CHAPTER III

VEHICLE-BRIDGE INTERACTION AND AXLE LOADS IDENTIFICATION

3.1 General

In this chapter, the concepts for the vehicle-bridge interaction as well as the axle loads identification are presented. The vehicle-bridge interaction is firstly considered in order to study the effect of various vehicle or bridge parameters and to use to simulate the dynamic response of the bridge and the vehicle for axle loads determination. Then the axle load identification methods from the dynamic responses (strain or moment) are outlined. The methods are related to optimization technique and finite element analysis of bridge structure. In this study, two methods are employed. The first method is the constant magnitude of moving axle loads assumption which utilizes the influence line and the optimization technique to identify the static axle load. The second method is the time-varying magnitude of moving axle loads assumption using the regularization with updated static component (USC) technique which utilizes the dynamic programming to identify the dynamic axle loads and using the iterative technique to decompose the identified load into static and dynamic component.

3.2 Vehicle-Bridge Interaction System

The dynamic response of bridge induced by a moving vehicle is an important data for bridge design and evaluation. Generally, there are two approaches to simulate the dynamic interaction response between bridge and vehicle. The first one is to simulate by the uncoupled iteration method. The bridge and vehicle system is solved separately and using an iterative process in each time step to find the equilibrium between the bridge and vehicle interaction. Another approach to simulate the vehicle-bridge interaction is to solve the fully coupled system of bridge and vehicle. In this research, the later approach is employed to simulate the dynamic interaction response because it can assemble the bridge and vehicle into one system which can be directly solved at each time step without any iteration.

3.2.1 Finite Element Method of Structural Formulation

The vehicle-bridge system can be simulated using modal decomposition analysis method but are subjected to modal truncation error in the dynamic response. Therefore, the dynamic response analysis for discrete system is preferred in this research since the bridge structure is simply-supported beam. The finite element method is then adopted to obtain the vehicle-bridge interaction model.

The modeling accuracy has been studied against the degree discretization of the structure for a moving load analysis (Rieker et al., 1996). It was noted that beams with various boundary conditions, and subjected to a moving load system with a general movement profile and external excitation can be successfully analyzed with accurate responses compared to those obtained from modal superposition analysis (Lin and Trethewy, 1990). In addition, the simulated responses are relatively not sensitive to the sampling frequency and number of data (Zhu and Law, 2004).

3.2.2 Vehicle Model

The vehicle-bridge model is present in Figure 3.1. A vehicle moving at a speed *v(t)* over a bridge. There are 4 degrees of freedom in the vehicle model consisting of vertical displacement, rotation of vehicle mass, vertical displacement of front and rear axle suspension mass. The equation of motion can be derived by dynamic equilibrium of vehicle in each degree of freedom as shown in Figure 3.2.

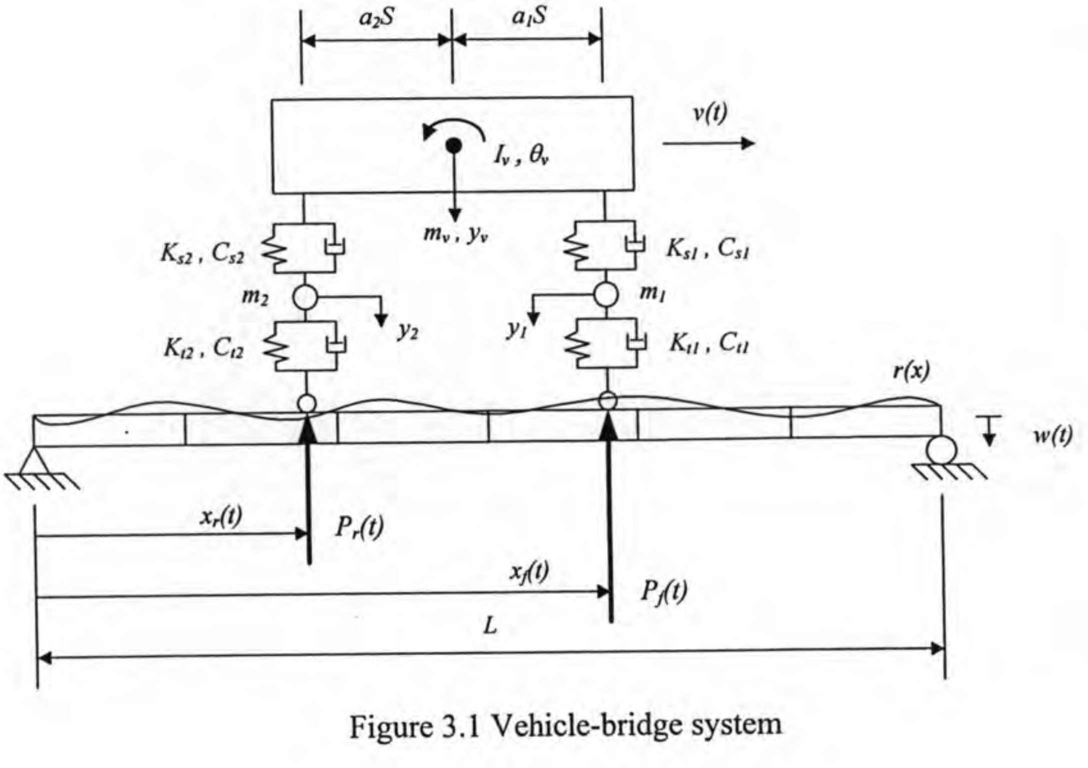


Figure 3.1 Vehicle-bridge system

Let m_v = mass of the vehicle

I_v = mass rotational moment of inertia of the vehicle

m_1 = mass of front axle suspension

m_2 = mass of rear axle suspension

K_{s1}, K_{s2} = suspension stiffness of front and rear axle

C_{s1}, C_{s2} = suspension damping of front and rear axle

K_{t1}, K_{t2} = tire stiffness of front and rear axle

C_{t1}, C_{t2} = tire damping of front and rear axle

S = axle spacing

L = span length of bridge

$x_f(t), x_r(t)$ = positions of the front and rear axle respectively at time t

$P_f(t), P_r(t)$ = front and rear axle force respectively at time t

v = velocity of vehicle

θ_v = rotation of vehicle mass

y_v = vertical displacement of vehicle

y_1, y_2 = vertical displacement of front and rear suspension mass

$w(t)$ = vertical dynamic deflection of bridge

$r(x)$ = road surface roughness at the location x

a_1, a_2 = center of gravity ratio of vehicle from front and rear axle.

Consider the vertical force equilibrium of vehicle mass:

$$\sum F = m_v \ddot{y}_v \quad ; \qquad -f_{s1} - f_{s2} = m_v \ddot{y}_v \tag{3.1}$$

where

$$f_{s1} = K_{s1}(y_v - \theta_v a_1 S - y_1) + C_{s1}(\dot{y}_v - \dot{\theta}_v a_1 S - \dot{y}_1)$$

$$f_{s2} = K_{s2}(y_v + \theta_v a_2 S - y_2) + C_{s2}(\dot{y}_v + \dot{\theta}_v a_2 S - \dot{y}_2).$$

Substituting f_{s1}, f_{s2} in Eq. (3.1), the equilibrium of vertical motion of vehicle mass becomes:

$$m_v\ddot{y}_v + (C_{s1} + C_{s2})\dot{y}_v + (K_{s1} + K_{s2})y_v$$

$$+(-C_{s1}a_1S + C_{s2}a_2S)\dot{\theta}_v + (-K_{s1}a_1S + K_{s2}a_2S)\theta_v \tag{3.2}$$

$$+(-C_{s1})\dot{y}_1 + (-K_{s1})y_1 + (-C_{s2})\dot{y}_2 + (-K_{s2})y_2 = 0$$

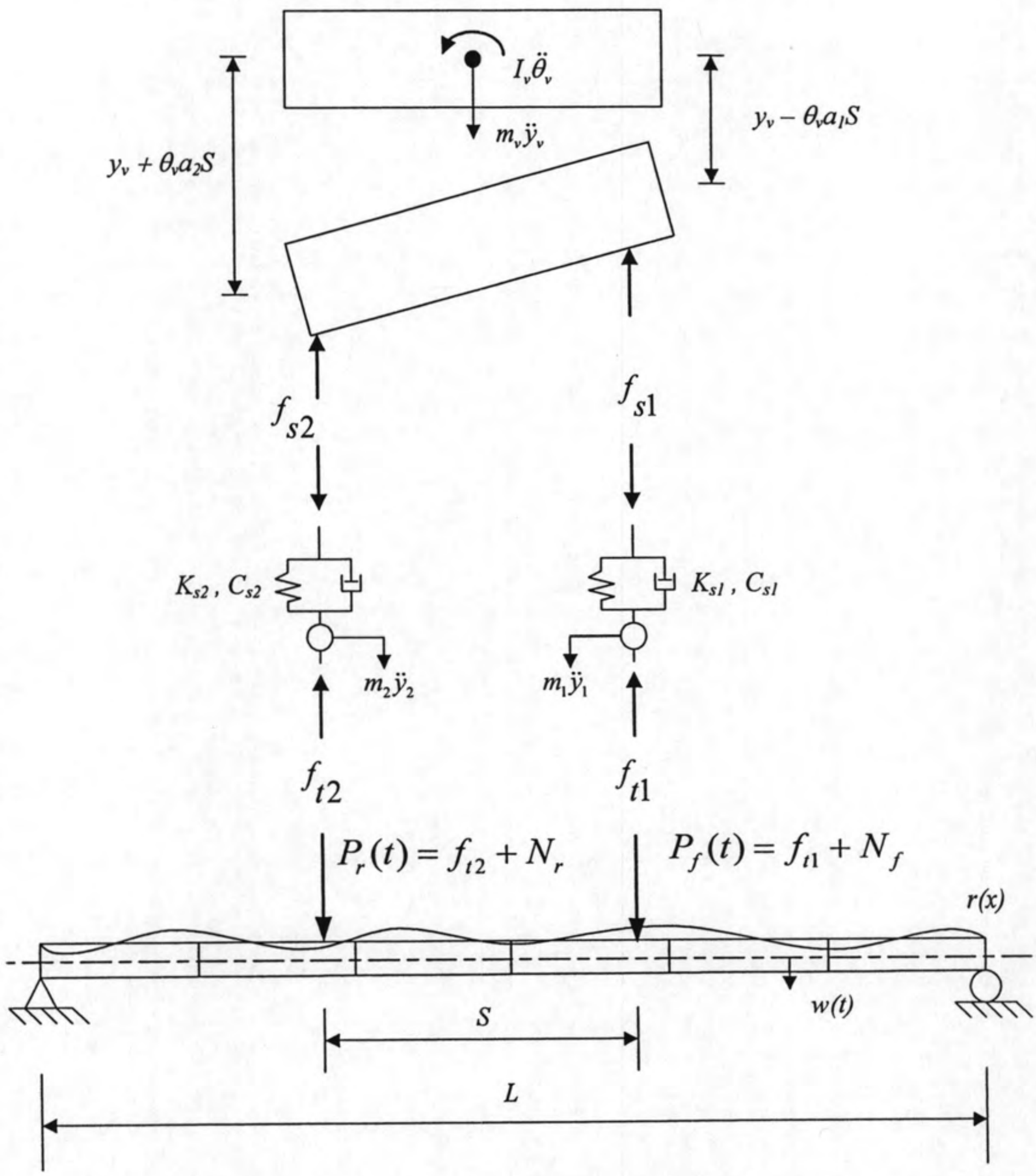


Figure 3.2 Free body diagram of vehicle-bridge system

Consider rotation of vehicle mass at center of gravity:

$$\sum M_c = I_v\ddot{\theta}_v \quad ; \qquad f_{s1}a_1S - f_{s2}a_2S = I_v\ddot{\theta}_v \tag{3.3}$$

Substituting f_{s1}, f_{s2} in Eq. (3.3), the equilibrium of rotation of vehicle mass becomes:

$$I_v\ddot{\theta}_v + (-C_{s1}a_1S + C_{s2}a_2S)\dot{y}_v + (-K_{s1}a_1S + K_{s2}a_2S)y_v$$

$$+(C_{s1}a_1^2S^2 + C_{s2}a_2^2S^2)\dot{\theta}_v + (K_{s1}a_1^2S^2 + K_{s2}a_2^2S^2)\theta_v \tag{3.4}$$

$$+(C_{s1}a_1S)\dot{y}_1 + (K_{s1}a_1S)y_1 + (-C_{s2}a_2S)\dot{y}_2 + (-K_{s2}a_2S)y_2 = 0$$

Consider the vertical equilibrium of suspension mass m_1:

$$\sum F = m_1\ddot{y}_1 \quad ; \qquad f_{s1} - f_{t1} = m_1\ddot{y}_1 \tag{3.5}$$

where

$$f_{t1} = K_{t1}(y_1 - \Delta_1) + C_{t1}(\dot{y}_1 - \dot{\Delta}_1)$$

$$\Delta_1 = (w_1(x_f(t),t) + r(x_f(t))) \tag{3.6}$$

$$\dot{\Delta}_1 = (\dot{w}_1(x_f(t),t) + \dot{r}(x_f(t)))$$

Substituting f_{s1}, f_{t1} in Eq. (3.5), the equilibrium of vertical motion of suspension mass m_1 becomes:

$$m_1\ddot{y}_1 + (-C_{s1})\dot{y}_v + (-K_{s1})\dot{y}_v + (C_{s1}a_1S)\dot{\theta}_v + (K_{s1}a_1S)\theta_v$$

$$+(C_{s1})\dot{y}_1 + (K_{s1})y_1 = -f_{t1} \tag{3.7}$$

Consider the vertical equilibrium of suspension mass m_2:

$$\sum F = m_2\ddot{y}_2 \quad ; \qquad f_{s2} - f_{t2} = m_2\ddot{y}_2 \tag{3.8}$$

where

$$f_{t2} = K_{t2}(y_2 - \Delta_2) + C_{t2}(\dot{y}_2 - \dot{\Delta}_2)$$

$$\Delta_2 = (w_2(x_r(t),t) + r(x_r(t))) \tag{3.9}$$

$$\dot{\Delta}_2 = (\dot{w}_1(x_r(t),t) + \dot{r}(x_r(t)))$$

Substituting f_{s2}, f_{t2} in Eq. (3.8), the equilibrium of vertical motion of suspension mass m_2 becomes:

$$m_2\ddot{y}_2+(-C_{s2})\dot{y}_v+(-K_{s2})y_v+(-C_{s2}a_2S)\dot{\theta}_v+(-K_{s2}a_2S)\theta_v$$
$$+(C_{s2})\dot{y}_2+(K_{s2})y_2=-f_{t2} \tag{3.10}$$

Thus, the equations of motion for the vehicle can be written in matrix form based on Eq. (3.2), (3.4), (3.7) and (3.10) as follow:

$$\mathbf{M_v}\ddot{\mathbf{Y}}(t)+\mathbf{C_v}\dot{\mathbf{Y}}(t)+\mathbf{K_v}\mathbf{Y}(t)=\mathbf{P_v}(t) \tag{3.11}$$

where

$$\mathbf{M_v}=\begin{bmatrix} m_v & 0 & 0 & 0 \\ 0 & I_v & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}$$

$$\mathbf{C_v}=\begin{bmatrix} C_{s1}+C_{s2} & (-C_{s1}a_1+C_{s2}a_2)S & -C_{s1} & -C_{s2} \\ (-C_{s1}a_1+C_{s2}a_2)S & (C_{s1}a_1^2+C_{s2}a_2^2)S & C_{s1}a_1S & -C_{s2}a_2S \\ -C_{s1} & C_{s1}a_1S & C_{s1} & 0 \\ -C_{s2} & -C_{s2}a_2S & 0 & C_{s2} \end{bmatrix}$$

$$\mathbf{K_v}=\begin{bmatrix} K_{s1}+K_{s2} & (-K_{s1}a_1+K_{s2}a_2)S & -K_{s1} & -K_{s2} \\ (-K_{s1}a_1+K_{s2}a_2)S & (K_{s1}a_1^2+K_{s2}a_2^2)S & K_{s1}a_1S & -K_{s2}a_2S \\ -K_{s1} & K_{s1}a_1S & K_{s1} & 0 \\ -K_{s2} & -K_{s2}a_2S & 0 & K_{s2} \end{bmatrix}$$

$$\mathbf{Y}(t)=\{y_v(t) \quad \theta_v(t) \quad y_1(t) \quad y_2(t)\}^T$$

$\mathbf{P_v}$ is the force terms containing the interaction force vector and static force vector as follows:

$$\mathbf{P}_v(t)=-\begin{Bmatrix} \mathbf{0} \\ \mathbf{P}_{int}(t) \end{Bmatrix}+\begin{Bmatrix} \mathbf{0} \\ \mathbf{M_s} \end{Bmatrix}=-\begin{Bmatrix} 0 \\ 0 \\ P_f(t) \\ P_r(t) \end{Bmatrix}+\begin{Bmatrix} 0 \\ 0 \\ N_f \\ N_r \end{Bmatrix}$$

$$P_f(t)=(f_{t1}(t)+N_f)=K_{t1}(y_1(t)-\Delta_1(t))+C_{t1}(\dot{y}_1(t)-\dot{\Delta}_1(t))+N_f$$

$$P_r(t)=(f_{t2}(t)+N_r)=K_{t2}(y_2(t)-\Delta_2(t))+C_{t2}(\dot{y}_2(t)-\dot{\Delta}_2(t))+N_r$$

$$N_f=(m_1+a_2m_v)g$$

$$N_r=(m_2+a_1m_v)g$$

3.2.3 Bridge Model

The bridge structure is considered as a simply supported bridge and is discretized by finite element method using beam elements as shown in Figure 3.2. The finite beam element has 2 nodes with respect to 4 degrees of freedom in vertical displacement and rotational displacement at both ends as shown in Figure 3.3.

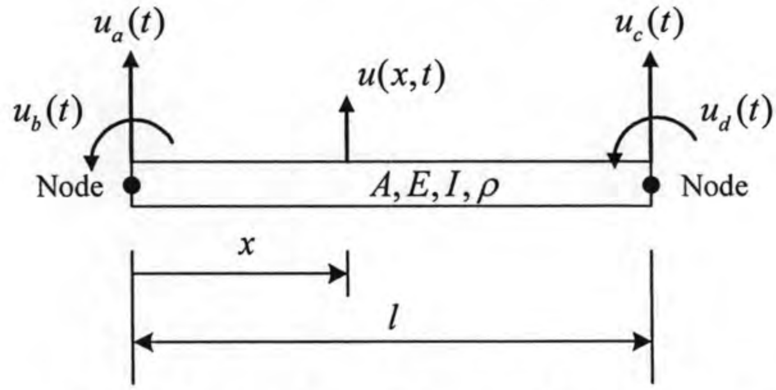


Figure 3.3 A finite beam element with 4 degrees of freedom

where

A	=	cross section area of beam element
E	=	modulus of elasticity of beam element
I	=	moment of inertia of beam element
ρ	=	mass per unit length of beam element
l	=	length of beam element.

Let $u(x,t)$ is the deflection of the bridge at distance x at time t. Thus, the governing equation of beam at position x and at time t can be expressed by:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0. \tag{3.12}$$

For the bridge having constant EI, Eq. (3.12) can be rewritten as:

$$\frac{\partial^4 u(x,t)}{\partial x^4} = 0 \tag{3.13}$$

The solution of Eq. (3.13) can be expressed in polynomial form as:

$$u(x,t) = c_1(t)x^3 + c_2(t)x^2 + c_3(t)x + c_4(t) \tag{3.14}$$

where $c_i(t)$ is the coefficient of the polynomial form with constant value.

The boundary conditions of beam element are:

$$u(0,t) = u_1(t) \qquad\qquad u(l,t) = u_3(t)$$

$$\frac{\partial u(0,t)}{\partial x} = u_2(t) \qquad\qquad \frac{\partial u(l,t)}{\partial x} = u_4(t) \tag{3.15}$$

Substituting (3.15) in Eq. (3.14), the constant values become:

$$c_4(t) = u_1(t)$$

$$c_3(t) = u_2(t)$$

$$c_2(t) = \frac{1}{l^2}\left[3(u_3 - u_1) - l(2u_2 + u_4)\right]$$

$$c_1(t) = \frac{1}{l^3}\left[2(u_1 - u_3) - l(u_2 + u_4)\right]. \tag{3.16}$$

Substituting (3.16) in Eq. (3.14), one can write the displacement equation of beam element at position x and at time t as follow:

$$u(x,t) = \left[1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right]u_1(t) + l\left[\frac{x}{l} - \frac{2x^2}{l^2} + \frac{x^3}{l^3}\right]u_2(t)$$
$$+\left[\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right]u_3(t) + l\left[-\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_4(t) \tag{3.17}$$

It is note that the coefficient terms in front of $u_i(t)$ are the shape functions of the displacements of the beam element.

The mass matrix of beam element can be formulated by introducing Eq. (3.17) into kinetic energy equation as:

$$T(t) = \frac{1}{2}\int_0^l \rho A\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx \tag{3.18}$$

Thus, Eq. (3.18) can be rewritten in the form:

$$T(t) = \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} \tag{3.19}$$

The matrix $\mathbf{M}$ is the elemental mass matrix and $\dot{\mathbf{u}}$ is the time derivative of the elemental displacement vector $\mathbf{u}(t)$ defined as:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} \tag{3.20}$$

Equating Eq. (3.18) to Eq. (3.19) with help of Eq. (3.20), one obtains the elemental mass matrix of beam element as:

$$\mathbf{M} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{3.21}$$

Likewise the mass matrix, one can calculate the stiffness matrix by replacing Eq. (3.17) into strain energy equation:

$$V(t) = \frac{1}{2}\int_0^l EI\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]^2 dx \tag{3.22}$$

The Eq. (3.22) can be rewritten as:

$$V(t) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}\,. \tag{3.23}$$

$\mathbf{u}(t)$ is define in (3.20), The stiffness matrix of beam element becomes:

$$\mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{3.24}$$

The elemental damping matrix of the bridge system is derived by free vibration system as follows:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0 \tag{3.25}$$

Multiplying both sides by the inverse of mass matrix, $\mathbf{M}^{-1}$, one gets:

$$\ddot{\mathbf{u}} + \overline{\mathbf{C}}\dot{\mathbf{u}} + \overline{\mathbf{K}}\mathbf{u} = 0 \tag{3.26}$$

where

$$\overline{\mathbf{C}} = \mathbf{M}^{-1}\mathbf{C}$$

$$\overline{\mathbf{K}} = \mathbf{M}^{-1}\mathbf{K}\,.$$

Transforming $\mathbf{u}$ to modal coordinate vector, $\mathbf{q}$, as :

$$\mathbf{u} = \mathbf{V}\mathbf{q} \tag{3.27}$$

In which, the vector $\mathbf{V}$ is the eigenvector of matrix $\overline{\mathbf{K}}$.

Substituting Eq. (3.27) into Eq. (3.26) and multiply by $\mathbf{V}^{-1}$, it is found that:

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{V}^{-1}\overline{\mathbf{C}}\mathbf{V}\dot{\mathbf{q}} + \mathbf{V}^{-1}\overline{\mathbf{K}}\mathbf{V}\mathbf{q} = 0 \tag{3.28}$$

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{C}^*\dot{\mathbf{q}} + \mathbf{K}^*\mathbf{q} = 0 \tag{3.29}$$

where

$$\mathbf{K}^* = \mathbf{V}^{-1}\overline{\mathbf{K}}\mathbf{V}$$
$$= \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_n^2 \end{bmatrix} \quad (3.30)$$

Assuming $\mathbf{C}^*$ has an orthogonality property like matrix $\mathbf{K}^*$, one gets:

$$\mathbf{C}^* = \mathbf{V}^{-1}\overline{\mathbf{C}}\mathbf{V}$$
$$= \begin{bmatrix} 2\xi_1\omega_1 & 0 & \cdots & 0 \\ 0 & 2\xi_2\omega_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\xi_n\omega_n \end{bmatrix} \quad (3.31)$$

where

ξ_i = damping ratio of the corresponding i^{th} mode shape

ω_i = natural frequency of the corresponding i^{th} mode shape.

Thus, the matrices $\overline{\mathbf{C}}$ and $\mathbf{C}^*$ can be obtained as follows:

$$\overline{\mathbf{C}} = \mathbf{V}\mathbf{C}^*\mathbf{V}^{-1} \quad (3.32)$$

$$\mathbf{C} = \mathbf{M}\overline{\mathbf{C}} \quad (3.33)$$

Therefore, the equation of motion for the bridge can be written as:

$$\mathbf{M_b}\ddot{\mathbf{R}}(t) + \mathbf{C_b}\dot{\mathbf{R}}(t) + \mathbf{K_b}\mathbf{R}(t) = \mathbf{P_b}(t) \quad (3.34)$$

where $\mathbf{M_b}$ = assembled mass matrix of the bridge

$\mathbf{C_b}$ = assembled damping matrix of the bridge

$\mathbf{K_b}$ = assembled stiffness matrix of the bridge

$\mathbf{R}(t)$ = global response vector of the bridge

and $\mathbf{P_b}(t)$ = external acting load vector of the bridge.

The external acting load vector of the bridge is the interaction force transformed to be the nodal loads of bridge's degrees of freedom.

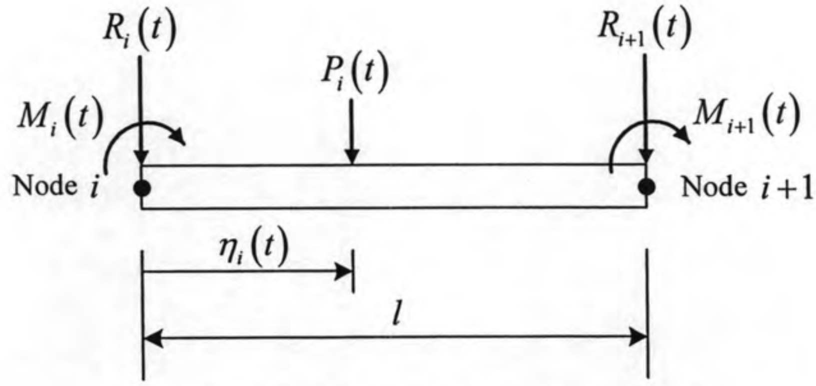


Figure 3.4 Nodal loads from external load

Figure 3.4 shows the beam element with an acting load and its equivalent nodal loads

where

$\eta_i(t)$ = the distance of the external acting load $P_i(t)$ from the left node of the beam element.

The nodal loads from external load as shown in Figure 3.4 can be expressed as

$$R_i(t)=\left(1-\frac{3\eta_i(t)^2}{l^2}+\frac{2\eta_i(t)^3}{l^3}\right)P_i(t)$$

$$M_i(t)=\left(\eta_i(t)-\frac{2\eta_i(t)^2}{l}+\frac{\eta_i(t)^3}{l^2}\right)P_i(t)$$

$$R_{i+1}(t)=\left(\frac{3\eta_i(t)^2}{l^2}-\frac{2\eta_i(t)^3}{l^3}\right)P_i(t)$$

$$M_{i+1}(t)=\left(\frac{\eta_i(t)^3}{l^2}-\frac{\eta_i(t)^2}{l}\right)P_i(t) \tag{3.35}$$

where

$R_i(t), R_{i+1}(t)$ = vertical load of node i^{th} and $i+1^{th}$ of element respectively

M_i, M_{i+1} = bending moment of node i^{th} and $i+1^{th}$ of element respectively.

From the above equations, the shape function of the j^{th} element used to transform the external acting load to the nodal load vector can be written as:

$$\mathbf{H}_j=\left\{1-3\left(\frac{\eta}{l}\right)^2+2\left(\frac{\eta}{l}\right)^3 \quad \eta\left(\frac{\eta}{l}-1\right)^2 \quad 3\left(\frac{\eta}{l}\right)^2-2\left(\frac{\eta}{l}\right)^3 \quad \eta\left(\frac{\eta}{l}\right)^2-\frac{\eta^2}{l}\right\}^T . \tag{3.36}$$

In case of the global external load shape function, the Eq. (3.36) can be expanded as the following form

$$\mathbf{H}_c = \begin{Bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & \mathbf{H}_1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \mathbf{H}_i & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \mathbf{H}_{N_p} & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{Bmatrix}^{\mathrm{T}} \tag{3.37}$$

where

$\mathbf{H}_c$ = an $NN \times N_p$ matrix with zero entries except at the degrees of freedom corresponding to the nodal displacements of the beam elements on which the load is acting,

NN = the number of degrees of freedom of the bridge after considering the boundary condition

and N_p = the number of external acting loads.

From the Eq. (3.37), the interaction force between the vehicle and the bridge can be transformed to be the nodal loads by using the relationship between the nodal load and the global load as follows:

$$\mathbf{P}_{\mathrm{b}}(t) = \mathbf{H}_c(x(t)) \cdot \mathbf{P}_{int}(t) \tag{3.38}$$

$$\mathbf{P}_{int}(t) = \left\{P_1(t), P_2(t), \ldots, P_{N_p}(t)\right\}^T \tag{3.39}$$

where

$\mathbf{P}_{\mathrm{b}}(\mathrm{t})$ = nodal load vector of bridge

$\mathbf{H}_c(x(t))$ = transformation vector from external loads to nodal loads

$\mathbf{P}_{int}(t)$ = vehicle-bridge interaction force vector with respect to number of axles.

Therefore, the equation of motion of the bridge can be rewritten as:

$$\mathbf{M}_{\mathrm{b}}\ddot{\mathbf{R}}(t) + \mathbf{C}_{\mathrm{b}}\dot{\mathbf{R}}(t) + \mathbf{K}_{\mathrm{b}}\mathbf{R}(t) = \mathbf{H}_c(x(t))\mathbf{P}_{int}(t) \tag{3.40}$$

3.2.4 Bridge Surface Roughness

In this research, the road surface roughness as given in the ISO-8606 specification is adopted. It is often related to the vehicle speed which is described by the velocity power spectrum density (PSD) and the displacement PSD. The general form of the displacement PSD of the road roughness surface is given as:

$$S_d(f) = S_d(f_0)\left(\frac{f}{f_0}\right)^{-\alpha} \tag{3.41}$$

where f_0 is the reference spatial frequency (= 0.1 cycles/m); α is an exponent of the PSD, and f is the spatial frequency (cycles/m). Eq. (3.41) gives an estimate on the degree of roughness of the road by the $S_d(f_0)$ value. This classification is made by assuming a constant vehicle velocity PSD and taking α =2. The ISO specification also gives the PSDs for different classes of roads.

Based on this ISO specification, the road surface roughness in the time domain can be simulated by applying the inverse fast Fourier transformation on $S_d(f_0)$ as follows:

$$r(x) = \sum_{i=1}^{N} \sqrt{4S(f_i)\Delta f} \cos\left(2\pi f_i x + \theta_i\right) \tag{3.42}$$

where $f_i = i\Delta f$ is the spatial frequency, $\Delta f = 1/(N\Delta)$, Δ is the distance interval between successive ordinates of the surface profile, N is the number of data points, and θ_i is a set of independent random phase angle uniformly distributed between 0 and 2π.

3.2.5 Vehicle-Bridge Interaction

To formulate the vehicle-bridge interaction as the equation of motion of the vehicle-bridge system, all degrees of freedom both vehicle and bridge must be solved simultaneously. Therefore, the equation of motion of the vehicle-bridge system is the combination of mass, damping, stiffness and interaction force terms corresponding to all degrees of freedom.

From the equation of motion of the vehicle and the bridge, in case of number of axles, $N_p = 2$, recalling the interaction force vector as:

$$\begin{aligned}\mathbf{P}_{int}(t) &= \begin{Bmatrix} P_f(t) \\ P_r(t) \end{Bmatrix} \\ &= \begin{Bmatrix} K_{t1}\left(y_1(t) - w_1\left(x_f(t),t\right) - r\left(x_f(t)\right)\right) + C_{t1}\left(\dot{y}_1(t) - \dot{w}_1\left(x_f(t),t\right) - \dot{r}\left(x_f(t)\right)\right) \\ K_{t2}\left(y_2(t) - w_2\left(x_r(t),t\right) - r\left(x_r(t)\right)\right) + C_{t2}\left(\dot{y}_2(t) - \dot{w}_2\left(x_r(t),t\right) - \dot{r}\left(x_r(t)\right)\right) \end{Bmatrix} \\ &\quad + \begin{Bmatrix} \left(m_1 + a_2 m_v\right) g \\ \left(m_2 + a_1 m_v\right) g \end{Bmatrix} \end{aligned} \tag{3.43}$$

It is noticed that the above interaction force term consists of degrees of freedom both vehicle and bridge. Thus the equation of motion of the vehicle and bridge must be rearranged as follows:

Once the response of the bridge, $\mathbf{R}(t)$ is obtained, the deflection of the bridge at position x and at time t can be calculated from:

$$w(x,t) = \mathbf{H}_c^T(x(t)) \cdot \mathbf{R}(t) \tag{3.44}$$

The time derivative of the bridge's deflection is

$$\dot{w}(x,t) = \frac{\partial \mathbf{H}_c^T(x(t))}{\partial x} \cdot \mathbf{R}(t) \cdot \dot{x}(t) + \mathbf{H}_c^T(x(t)) \cdot \dot{\mathbf{R}}(t). \tag{3.45}$$

Substituting Eq. (3.44) and (3.45) in Eq. (3.43) yields

$$\begin{aligned} P_f(t) &= K_{t1}\left(y_1(t) - \mathbf{H}_c^T(x_f(t)) \cdot \mathbf{R}(t) - r(x_f(t))\right) \\ &\quad + C_{t1}\left(\dot{y}_1(t) - \frac{\partial \mathbf{H}_c^T(x_f(t))}{\partial x} \cdot \mathbf{R}(t) \cdot v(t) - \mathbf{H}_c^T(x_f(t)) \cdot \dot{\mathbf{R}}(t)\right) + (m_1 + a_2 m_v) g \end{aligned}$$

$$\begin{aligned} P_r(t) &= K_{t2}\left(y_2(t) - \mathbf{H}_c^T(x_r(t)) \cdot \mathbf{R}(t) - r(x_r(t))\right) \\ &\quad + C_{t2}\left(\dot{y}_2(t) - \frac{\partial \mathbf{H}_c^T(x_r(t))}{\partial x} \cdot \mathbf{R}(t) \cdot v(t) - \mathbf{H}_c^T(x_r(t)) \cdot \dot{\mathbf{R}}(t)\right) + (m_2 + a_1 m_v) g \end{aligned} \tag{3.46}$$

The Eq. (3.46) can be rewritten in matrix form as:

$$\begin{aligned} \begin{Bmatrix} P_f(t) \\ P_r(t) \end{Bmatrix} &= \begin{bmatrix} K_{t1} & 0 \\ 0 & K_{t2} \end{bmatrix} \cdot \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} + \begin{bmatrix} C_{t1} & 0 \\ 0 & C_{t2} \end{bmatrix} \cdot \begin{Bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{Bmatrix} \\ &\quad - \begin{bmatrix} K_{t1} \cdot \mathbf{H}_c^T(x_f(t)) + C_{t1} \cdot v(t) \cdot \dfrac{\partial \mathbf{H}_c^T(x_f(t))}{\partial x} \\ K_{t2} \cdot \mathbf{H}_c^T(x_r(t)) + C_{t2} \cdot v(t) \cdot \dfrac{\partial \mathbf{H}_c^T(x_r(t))}{\partial x} \end{bmatrix} \cdot \{\mathbf{R}(t)\} \\ &\quad - \begin{bmatrix} C_{t1} \cdot \mathbf{H}_c^T(x_f(t)) \\ C_{t2} \cdot \mathbf{H}_c^T(x_r(t)) \end{bmatrix} \cdot \{\dot{\mathbf{R}}(t)\} - \begin{Bmatrix} K_{t1} \cdot r(x_f(t)) \\ K_{t2} \cdot r(x_r(t)) \end{Bmatrix} + \begin{Bmatrix} (m_1 + a_2 m_v) \cdot g \\ (m_2 + a_1 m_v) \cdot g \end{Bmatrix} \end{aligned} \tag{3.47}$$

Introducing Eq. (3.47) into the vehicle's equation of motion (3.11), the equilibrium for the vehicle degrees of freedom becomes

$$\left[\begin{array}{c|cc}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{M}_{v1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{v2}\end{array}\right]\left\{\begin{array}{c}\ddot{\mathbf{R}}\\ \hline \ddot{\mathbf{Y}}\end{array}\right\}+\left[\begin{array}{c|cc}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{C}_{v11} & \mathbf{C}_{v12}\\ -\mathbf{C}_t\cdot\mathbf{H}^T(x) & \mathbf{C}_{v21} & \mathbf{C}_{v22}+\mathbf{C}_t\end{array}\right]\left\{\begin{array}{c}\dot{\mathbf{R}}\\ \hline \dot{\mathbf{Y}}\end{array}\right\}$$

$$+\left[\begin{array}{c|cc}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{K}_{v11} & \mathbf{K}_{v12}\\ -\mathbf{K}_t\cdot\mathbf{H}^T-\mathbf{C}_t\cdot v\cdot\partial\mathbf{H}^T(x)/\partial x & \mathbf{K}_{v21} & \mathbf{K}_{v22}+\mathbf{K}_t\end{array}\right]\left\{\begin{array}{c}\mathbf{R}\\ \hline \mathbf{Y}\end{array}\right\}=\left\{\begin{array}{c}\mathbf{0}\\ \hline \mathbf{0}\\ -\mathbf{K}_t\cdot\mathbf{r}(x)\end{array}\right\} \tag{3.48}$$

where

$$\mathbf{M}_{v1}=\begin{bmatrix} m_v & 0\\ 0 & I_v\end{bmatrix};\mathbf{M}_{v2}=\begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix}$$

$$\mathbf{C}_{v11}=\begin{bmatrix} C_{s1}+C_{s2} & \left(-C_{s1}a_1+C_{s2}a_2\right)S\\ \left(-C_{s1}a_1+C_{s2}a_2\right)S & \left(C_{s1}a_1^2+C_{s2}a_2^2\right)S^2\end{bmatrix};$$

$$\mathbf{C}_{v12}=\begin{bmatrix} -C_{s1} & -C_{s2}\\ C_{s1}a_1S & -C_{s2}a_2S\end{bmatrix};$$

$$\mathbf{C}_{v21}=\begin{bmatrix} -C_{s1} & C_{s1}a_1S\\ -C_{s2} & -C_{s2}a_2S\end{bmatrix};\mathbf{C}_{v22}=\begin{bmatrix} C_{s1} & 0\\ 0 & C_{s2}\end{bmatrix};$$

$$\mathbf{K}_{v11}=\begin{bmatrix} K_{s1}+K_{s2} & \left(-K_{s1}a_1+K_{s2}a_2\right)S\\ \left(-K_{s1}a_1+K_{s2}a_2\right)S & \left(K_{s1}a_1^2+K_{s2}a_2^2\right)S^2\end{bmatrix};$$

$$\mathbf{K}_{v12}=\begin{bmatrix} -K_{s1} & -K_{s2}\\ K_{s1}a_1S & -K_{s2}a_2S\end{bmatrix};$$

$$\mathbf{K}_{v21}=\begin{bmatrix} -K_{s1} & K_{s1}a_1S\\ -K_{s2} & -K_{s2}a_2S\end{bmatrix};\mathbf{K}_{v22}=\begin{bmatrix} K_{s1} & 0\\ 0 & K_{s2}\end{bmatrix};$$

$$\mathbf{C}_t=\begin{bmatrix} C_{t1} & 0\\ 0 & C_{t2}\end{bmatrix};\mathbf{K}_t=\begin{bmatrix} K_{t1} & 0\\ 0 & K_{t2}\end{bmatrix};$$

$$\mathbf{r}=\begin{Bmatrix} r\left(x_f(t)\right)\\ r\left(x_r(t)\right)\end{Bmatrix};\mathbf{Y}=\{y_v \quad \theta_v \quad y_1 \quad y_2\}^T.$$

Similar to Eq. (3.48), introducing Eq. (3.47) into the bridge's equation of motion Eq. (3.40), the equilibrium for the bridge degrees of freedom becomes

$$\left[\begin{array}{c|cc}\mathbf{M_b} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]\left\{\frac{\ddot{\mathbf{R}}}{\ddot{\mathbf{Y}}}\right\}+\left[\begin{array}{c|cc}\mathbf{C_b}+\mathbf{H}(x)\cdot\mathbf{C}_t\cdot\mathbf{H}^T(x) & \mathbf{0} & -\mathbf{H}(x)\cdot\mathbf{C}_t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]\left\{\frac{\dot{\mathbf{R}}}{\dot{\mathbf{Y}}}\right\}$$
$$+\left[\begin{array}{c|cc}\mathbf{K_b}+\mathbf{H}(x)\cdot\mathbf{K}_t\cdot\mathbf{H}^T(x) & \mathbf{0} & -\mathbf{H}(x)\cdot\mathbf{K}_t-v\cdot\partial\mathbf{H}(x)/\partial x\cdot\mathbf{C}_t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]\left\{\frac{\mathbf{R}}{\mathbf{Y}}\right\}$$
$$=\left\{\begin{array}{c}-\mathbf{H}(x)\cdot\mathbf{K}_t\cdot\mathbf{r}(x)+\mathbf{H}(x)\cdot\mathbf{M}_s \\ \hline \mathbf{0} \\ \mathbf{0}\end{array}\right\} \tag{3.49}$$

where

$$\mathbf{M}_s=\left\{\begin{array}{c}(m_1+a_2m_v)g \\ (m_2+a_1m_v)g\end{array}\right\}.$$

From the combination of Eq. (3.48) and Eq. (3.49), the global equation of motion of vehicle-bridge interaction system is expressed as follow:

$$\left[\begin{array}{c|cc}\mathbf{M_b} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{M}_{v1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{v2}\end{array}\right]\left\{\frac{\ddot{\mathbf{R}}}{\ddot{\mathbf{Y}}}\right\}+\left[\begin{array}{c|cc}\mathbf{C_b}+\mathbf{H}(x)\cdot\mathbf{C}_t\cdot\mathbf{H}^T(x) & \mathbf{0} & -\mathbf{H}(x)\cdot\mathbf{C}_t \\ \hline \mathbf{0} & \mathbf{C}_{v11} & \mathbf{C}_{v12} \\ -\mathbf{C}_t\cdot\mathbf{H}^T(x) & \mathbf{C}_{v21} & \mathbf{C}_{v22}+\mathbf{C}_t\end{array}\right]\left\{\frac{\dot{\mathbf{R}}}{\dot{\mathbf{Y}}}\right\}$$
$$+\left[\begin{array}{c|cc}\mathbf{K_b}+\mathbf{H}(x)\cdot\mathbf{K}_t\cdot\mathbf{H}^T(x) & \mathbf{0} & -\mathbf{H}(x)\cdot\mathbf{K}_t-v\cdot\partial\mathbf{H}(x)/\partial x\cdot\mathbf{C}_t \\ \hline \mathbf{0} & \mathbf{K}_{v11} & \mathbf{K}_{v12} \\ \mathbf{K}_t\cdot\mathbf{H}^T-\mathbf{C}_t\cdot v\cdot\partial\mathbf{H}^T(x)/\partial x & \mathbf{K}_{v21} & \mathbf{K}_{v22}+\mathbf{K}_t\end{array}\right]\left\{\frac{\mathbf{R}}{\mathbf{Y}}\right\}$$
$$=\left\{\begin{array}{c}-\mathbf{H}(x)\cdot\mathbf{K}_t\cdot\mathbf{r}(x)+\mathbf{H}(x)\cdot\mathbf{M}_s \\ \hline \mathbf{0} \\ -\mathbf{K}_t\cdot\mathbf{r}(x)\end{array}\right\} \tag{3.50}$$

The Eq. (3.50) is the vehicle-bridge interaction equation, and the Eq. (3.47) is the front and rear axle load equations which are compose of static load of vehicle and dynamic interaction force between vehicle and bridge. The vehicle-bridge interaction equation can be solved step-by-step using either direct integration method such as Newmark's β method or discretization method by state-space formulation.

3.3 Relationship of Axle Loads and Bridge Bending Moments

The strain occurring at any measuring point at the bottom surface of the beam element in Figure 3.5 can be determined from:

$$\varepsilon_j(x_j,t) = -\gamma_j \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=x_j} \tag{3.51}$$

where γ_j is a distance between the bottom surface of the bridge and the neutral axis of the bridge section at the measuring location x_j.

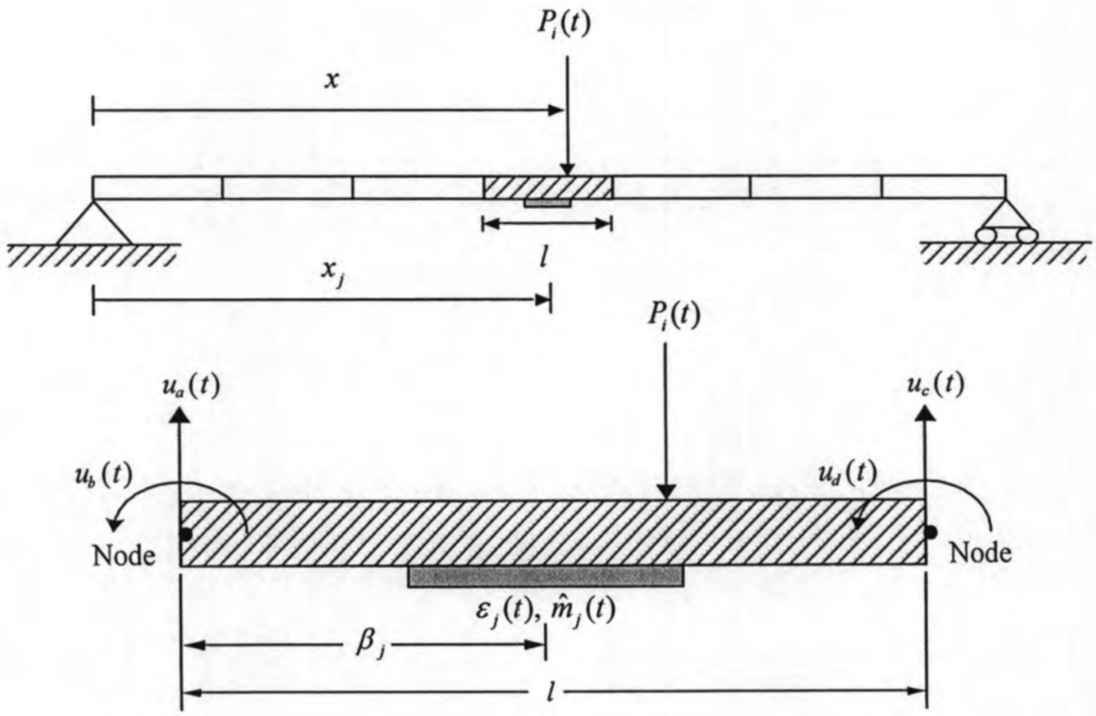


Figure 3.5 Measuring point of strain in beam element

Substituting $w(x,t)$ into Eq. (3.51) yields

$$\varepsilon_j(\beta_i,t) = -\left(\frac{\gamma_j}{l^3}\right)\cdot\left[(12\beta_j - 6l)u_a(t) + l(6\beta_j - 4l)u_b(t) - (12\beta_j - 6l)u_c(t) + l(6\beta_j - 2l)u_d(t)\right] \tag{3.52}$$

From the relationship between bending moment $\hat{m}_j(t)$ and strain $\varepsilon_j(t)$, the Eq. (3.52) can be converted into bending moment as:

$$\hat{m}_j(t) = EI \cdot \frac{\varepsilon_j(t)}{\gamma_j} \tag{3.53}$$

Therefore, Eq. (3.53) can be rewritten in the following form as:

$$\hat{m}_j(\beta_j,t) = -\left(\frac{EI}{l^3}\right)\left\{(12\beta_j - 6l) \quad l(6\beta_j - 4l) \quad -(12\beta_j - 6l) \quad l(6\beta_j - 2l)\right\}\begin{Bmatrix} u_a(t) \\ u_b(t) \\ u_c(t) \\ u_d(t) \end{Bmatrix}$$

(3.54)

where $u_a(t)$, $u_b(t)$, $u_c(t)$ and $u_d(t)$ are the nodal displacements of the corresponding beam element and β_j is the local location of the measuring point determined from the global location x_j.

3.4 Axle Loads Identification

In loads identification system, the interaction between external acting loads and bridge responses is considered. The problem is to identify the external loading in time histories without investigation of others vehicle's properties such as mass or suspension characteristics, etc. The input for the identification system is only measured bridge responses (bending moments) and location of moving axles on the bridge with respect to time. Therefore, only bridge's equation of motion is focused.

The concept of axle load identification of a passing vehicle is to minimize the error from output between measured and identified bridge responses. In this paper, the measured bending moment vector Z(t) at N selected measuring points of the bridge under a moving vehicle are assumed to be simulated by solving the vehicle-bridge dynamic interaction, Eq. (3.50) and introducing the obtained nodal bridge responses into Eq. (3.40). While the corresponding identified bending moment vector, $\hat{\mathbf{Z}}(t)$, at the same bridge sections are approximated by solving equations of motion of the bridge subjected to a pair of moving axle loads. In general, these axle loads are assumed to be either constant or time-varying magnitudes as will be discussed in the following sections. Then the moving axle loads of the vehicle are identified through the minimization of the square error of the bending moments of the bridge.

$$E = \sum_{i=1}^{NT}[(\mathbf{Z}_i - \hat{\mathbf{Z}}_i)^T \mathbf{B}(\mathbf{Z}_i - \hat{\mathbf{Z}}_i)] \tag{3.55}$$

where $\mathbf{Z}_i = \{m_1 \quad m_2 \quad \dots \quad m_N\}_i^T$ and $\hat{\mathbf{Z}}_i = \{\hat{m}_1 \quad \hat{m}_2 \quad \dots \quad \hat{m}_N\}_i^T$ are the discrete forms of the measured and the estimated bending moment vectors of the bridge at time step i, respectively.

B is the positive-definite weighting matrix and *NT* is the total number of data points (in time).

3.4.1 Method I: Constant Magnitude of Moving Axle Loads Assumption

With constant magnitude of moving axle loads assumption, the dynamic interaction loads between the vehicle and the bridge, i.e. $f_f(t)$ and $f_r(t)$, as previously defined in Eq.(3.43), are assumed to be sinusoidal and can be omitted for static weight estimation. Therefore the vehicle is simply replaced by a pair of constant static weights of moving axle loads as in Figure 3.6. To identify the axle loads of the vehicle, the estimated bending moments of the bridge are calculated from the static influence lines of moving loads and are compared with their corresponding measured bending moments. This method is simple but its main disadvantage is that it cannot provide any dynamic information of the axle loads of vehicles. The influence line of the bending moment of a simply-supported bridge at a section *j* as in Figure 3.6 can be expressed by a triangular function.

$$IL_j(x(t)) = \begin{Bmatrix} x(t) - \dfrac{x(t)\cdot x_j}{L} \;, & x(t) \le x_j \\ x_j - \dfrac{x(t)\cdot x_j}{L} \;, & x(t) > x_j \end{Bmatrix} \tag{3.56}$$

where $IL_j(x(t))$ is the static influence line of bridge bending moment at measuring section *j* defined according to the load location, $x(t)$.

$x(t)$ is the distance of the moving front axle load, $x_f(t)$, or rear axle load, $x_r(t)$, from the left support of the bridge,

and x_j is the distance of measuring section *j* from the left support of the bridge.

Consequently, the bending moment vector of the bridge at *N* measuring sections induced by the movement of the front and rear axle loads can be computed by the method of superposition as

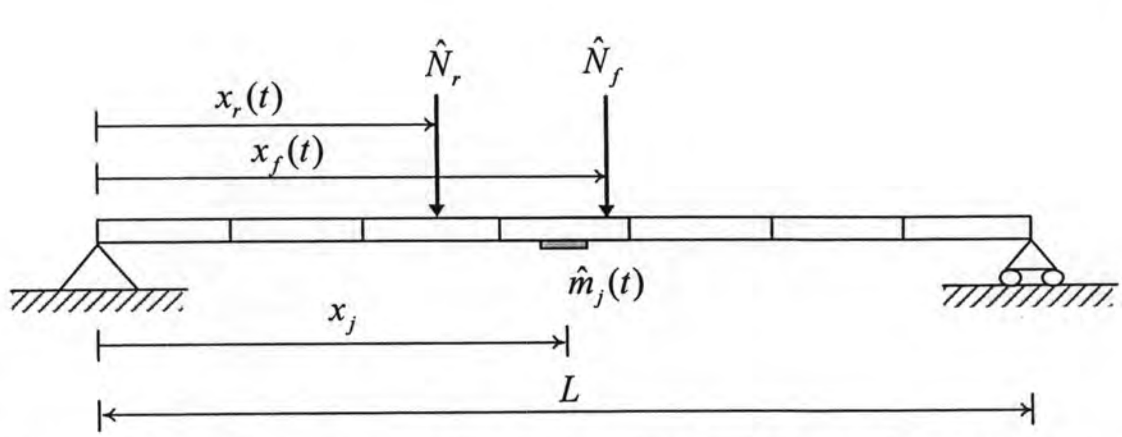


Figure 3.6 Simplified vehicle-bridge model used for weight estimation with constant magnitude of moving loads assumption.

$$\hat{\mathbf{Z}}(t) = \begin{Bmatrix} \hat{m}_1(t) \\ \hat{m}_2(t) \\ \vdots \\ \hat{m}_N(t) \end{Bmatrix} = \begin{Bmatrix} \hat{N}_f \cdot IL_1(x_f(t)) + \hat{N}_r \cdot IL_1(x_r(t)) \\ \hat{N}_f \cdot IL_2(x_f(t)) + \hat{N}_r \cdot IL_2(x_r(t)) \\ \vdots \\ \hat{N}_f \cdot IL_N(x_f(t)) + \hat{N}_r \cdot IL_N(x_r(t)) \end{Bmatrix} \tag{3.57}$$

Since the estimated bending moment of the bridge, $\hat{\mathbf{Z}}(t)$, depends on only a set of estimated moving axle weights $\hat{N} = \{\hat{N}_f \quad \hat{N}_r\}$. Recalling Eq. (3.55), the objective function in term of time discretization can be rewritten as

$$E(\hat{N}) = \sum_{i=1}^{NT} [(\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\hat{N}))^T \mathbf{B}(\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\hat{N}))] \tag{3.58}$$

To minimize above objective function, $E(\hat{N})$, a Sequential Quadratic Programming (SQP) method is used. This method iteratively solve a series of quadratic programming (QP) problems, using the Broyden, Fletcher, Goldfarb, and Shanno (BFGS) formula to update the Hessian of the Lagrangian, L, and $g_k(x)$ are the constraints including the upper and lower bounds of the axle weights, and γ_k are the Lagrange multipliers.

$$L(\hat{N}) = E(\hat{N}) + \sum_{k=1}^{2} \gamma_k g_k(\hat{N}) \tag{3.59}$$

The MATLAB's optimization function *fmincon* (Mathworks, 2007), which utilizes previously mentioned procedures, is adopted to find the constrained minimum of the nonlinear objective function, Eq. (3.58). Minimization of this objective function results in the optimal axle weights, $\hat{N}=\left[\hat{N}_f \quad \hat{N}_r\right]^T$, which yields minimum error between the measured and the estimated bending moment vectors. These obtained optimal axle weights are constants and are assumed to be the best estimated axle weights for the passing vehicle.

3.4.2 Method II: Time-Varying Magnitude of Moving Axle Loads Assumption

Unlike the constant magnitudes of moving axle loads assumption, the axle loads of the vehicle are assumed to be time-varying. With this assumption, the dynamic interaction loads between the vehicle and the bridge, i.e. $f_f(t)$ and, $f_r(t)$ as well as the static axle weights of the vehicle, i.e. $\hat{N}_f$ and, $\hat{N}_r$ are taken into account. Therefore the vehicle is replaced by a pair of time-varying magnitudes of moving axle loads similar to the system shown in Figure 3.5. To identify the axle loads of the vehicle, the identified bending moments of the bridge are calculated solely from the bridge's equation of motion in Eq. (3.40). The force term of the equation is then replaced by the unknown acting load vector expressed as:

$$\mathbf{M}_\mathbf{b}\ddot{\mathbf{R}}(t)+\mathbf{C}_\mathbf{b}\dot{\mathbf{R}}(t)+\mathbf{K}_\mathbf{b}\mathbf{R}(t)=\mathbf{P}_\mathbf{b}(t) \tag{3.60}$$

where $\mathbf{P}_\mathbf{b}(t)$ is the unknown external nodal load vector which can be rewritten as

$$\mathbf{P}_\mathbf{b}(t)=\mathbf{H}_c(x(t))\hat{\mathbf{P}}(t) \tag{3.61}$$

in which $\hat{\mathbf{P}}(t)$ is a $N_p\times 1$ vector of the unknown applied loads.

Using the state-space formulation, Eq. (3.60) is converted into a set of first order differential equation as follows:

$$\dot{\mathbf{X}}(t)=\mathbf{K}^*\mathbf{X}(t)+\overline{\mathbf{P}}(t) \tag{3.62}$$

where $\mathbf{X}(t)=\begin{bmatrix}\mathbf{R}(t)\\ \dot{\mathbf{R}}(t)\end{bmatrix}_{2NN\times 1}$, $\mathbf{K}^*=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}_\mathbf{b}^{-1}\mathbf{K}_\mathbf{b} & -\mathbf{M}_\mathbf{b}^{-1}\mathbf{C}_\mathbf{b}\end{bmatrix}_{2NN\times 2NN}$,

$$\begin{aligned}\overline{\mathbf{P}}(t)&=\begin{bmatrix}\mathbf{0}\\ -\mathbf{M}_\mathbf{b}^{-1}\mathbf{P}_\mathbf{b}(t)\end{bmatrix}_{2NN\times 1}\\ &=\begin{bmatrix}\mathbf{0}\\ -\mathbf{M}_\mathbf{b}^{-1}\mathbf{H}_c\end{bmatrix}_{2NN\times N_p}\cdot\hat{\mathbf{P}}(t)_{N_p\times 1}\end{aligned} \tag{3.63}$$

where $\mathbf{X}$ represents a vector of state variables of length twice of total degrees of freedom of the bridge (2*NN*) containing the displacements and velocities of the nodes, and N_p is the number of acting loads. These differential equations are then rewritten as discrete equations using the standard exponential matrix representation.

$$\mathbf{X}_{j+1}(t) = \mathbf{F}\mathbf{X}_j(t) + \overline{\mathbf{G}}_{j+1}(t)\overline{\mathbf{P}}_j(t), \tag{3.64}$$

$$\mathbf{F} = \mathbf{e}^{\mathbf{K}^* h}, \tag{3.65}$$

$$\overline{\mathbf{G}} = \mathbf{K}^{*^{-1}}(\mathbf{F} - \mathbf{I}), \tag{3.66}$$

where matrix $\mathbf{F}$ is the exponential matrix, and together with matrix $\overline{\mathbf{G}}$ represents the dynamics of the system, *j*+1 denoted the value at the *j*+1th time step of computation, the time step *h* represents the time difference between the variable states $\mathbf{X}_j$ and $\mathbf{X}_{j+1}$ in the computation, and $\overline{\mathbf{G}}$ is a matrix relating the forces to the system.

Substituting Eq. (3.63) and (3.66) into Eq. (3.64), one has

$$\mathbf{X}_{j+1}(t) = \mathbf{F}\mathbf{X}_j(t) + \mathbf{G}_{j+1}(t)\hat{\mathbf{P}}_j(t), \tag{3.67}$$

where
$$\mathbf{G} = \overline{\mathbf{G}}_{2NN\times 2NN}\begin{bmatrix} \mathbf{0} \\ -\mathbf{M_b}^{-1}\mathbf{H}_c \end{bmatrix}_{2NN\times N_p} \tag{3.68}$$

The matrix F can be obtained using exponential expansion as follows:

$$\mathbf{F} = \mathbf{e}^{\mathbf{K}^* h} = \mathbf{I} + h\mathbf{K}^* + \frac{h^2}{2!}\mathbf{K}^{*^2} + \frac{h^3}{3!}\mathbf{K}^{*^3} + \cdots \tag{3.69}$$

where
$$\mathbf{K}^* = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}. \tag{3.70}$$

Substituting Eq. (3.70) into Eq. (3.69) becomes

$$\begin{aligned}
\mathbf{F} = \mathbf{e}^{\mathbf{K}^* h} &= \mathbf{V}\mathbf{V}^{-1} + h\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{h^2}{2!}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{h^3}{3!}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \ldots \\
&= \mathbf{V}\mathbf{V}^{-1} + h\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \frac{h^2}{2!}\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1} + \frac{h^3}{3!}\mathbf{V}\mathbf{\Lambda}^3\mathbf{V}^{-1} + \ldots \\
&= \mathbf{V}\left(\mathbf{I} + h\mathbf{\Lambda} + \frac{h^2}{2!}\mathbf{\Lambda}^2 + \frac{h^3}{3!}\mathbf{\Lambda}^3 + \ldots\right)\mathbf{V}^{-1} \\
&= \mathbf{V}\mathbf{e}^{\mathbf{\Lambda}h}\mathbf{V}^{-1}
\end{aligned} \tag{3.71}$$

From the previous research, they found this simple least-square method is not effective. This is because of the obtained solution usually exhibits large fluctuation due to the measurement noises. In order to avoid this phenomenon, the smoothening term is added to the least-squares error. Then the moving axle loads can be identified through the minimization of the square error of the bending moments of the bridge, E, with regularization. This conventional regularization can be expressed in time-discretization form (Law and Fang, 2001 and Zhu and Law, 2002) as:

$$E = \sum_{i=1}^{N} [(\mathbf{Z}_i - \hat{\mathbf{Z}}_i)^T \mathbf{B}(\mathbf{Z}_i - \hat{\mathbf{Z}}_i) + \hat{P}_i^T \mathbf{D} \hat{P}_i] \tag{3.72}$$

where the additional term $\hat{P}_i^T \mathbf{D} \hat{P}_i$ is a smoothening term known as the regularization and the method is called the Tikhonov regularization method. Matrix **D** is a diagonal matrix with positive definite represented as:

$$\mathbf{D} = \lambda \mathbf{I} \tag{3.73}$$

where λ is the regularization parameter.

Substituting Eq. (3.73) into Eq. (3.72), one gets

$$E(\hat{P}) = \sum_{i=1}^{NT} (\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\hat{P}))^T \mathbf{B}(\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\hat{P})) + \lambda \hat{P}_i^T \hat{P}_i \tag{3.74}$$

where $\hat{P} = \begin{bmatrix} \hat{P}_f & \hat{P}_r \end{bmatrix}^T$ is the unknown moving loads vector with time-varying magnitude at time step i . A small value of λ causes the solution to match the data closely but produces large oscillatory deviations while a large one produces smooth forces that may not match the data well. When λ is zero, the solution becomes that for the least square problem.

In this research, the dynamic programming method with updated static component technique as proposed by Pinkaew (2006) is employed to minimize above error equation. This is because the method yields better solution and robustness against the choice of λ than the conventional regularization method.

In this method, we defined the identified bending moment vector that:

$$\hat{\mathbf{Z}}_i = \mathbf{Q}\mathbf{X}_i \tag{3.75}$$

where

Q	=	transformation matrix between bending moment and response variables.
$\mathbf{X}_i$	=	response variables compose of nodal displacements and nodal rotations.

Eq. (3.74) becomes:

$$E = \sum_{i=1}^{N} \left((\mathbf{Z}_i - \mathbf{Q}\mathbf{X}_i), \mathbf{D}(\mathbf{Z}_i - \mathbf{Q}\mathbf{X}_i) + (\mathbf{P}_i, \mathbf{E}\mathbf{P}_i) \right) \tag{3.76}$$

where matrix **D** is the identity matrix, and matrix **E** is the diagonal matrix with regularization value (λ).

For minimize the function E, the dynamic programming is adopted to find minimum E at step n. It can be written as:

$$\mathbf{g}_n(\mathbf{X}) = \min_{\mathbf{f}_j} E_n(\mathbf{X}, \mathbf{P}_j) \tag{3.77}$$

From the Bellman's principle of optimality, it can be proven that:

$$\mathbf{g}_{n-1}(\mathbf{X}) = \min_{\mathbf{f}_{n-1}}\left(\left(\mathbf{Z}_{n-1} - \mathbf{QX}\right), \mathbf{D}\left(\mathbf{Z}_{n-1} - \mathbf{QX}\right) + \left(\mathbf{P}_{n-1}, \mathbf{EP}_{n-1}\right) + \mathbf{g}_n\left(\mathbf{FX} + \mathbf{G}_n\mathbf{P}_{n-1}\right)\right) \tag{3.78}$$

where $\mathbf{P}_n$ and $\mathbf{g}_n$ are optimal loading term and optimal cost term respectively. This method is solved by start at step $n = N$ and backward calculated into step $n = 1$. At the final step, the minimum value can be found that:

$$\mathbf{g}_N(\mathbf{X}) = \min_{\mathbf{f}_N}\left[\left(\mathbf{Z}_N - \mathbf{QX}\right), \mathbf{D}\left(\mathbf{Z}_N - \mathbf{QX}\right) + \left(\mathbf{P}_N, \mathbf{EP}_N\right)\right] \tag{3.79}$$

when $\mathbf{P}_N = 0$, Eq. (3.79) becomes:

$$\mathbf{g}_N(\mathbf{X}) = \mathbf{q}_N + (\mathbf{X}, \mathbf{S}_N) + (\mathbf{X}, \mathbf{R}_N\mathbf{X}) \tag{3.80}$$

where

$$\begin{aligned} \mathbf{q}_N &= \mathbf{Z}_N^T\mathbf{D}\mathbf{Z}_N \\ \mathbf{S}_N &= -2\mathbf{Q}_N^T\mathbf{D}\mathbf{Z}_N \\ \mathbf{R}_N &= \mathbf{Q}_N^T\mathbf{D}\mathbf{Q}_N \end{aligned} \tag{3.81}$$

Eq. (3.81) is the initial value for backward calculation at step $n = N$, by substituting Eq. (3.80) at step n and n-1 into Eq. (3.78) and expand the right term of equation:

$$\mathbf{q}_{n-1} + (\mathbf{X}, \mathbf{S}_{n-1}) + (\mathbf{X}, \mathbf{R}_{n-1}\mathbf{X}) = \min_{\mathbf{f}_{n-1}}\left[(\mathbf{P}_{n-1} + \mathbf{V}_n\mathbf{X} + \mathbf{U}_n), \mathbf{H}_n(\mathbf{P}_{n-1} + \mathbf{V}_n\mathbf{X} + \mathbf{U}_n) + \mathbf{r}_{n-1}(\mathbf{X})\right]$$

where

$$\begin{aligned} &\mathbf{H}_n = \mathbf{E} + \mathbf{G}_n^T\mathbf{R}_n\mathbf{G}_n,\ 2\mathbf{H}_n\mathbf{V}_n = 2\mathbf{G}_n^T\mathbf{R}_n\mathbf{F},\ \mathbf{V}_n = \mathbf{H}_n^{-1}\,\mathbf{G}_n^T\mathbf{R}_n\mathbf{F} \\ &2\mathbf{H}_n\mathbf{U}_n = \mathbf{G}_n^T\mathbf{S}_n,\ \mathbf{U}_n = (\mathbf{H}_n^{-1}\mathbf{G}_n^T\mathbf{S}_n)/2 \\ &\mathbf{r}_{n-1}(\mathbf{x}) = (\mathbf{q}_n + \mathbf{Z}_{n-1}^T\mathbf{D}\mathbf{Z}_{n-1}) + \mathbf{X}^T(\mathbf{Q}^T\mathbf{DQ} + \mathbf{F}^T\mathbf{R}_n\mathbf{F})\mathbf{X} + \mathbf{X}^T(\mathbf{F}^T\mathbf{S}_n - 2\mathbf{Q}^T\mathbf{D}\mathbf{Z}_{n-1}) \\ &\quad - \mathbf{X}^T\mathbf{V}_n^T\mathbf{H}_n\mathbf{V}_n\mathbf{X} - \mathbf{U}_n^T\mathbf{H}_n\mathbf{U}_n - 2\mathbf{X}^T\mathbf{V}_n^T\mathbf{H}_n\mathbf{U}_n \end{aligned} \tag{3.82}$$

and **F** is the exponential matrix.

Minimize the right term of Eq. (3.82), lead to the optimal load that:

$$\mathbf{P}_{n\text{-}1} = -\mathbf{H}_n^{-1}\mathbf{G}_n^T\left[\mathbf{R}_n\mathbf{F}\mathbf{X}_{n\text{-}1} + \frac{\mathbf{S}_n}{2}\right] \tag{3.83}$$

Eq. (3.82) becomes:

$$\mathbf{q}_{n\text{-}1} + (\mathbf{X},\mathbf{S}_{n\text{-}1}) + (\mathbf{X},\mathbf{R}_{n\text{-}1}\mathbf{X}) = \mathbf{r}_{n\text{-}1}(\mathbf{X}) \tag{3.84}$$

From spreading term in Eq. (3.84), and by coefficient comparison method, one obtains:

$$\mathbf{R}_{n\text{-}1} = \mathbf{Q}^T\mathbf{D}\mathbf{Q} + \mathbf{F}^T\left[\mathbf{I} - \mathbf{R}_n^T\mathbf{G}_n\mathbf{H}_n^{-1}\mathbf{G}_n^T\right]\mathbf{R}_n\mathbf{F}$$

$$\mathbf{S}_{n\text{-}1} = -2\mathbf{Q}^T\mathbf{D}\mathbf{Z}_{n\text{-}1} + \mathbf{F}^T\left[\mathbf{I} - \mathbf{R}_n^T\mathbf{G}_n\mathbf{H}_n^{-1}\mathbf{G}_n^T\right]\mathbf{S}_n \tag{3.85}$$

It is found that the regularization method outlined in the previous section requires an optimal regularization parameter, λ, to identify the axle loads accurately. This optimal parameter is rather difficult to pre-assign in real applications, because it depends on the configuration, speed and weight of the identified vehicle. The L-curve method (Hansen, 1992) or generalized cross-validation (Golub et al., 1979) might be employed to determine this optimal parameter, but large computing time is required. Moreover, the obtained parameter is a sub-optimal value which does not guarantee accurate identification results. To overcome such a difficulty and to enhance identification accuracy, the regularization with updated static component (USC) technique (Pinkaew, 2006) is adopted. Since the bridge responses are composed of two components, which are the static (quasi-static) and the dynamic components, they theoretically require different values of optimal regularization parameters. Therefore, the USC technique decomposes the bridge responses into static and dynamic components. The static component is identified separately, while only the dynamic component remains in the regularization process. With iteration, the regularization method using dynamic programming is employed to identify the dynamic component. Then, the obtained identified result is used to update the associated static component until the convergent solution is achieved. The computational diagram of the proposed regularization with updated static component technique is shown in Figure 3.7. It is noted that the static component of the time-varying quantity is simply calculated using time averaging.

where $\hat{\mathbf{P}}_s$ is static load component obtained by time-averaging of identified load,

$\hat{\mathbf{Z}}_s$ is static bending moment constructed from static load component,

$\hat{\mathbf{Z}}_d$ is dynamic bending moment obtained from subtracting static bending moment from the measured bending moment,

$\hat{\mathbf{P}}_d$ is dynamic load component obtained from identification of remaining dynamic bending moment,

δ is iterative error between updated and previous identified load.

To investigate the accuracy of the identified results, the identification error is defined as:

$$\begin{aligned} \textit{Identification Error, Front Axle} &= \frac{\left\| \hat{\mathbf{P}}_f(t) - \mathbf{P}_f(t) \right\|}{\left\| \mathbf{P}_f(t) \right\|} \times 100\% \\ \textit{Identification Error, Rear Axle} &= \frac{\left\| \hat{\mathbf{P}}_r(t) - \mathbf{P}_r(t) \right\|}{\left\| \mathbf{P}_r(t) \right\|} \times 100\% \end{aligned} \tag{3.86}$$

where $\|\cdot\|$ is the Euclidean norm of a vector. $\mathbf{P}(t)$ and $\hat{\mathbf{P}}(t)$ are the actual and the identified axle loads of the vehicle for either front or rear axle, respectively.

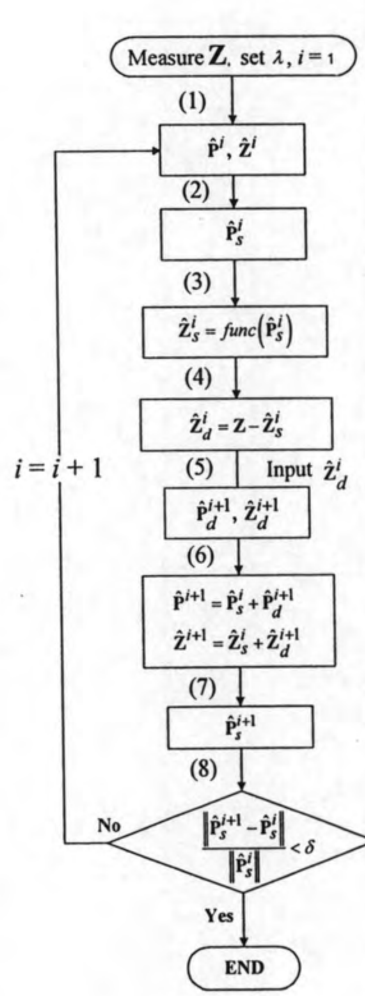
It is noted that the obtained results from Eq. (3.76) are dynamic axle loads. Therefore, to estimate the corresponding axle weights of the vehicle, the simple time-averaging is adopted.

$$\begin{aligned} \hat{N}_f &= \frac{1}{NT}\sum_{i=1}^{NT}[\hat{P}_f]_i \quad \text{and} \quad \hat{N}_r = \frac{1}{NT}\sum_{i=1}^{NT}[\hat{P}_r]_i \\ \hat{N}_g &= \hat{N}_f + \hat{N}_r \end{aligned} \tag{3.87}$$

in which $\hat{N}_f$, $\hat{N}_r$ and $\hat{N}_g$ are the estimated front axle, rear axle and gross weights, respectively.

However, for the full-scale tests, the dynamic axle loads of the vehicles are not measured. Then, the obtained accuracies of the identified results from the two methods are indirectly evaluated through the identification errors of the axle weights as

$$\begin{aligned}
\textit{Estimation Error, Front Axle Weight} &= \frac{\hat{N}_f - N_f}{N_f} \times 100\% \\
\textit{Estimation Error, Rear Axle Weight} &= \frac{\hat{N}_r - N_r}{N_r} \times 100\% \\
\textit{Estimation Error, Gross Weight} &= \frac{\hat{N}_g - N_g}{N_g} \times 100\%
\end{aligned} \tag{3.88}$$



Steps of computation for USC technique

(1) Load identification using optimization of Eq. (3.76).

(2) Load decomposition to obtain the static component (quasi-static) by time-averaging.

(3) Calculate the corresponding static bending moments.

(4) Compute the remaining dynamic bending moments.

(5) Repeat the load identification using optimization as in step (1) by input only dynamic bending moments from step (4).

(6) Update the identified result from the combination of the previous and the latest identified result.

(7) Load decomposition again to obtain the updated static component (quasi-static) by time-averaging.

(8) Perform the convergent check of the updated result.

Figure 3.7 Computational diagram of load identification through regularization with update static component (USC) technique