^n y)| \le 2\|f\| \Bigg[&\frac{\sqrt{2}}{9}(|x_1 - y_1 + x_3 - y_3| + |x_2 - y_2 + x_4 - y_4|) \\ &+ \frac{\sqrt{2}\,(|x_1 - y_1| + |x_2 - y_2| + |x_1 - y_1 + x_3 - y_3| + |x_2 - y_2 + x_4 - y_4|)}{9 - 6\sqrt{2}} \Bigg] \\ &+ \|f\| \left(\frac{1}{3}|x_1 - y_1| + |x_1 - y_1 + x_3 - y_3| + \frac{1}{3}|x_2 - y_2| + |x_2 - y_2 + x_4 - y_4| \right) \\ &+ \|f\||x_1 - y_1| + \|f\||x_2 - y_2| + |f(x - y)|,\end{aligned}$$

for each $f \in \ell_2$, $n \in \mathbb{N}$,$x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots) \in \ell_2$.

By letting $J : \ell_2 \to \mathcal{P}^f(\ell_2)$ be defined by $J(f) = \{f, g_1, g_2, g_3, g_4\}$ for each $f \in \ell_2$, where

$$g_1(x) = \|f\|\big(\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9 - 6\sqrt{2}} + 1\big)(x_1 + x_3), g_2(x) = \|f\|\big(\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9 - 6\sqrt{2}} + 1\big)(x_2 + x_4),$$

$$g_3(x) = \|f\|\big(\frac{2\sqrt{2}}{9 - 6\sqrt{2}} + \frac{4}{3}\big)x_1, g_4(x) = \|f\|\big(\frac{2\sqrt{2}}{9 - 6\sqrt{2}} + \frac{4}{3}\big)x_2,$$

for each $x = (x_1, x_2, \dots) \in \ell_2$, it follows that T is J-nonexpansive.

Notice that $(0, 0, \dots)$ is a fixed point of T, and for each $f \in \ell_2$ and $n, m \in \mathbb{N}$, $\pi_n(A(f)) = \pi_m(A(f))$. Then, by Theorem 5.1, T is uniformly virtually stable with respect to the sequence of all natural numbers and hence the fixed point set of T is a retract of the weak convergence set of T. Moreover, the fixed point set is not convex because $x = (1, 1, 2, 2, 0, \dots)$ and $y = (1, 1, -4, -4, 0, \dots)$ are fixed points of T, while the convex combination $\frac{1}{2}x + \frac{1}{2}y = (1, 1, -1, -1, 0, \dots)$ is not. Therefore, since ℓ_2 is uniformly convex, T is not nonexpansive.

Example 5.5. Let $X = \big\{(x_n) \in \ell_2 : |x_1|, |x_2| \le 10 \quad \text{and for any } i \ge 3, \quad |x_i| \le \frac{10}{2^{i-2}}\big\}$ and fix $c = (10, 10, 15, 4, 8, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots) \in \ell_2$. Notice that X is weakly-compact and convex. Consider $T : X \to X$ defined by

$$T(x_1, x_2, \dots) = (\sin(g(x)), \cos(g(x)), x_3, x_4, \dots),$$

where $g(x) = \frac{1}{2\|c\|_2+1}\sum_{n=1}^{\infty} c_n x_n$ for all $x = (x_1, x_2, \dots) \in \ell_2$. By letting $l = 2$, $g_1 = g_2 = g$, $h_1 = \sin$, $h_2 = \cos$, $a = 1$ and $0 = b_3 = b_4 = \dots$, we have $\|g_1\| = \|g_2\| = \|g\| = \frac{\|c\|_2}{2\|c\|_2+1}$ and $L(h_1)\|g_1\| + L(h_2)\|g_2\| < 1$. By Corollary 3.7 in [12], T is functionally uniformly lipschitzian, so it is J-nonexpansive. Then for each $n, m \in \mathbb{N}$, $\pi_n(A(f)) = \pi_m(A(f))$ because of the definition of a functionally uniformly lipschitzian selfmap. By Theorem 5.1, T is uniformly virtually stable with respect to the sequence of all natural numbers and hence the fixed point set of T is a retract of the weak convergence set of T. Notice also that $x = (0, 1, 0, 0, -\frac{5}{4}, 0, \dots)$, $y = (1, 0, \frac{(2\|c\|_2+1)\pi}{30}, -\frac{5}{2}, 0, \dots) \in F(T)$ but $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, \frac{1}{2}, \frac{(2\|c\|_2+1)\pi}{60}, -\frac{5}{4}, -\frac{5}{8}, 0, \dots) \notin F(T)$; i.e., $F(T)$ is not convex. Therefore, since ℓ_2 is uniformly convex, T is not nonexpansive.

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VITA

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