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GENERALIZED TRANSFORMATION SEMIGROUPS AND LINEAR TRANSFORMATION SEMIGROUPS WHOSE BI-IDEALS ARE QUASI-IDEALS

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ชัยวัฒน์ นามนาค :เซมิกรุปการแปลงนัยทั่วไปและเซมิกรุปการแปลงเชิงเส้นซึ่งไบ-ไอเดียลเป็นควอซี-ไอเดียล (GENERALIZED TRANSFORMATION SEMIGROUPS AND LINEAR TRANSFORMATION SEMIGROUPS WHOSE BI-IDEALS ARE QUASI-IDEALS)
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เราเรียกเซมิกรุปย่อย Q ของเซมิกรุป S ว่า *ควอซี-ไอเดียล* ของ S ถ้า $SQ \cap QS \subseteq Q$ และ *ไบ-ไอเดียล*ของ S หมายถึง เซมิกรุปย่อย B ของ S ซึ่ง $BSB \subseteq B$ ควอซี-ไอเดียลเป็นนัยทั่วไปของไอเดียลซ้ายและไอเดียลขวา และไบ-ไอเดียลให้นัยทั่วไปของควอซี-ไอเดียล แนวคิดของไบ-ไอเดียลและแนวคิดของควอซี-ไอเดียลสำหรับเซมิกรุปแนะนำโดย อาร์ เอ กูด และ ดี อาร์ ฮิวส์ ในปี 1952 และโดย โอ สตีนเฟลด์ ในปี 1956 ตามลำดับ หลังจากนั้นได้มีการศึกษาเรื่องควอซี-ไอเดียลและไบ-ไอเดียลของเซมิกรุปกันอย่างกว้างขวาง สิ่งที่เราสนใจในการวิจัยนี้คือเซมิกรุปซึ่งมีไบ-ไอเดียลและควอซี-ไอเดียลเป็นสิ่งเดียวกัน และ เราเรียกเซมิกรุปเช่นนี้ว่า *BQ-เซมิกรุป*

สำหรับเซต X และ Y ให้ $P(X, Y)$ เป็นเซตของการส่ง $\alpha : A \to Y$ ทั้งหมด โดยที่ $A \subseteq X$ สำหรับ $\theta \in P(Y, X)$ ให้ $(P(X, Y), \theta)$ แทนเซมิกรุป $(P(X, Y), *)$ โดย $\alpha * \beta = \alpha\theta\beta$ สำหรับ $\alpha, \beta \in P(X, Y)$ ทั้งหมด จุดประสงค์แรกของการวิจัยนี้คือให้ลักษณะว่าเซมิกรุปย่อยบางชนิดของ $(P(X, Y), \theta)$ สำหรับ θ ที่เจาะจงจะเป็น BQ-เซมิกรุปเมื่อใดในเทอมของขนาดของ X และ Y

สำหรับปริภูมิเวกเตอร์ V บนริงการหาร ให้ $L(V)$ เป็นเซมิกรุปภายใต้การประกอบของการแปลงเชิงเส้น $\alpha : V \to V$ ทั้งหมด เราศึกษาเซมิกรุปย่อยหลากหลายของ $L(V)$ ที่นิยามโดยเคอร์เนลและภาพของการแปลงเชิงเส้น เพื่อจุดประสงค์ที่สอง เราให้ลักษณะว่าเซมิกรุปการแปลงเชิงเส้นเหล่านี้จะเป็น BQ-เซมิกรุปเมื่อใดในเทอมของมิติของ V

สุดท้ายเราศึกษาเซมิกรุปการแปลงเต็มที่รักษาอันดับ $T_{OP}(I)$ บนช่วง I ของจำนวนจริง เราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับ I ที่ทำให้ $T_{OP}(I)$ เป็น BQ-เซมิกรุป

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต..
สาขา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา..
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CHAIWAT NAMNAK : GENERALIZED TRANSFORMATION SEMIGROUPS AND LINEAR TRANSFORMATION SEMIGROUPS WHOSE BI-IDEALS ARE QUASI-IDEALS. THESIS ADVISOR : ASSOCIATE PROFESSOR YUPAPORN KEMPRASIT, Ph.D., 63 pp.
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A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. By a *bi-ideal* of S we mean a subsemigroup B of S such that $BSB \subseteq B$. Quasi-ideals are a generalization of left ideals and right ideals and bi-ideals generalize quasi-ideals. The notion of bi-ideal and the notion of quasi-ideal for semigroups were introduced respectively by R. A. Good and D. R. Huges in 1952 and O. Steinfeld in 1956. Since then, both quasi-ideals and bi-ideals of semigroups have been widely studied. Semigroups whose bi-ideals and quasi-ideals coincide are of our interest in this research. One calls such semigroups *BQ-semigroups*.

For sets X and Y, let $P(X, Y)$ be the set of all mappings $\alpha : A \to Y$ where $A \subseteq X$. For $\theta \in P(Y, X)$, let $(P(X, Y), \theta)$ denote the semigroup $(P(X, Y), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in P(X, Y)$. The first purpose of this research is to characterize when certain subsemigroups of $(P(X, Y), \theta)$ with a particular θ are BQ-semigroups in terms of the cardinalities of X and Y.

For a vector space V over a division ring, let $L(V)$ be the semigroup under composition of all linear transformations $\alpha : V \to V$. Various subsemigroups of $L(V)$ defined by kernels and images of linear transformations are studied for our second purpose. We characterize when these linear transformation semigroups are BQ-semigroups in terms of the dimensions of V.

Finally, we study the full order-preserving transformation semigroup $T_{OP}(I)$ on an interval I of real numbers. Necessary and sufficient conditions for I so that $T_{OP}(I)$ is a BQ-semigroup are given.

Department **Mathematics** Student's signature...

Field of study **Mathematics** Advisor's signature...

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

In this introductory chapter, we present a number of elementary concepts, notations and propositions on semigroups most of which will be indispensable for the remainder of this research.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of natural numbers (positive integers), the set of integers and the set of real numbers. For any set X, let $|X|$ denote the cardinality of X.

An element e of a semigroup S is called an *idempotent* if $e^2 = e$. For a semigroup S, let $E(S)$ be the set of all idempotents of S. A semigroup S with zero 0 is called a *zero semigroup* if $xy = 0$ for all $x, y \in S$. An element x of a semigroup S is *regular* if $x = xyx$ for some $y \in S$, and S is called a *regular semigroup* if every element of S is regular.

A nonempty subset A of a semigroup S is called a *left* [*right*] *ideal* of S if $SA \subseteq A$ [$AS \subseteq A$], and A is called an *ideal* of S if A is both a left and a right ideal of S. We call a semigroup S a *left* [*right*] *simple semigroup* if S is the only left [right] ideal of S. Likewise a semigroup S is called a *simple semigroup* if S is the only ideal of S. The following known result will be used later.

Proposition 1.1. *A semigroup S is left [right] simple if and only if $Sx = S$ [$xS = S$] for all $x \in S$.*

A semigroup S with zero 0 is called a *left* [*right*] *0-simple semigroup* if (i) $S^2 \neq \{0\}$ and (ii) $\{0\}$ and S are the only left [right] ideals of S. A *0-simple semigroup* is a semigroup S with zero 0 such that (i) $S^2 \neq \{0\}$ and (ii) $\{0\}$ and S are the only ideals of S.

For a semigroup S, let S^1 be S if S has an identity, otherwise, let S^1 be the semigroup $S \cup \{1\}$ where $1 \notin S$ with the operation extended from the operation on S by defining $1x = x1 = x$ for all $x \in S \cup \{1\}$.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$, and by a *bi-ideal* of S we mean a subsemigroup B of S such that $BSB \subseteq B$. Clearly, every left ideal and every right ideal of S is a quasi-ideal of S and every quasi-ideal of S is a bi-ideal of S. The notion of quasi-ideal for semigroups was introduced by O. Steinfeld [16] in 1956. In fact, the notion of bi-ideal for semigroups was introduced earlier by R. A. Good and D. R. Huges [3] in 1952.

Example 1.2. (1) Let R be a division ring, $n \in \mathbb{N}$ and $M_n(R)$ the semigroup of all $n \times n$ matrices over R under the usual multiplication of matrices. For each $C \in M_n(R)$, let C_{ij} denote the entry of C in the i^{th} row and the j^{th} column. For $k, l \in \{1, 2, \ldots, n\}$, let $Q_n^{kl}(R)$ be the subset of $M_n(R)$ consisting of all matrices $C \in M_n(R)$ such that

$$C_{ij} = 0 \quad \text{if} \quad i \neq k \quad \text{or} \quad j \neq l.$$

Then for $k, l \in \{1, 2, \ldots, n\}$,

$$M_n(R)Q_n^{kl}(R) = \left\{ \begin{array}{c} \begin{array}{ccccccc} & & & l^{th} & & & \\ & & & \downarrow & & & \end{array} \\ \begin{bmatrix} 0 & \ldots & 0 & x_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & x_2 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & x_n & 0 & \ldots & 0 \end{bmatrix} \end{array} \;\middle|\; x_1, x_2, \ldots, x_n \in R \right\}$$

and

$$Q_n^{kl}(R)M_n(R) = \left\{ \begin{array}{c} \\ \\ \\ k^{th} \rightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ x_1 & x_2 & \dots & x_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \;\middle|\; x_1, x_2, \dots, x_n \in R \right\}$$

which implies that $M_n(R)Q_n^{kl}(R) \cap Q_n^{kl}(R)M_n(R) = Q_n^{kl}(R)$, so $Q_n^{kl}(R)$ is a quasi-ideal of $M_n(R)$. Moreover, if $n > 1$, then for all $k, l \in \{1, 2, \dots, n\}$, $Q_n^{kl}(R)$ is neither a left ideal nor a right ideal of $M_n(R)$.

(2) Let R be a division ring, $n \in \mathbb{N}, n \geq 4$ and $SU_n(R)$ the semigroup of all strictly upper triangular matrices over R under the usual multiplication of matrices. Let

$$B = \left\{ \begin{bmatrix} 0 & \dots & 0 & x & 0 \\ 0 & \dots & 0 & 0 & y \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \;\middle|\; x,\ y \in R \right\}.$$

Then $B^2 = \{0\}$, so B is a subsemigroup of $SU_n(R)$. Moreover, $BSU_n(R)B = \{0\} \subseteq B$. But

$$\begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\in (SU_n(R)B \cap BSU_n(R)) \smallsetminus B,$$

so B is a bi-ideal but not a quasi-ideal of $SU_n(R)$.

Example 1.2(1) shows that quasi-ideals of semigroups are a generalization of one-sided ideals. It is shown in Example 1.2(2) that bi-ideals of semigroups generalize quasi-ideals.

We know that the intersection of a set of subsemigroups of a semigroup S is a subsemigroup of S if it is nonempty. It is known that the intersection of a set of quasi-ideals of a semigroup S is either $\varnothing$ or a quasi-ideal of S and this is also true for bi-ideals of S ([15], page 10 and 12). For a nonempty subset X of a semigroup S, let $(X)_q$ and $(X)_b$ denote the intersection of all quasi-ideals of S containing X and the intersection of all bi-ideals of S containing X, respectively. Then $(X)_q$ $[(X)_b]$ is the smallest quasi-ideal [bi-ideal] of S containing X and $(X)_q$ $[(X)_b]$ is called the *quasi-ideal* [*bi-ideal*] *of* S *generated by* X. For $x_1, x_2, \dots, x_n \in S$, let $(x_1, x_2, \dots, x_n)_q$ and $(x_1, x_2, \dots, x_n)_b$ denote respectively $(\{x_1, x_2, \dots, x_n\})_q$ and $(\{x_1, x_2, \dots, x_n\})_b$. Since every quasi-ideal of S is a bi-ideal of S, we have

Proposition 1.3. *For every nonempty subset X of a semigroup S, $(X)_b \subseteq (X)_q$.*

The following facts are well-known.

Proposition 1.4. ([2], page 84–85). *For any nonempty subset X of a semigroup S,*

$$(X)_q = S^1X \cap XS^1 = (SX \cap XS) \cup X$$

and

$$(X)_b = XS^1X \cup X = XSX \cup X \cup X^2.$$

Let $\boldsymbol{BQ}$ denote the class of all semigroups whose bi-ideals and quasi-ideals coincide. Then a semigroup S is in $\boldsymbol{BQ}$ if and only if every bi-ideal of S is a quasi-ideal. One call a semigroup in $\boldsymbol{BQ}$ a *BQ-semigroup*. The following two propositions give some significant subclasses of $\boldsymbol{BQ}$.

Proposition 1.5. ([11]). *Every regular semigroup is in $\boldsymbol{BQ}$.*

Proposition 1.6. ([7]). *Every left [right] simple semigroup and every left [right] 0-simple semigroup belongs to $\boldsymbol{BQ}$.*

Not only these kinds of semigroups belong to $\boldsymbol{BQ}$. Zero semigroups containing more than one element are obvious examples. Some other significant examples can be seen in this research. However, J. Calais [1] has characterized the semigroups in $\boldsymbol{BQ}$ as follows:

Proposition 1.7. ([1]). *A semigroup S is in $\boldsymbol{BQ}$ if and only if $(x, y)_q = (x, y)_b$ for all $x, y \in S$.*

If S is a BQ-semigroup, then for a nonempty subset X of S, $(X)_b$ is a quasi-ideal of S containing X which implies that $(X)_q \subseteq (X)_b$. We thus deduce from Proposition 1.3 that

Proposition 1.8. *If $S \in \boldsymbol{BQ}$, then $(X)_b = (X)_q$ for every nonempty subset X of S. Hence if S is a semigroup such that $(x)_b \neq (x)_q$ $\big((x)_b \subsetneq (x)_q\big)$ for some $x \in S$, then $S \notin \boldsymbol{BQ}$.*

Next, let X be a set. A *partial transformation* of X is a map from a subset of X into X. By a *transformation* of X is a map from X into X. The *empty transformation* is the partial transformation 0 with empty domain. Let P_X be the set of all partial transformations of X. For $\alpha \in P_X$, let $\operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha$ denote respectively the domain and the image (range) of α. Then P_X is a semigroup under the composition of maps, that is, for $\alpha, \beta \in P_X$

$$\operatorname{Dom}\alpha\beta = \{\, x \in \operatorname{Dom}\alpha \mid x\alpha \in \operatorname{Dom}\beta \,\},$$

$$x(\alpha\beta) = (x\alpha)\beta \text{ for all } x \in \operatorname{Dom}\alpha\beta.$$

This implies that for $\alpha, \beta \in P_X$, $\operatorname{Dom}\alpha\beta \subseteq \operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$. Let

$$T_X = \{\, \alpha \in P_X \mid \operatorname{Dom}\alpha = X \,\},$$

$$I_X = \{\, \alpha \in P_X \mid \alpha \text{ is one-to-one} \,\},$$

$$M_X = \{\, \alpha \in T_X \mid \alpha \text{ is one-to-one} \,\},$$

$$E_X = \{\, \alpha \in T_X \mid \operatorname{Im}\alpha = X \,\},$$

$$G_X = \{\, \alpha \in T_X \mid \alpha \text{ is one-to-one and } \operatorname{Im}\alpha = X \,\}.$$

Then all T_X, I_X, M_X and E_X are subsemigroups of P_X, $G_X = M_X \cap E_X$, $G_X \subseteq E_X \subseteq T_X \subseteq P_X$ and $G_X \subseteq M_X \subseteq I_X \subseteq P_X$. In particular, G_X is a subgroup of P_X, that is, G_X is a subsemigroup of P_X which also forms a group. We call P_X, T_X, I_X and G_X , the *partial transformation semigroup* on X, the *full transformation* *semigroup* on X, the *one-to-one partial transformation semigroup* on X and the *symmetric group* on X, respectively. It is well-known that P_X, T_X and I_X are regular semigroups ([5], page 4). By Proposition 1.5, P_X, T_X, I_X and G_X are BQ-semigroups for any set X. Due to the fact that for an infinite set X, for every $a \in X$, $|X| = |X \smallsetminus \{a\}|$, we have that $|X| < \infty$ if and only if $M_X = E_X = G_X$, hence M_X and E_X are in $\boldsymbol{BQ}$ if $|X| < \infty$. The semigroups M_X and E_X have the following special properties which can be proved easily.

Proposition 1.9. *Let X be a set.*

(*i*) *For α, β, $\gamma \in T_X$, if $\beta\alpha = \gamma\alpha$ [$\alpha\beta = \alpha\gamma$] and $\alpha \in M_X$ [E_X], then $\beta = \gamma$. Hence M_X [E_X] is right [left] cancellative.*

(*ii*) *For $\alpha \in M_X$ [E_X], α is regular in M_X [E_X] if and only if $\alpha \in G_X$. Hence M_X [E_X] is regular if and only if $|X| < \infty$.*

(*iii*) *If X is infinite, then $M_X \smallsetminus G_X$ [$E_X \smallsetminus G_X$] is a unique maximal proper ideal of M_X [E_X]. Hence M_X [E_X] is left simple if and only if $|X| < \infty$ and M_X [E_X] is right simple if and only if $|X| < \infty$.*

From Proposition 1.9(ii) and (iii), it follow that M_X and E_X are neither regular nor left [right] simple if X is infinite. However, we cannot conclude from Proposition 1.5 or Proposition 1.6 that M_X and E_X do not belong to $\boldsymbol{BQ}$ when X is infinite. It was proved in [8] that M_X [E_X] belongs to $\boldsymbol{BQ}$ if and only if $|X| < \infty$.

Proposition 1.10. ([8]). *For a set X,*

(*i*) *$M_X \in \boldsymbol{BQ}$ if and only if $|X| < \infty$,*

(*ii*) *$E_X \in \boldsymbol{BQ}$ if and only if $|X| < \infty$.*

For an infinite set X, let

$$OE_X = \{\, \alpha \in T_X \mid X \smallsetminus \operatorname{Im}\alpha \text{ is infinite} \,\}.$$

Let $A \subseteq X$ be such that $|X \smallsetminus A| = |A| = |X|$ and let $\lambda : X \to X \smallsetminus A$ be a bijection. Then $\lambda \in OE_X$. Since $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$ for all $\alpha, \beta \in T_X$, it follows that OE_X is a subsemigroup of T_X. It was shown by Y. Kemprasit [9] that OE_X is a BQ-semigroup for every infinite set X but OE_X is neither regular nor left [right] simple. We can consider OE_X as the "opposite semigroup"of E_X. It was proved by P. M. Higgins [4] that OE_X is dense in T_X in the following sense:

$$\text{for any semigroup } S \text{ and any homomorphisms } \varphi, \psi \colon T_X \to S,$$
$$\varphi|_{OE_X} = \psi|_{OE_X} \text{ implies that } \varphi = \psi.$$

Next, let X be an infinite set and

$$BL_X = \{\, \alpha \in T_X \mid \alpha \text{ is one-to-one and } X \smallsetminus \operatorname{Im}\alpha \text{ is infinite} \,\}.$$

Then $BL_X = M_X \cap OE_X$. Clearly, λ defined above is in BL_X.We then deduce that BL_X is a subsemigroup of T_X. By Proposition 1.9(i), BL_X is right cancellative. This implies that $E(BL_X) = \varnothing$ since $1_X \notin BL_X$. If X is countably infinite, BL_X is called the *Baer-Levi semigroup* on X ([5], page 14). The Baer-Levi semigroup on a countably infinite set is known to be right simple ([5], page 14), so it is in $\boldsymbol{BQ}$ by Proposition 1.6. It was proved by Y. Kemprasit in [9] that countable infiniteness of X is also necessary for BL_X to be in $\boldsymbol{BQ}$.

Proposition 1.11. ([9]). *For an infinite set X, $BL_X \in \boldsymbol{BQ}$ if and only if X is countably infinite.*

K. D. Magill ([12] and [13]) generalized the notion of transformation semigroups as follows: Let X and Y be sets and let $T(X, Y)$ denote the set of all transformations $\alpha : X \to Y$. Then for a fixed $\theta \in T(Y,X)$, define an operation "$*$" on $T(X,Y)$ by

$$\alpha * \beta = \alpha\theta\beta \quad \text{for all } \alpha, \beta \in T(X,Y).$$

Under this operation, $T(X,Y)$ becomes a semigroup which is denoted by $(T(X,Y),\theta)$. Moreover, the semigroup $((T(X,Y),\theta)$ need not be regular.

R. P. Sullivan ([17]) generalized one step further by considering the set $P(X,Y)$ of all partial transformations from X into Y, that is, $P(X,Y) = \{\, \alpha : A \to Y \mid A \subseteq X \,\}$, and generalizing the above semigroup as follows: For a nonempty subset S of $P(X,Y)$ and $\theta \in P(Y,X)$, if $\alpha\theta\beta \in S$ for all $\alpha, \beta \in S$, let (S,θ) denote the semigroup $(S,*)$ with $*$ defined as above. In the same way, we define $I(X,Y), M(X,Y), E(X,Y)$ and define the corresponding semigroups $(I(X,Y),\theta)$ where $\theta \in I(Y,X)$, $(M(X,Y),\theta)$ where $\theta \in M(Y,X)$ and $(E(X,Y),\ \theta)$ where $\theta \in E(Y,X)$, respectively. We remark here that

$$(P(X,X),1_X) = P_X, (T(X,X),1_X) = T_X, (I(X,X),1_X) = I_X,$$
$$(M(X,X),1_X) = M_X, (E(X,X),1_X) = E_X.$$

In Chapter II, we prove that $(S(X,Y),\theta)$ always belongs to $\boldsymbol{BQ}$ if $S(X,Y)$ is any of $P(X,Y)$, $T(X,Y)$ and $I(X,Y)$ where $\theta \in S(Y,X)$. In particular, we also show that these three semigroups need not be regular. Moreover, by the help of Proposition 1.10, we shall prove that the condition that $|X| = |Y| < \infty$ is necessary and sufficient for $(M(X,Y),\theta)$ and $(E(X,Y),\theta)$ to be in $\boldsymbol{BQ}$.

Let V be a vector space over a division ring. For $A \subseteq V$, we let $\langle A\rangle$ denote the subspace of V spanned by A. To introduce various linear transformation semigroups for Chapter III, we first give some basic properties of vector spaces.

Proposition 1.12. *Let V and W be vector spaces over a division ring.*

(i) *If $\alpha : V \to W$ is a linear transformation, then*

$$dim\,V = dim\,Ker\,\alpha + dim\,Im\,\alpha.$$

(ii) *If U is a subspace of V, then*

$$dim\,V = dim\,U + dim\,(V/U).$$

(iii) *If U and Z are subspaces of V such that $Z \subseteq U$, then*

$$dim\,(V/U) = dim\,(V/Z/U/Z) \leq dim\,(V/Z).$$

(iv) *If B is a basis of V and $A \subseteq B$, then $\{\, v + \langle A \rangle \mid v \in B \smallsetminus A \,\}$ is a basis of $V/\langle A \rangle$ and $v + \langle A \rangle \neq v' + \langle A \rangle$ for distinct $v, v' \in B \smallsetminus A$. Hence*

$$dim\,(V/\langle A \rangle) = |B \smallsetminus A|.$$

(v) *Let $\alpha : V \to W$ be a linear transformation. If $w_1, w_2, \ldots, w_n \in W$ are linearly independent and $v_1, v_2, \ldots, v_n \in V$ are such that $v_i\alpha = w_i$ for all $i \in \{1, 2, \ldots, n\}$, then $v_1, v_2, \ldots, v_n$ are linearly independent.*

(vi) *Let $\alpha : V \to W$ be a linear transformation, B_1 is a basis of $Ker\,\alpha$ and B_2 is a basis of $Im\,\alpha$. If for each $v \in B_2$, $u_v \in V$ is such that $u_v\alpha = v$, then $B_1 \cup \{\, u_v \mid v \in B_2 \,\}$ is a basis of V.*

(vii) *Let $\alpha : V \to W$ be a linear transformation and B a basis of V. If $B\alpha$ is a linearly independent subset of W and $\alpha|_B$ is one-to-one, then α is one-to-one, that is, $Ker\,\alpha = \{0\}$.*

(viii) *Let $\alpha : V \to W$ be a linear transformation, B a basis of V and $A \subseteq B$. If $A\alpha$ is a linearly independent subset of W, $\alpha|_A$ is one-to-one and $(B \smallsetminus A)\alpha = \{0\}$, then $Ker\,\alpha = \langle B \smallsetminus A \rangle$.*

For a vector space V over a division ring, let $L(V)$ be the semigroup under composition of all linear transformations $\alpha\colon V \to V$. It is known that $L(V)$ is regular ([6], page 443). Let

$$G(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is an isomorphism} \,\}.$$

Then $G(V)$ is a subgroup of $L(V)$. By Proposition 1.5, both $L(V)$ and $G(V)$ are in $\boldsymbol{BQ}$. The following subsets of $L(V)$ are considered:

$$M(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one} \,\} \text{ and}$$

$$E(V) = \{\, \alpha \in L(V) \mid \operatorname{Im}\alpha = V \,\}.$$

Both $M(V)$ and $E(V)$ are clearly subsemigroups of $L(V)$ containing $G(V)$ and $M(V)$ $[E(V)] = G(V)$ if and only if $\dim V < \infty$. Next, we define the "opposite semigroups" of $M(V)$ and $E(V)$ to be respectively by

$$OM(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker}\alpha \text{ is infinite} \,\} \text{ and}$$

$$OE(V) = \{\, \alpha \in L(V) \mid \dim\,(V/\operatorname{Im}\alpha) \text{ is infinite} \,\}$$

where $\dim V$ is infinite. To show that $OM(V)$ and $OE(V)$ are indeed subsemigroups of $L(V)$, suppose that $\dim V$ is infinite. Then we have that $0 \in OM(V)$ and $0 \in OE(V)$ where 0 is the zero map on V. Since for all $\alpha, \beta \in L(V)$, $\operatorname{Ker}\alpha\beta \supseteq \operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$, it follows that $OM(V)$ and $OE(V)$ are subsemigroups of $L(V)$, respectively (see Proposition 1.12(iii)). The following subset of $L(V)$ is also considered:

$$OME(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker}\alpha \text{ and } \dim\,(V/\operatorname{Im}\alpha) \text{ are infinite} \,\}$$

where $\dim V$ is infinite. Since $0 \in OM(V) \cap OE(V) = OME(V)$ and both $OM(V)$ and $OE(V)$ are subsemigroups of $L(V)$, we have that $OME(V)$ is a subsemigroup of $L(V)$ containing 0.

The semigroup BL_X where X is infinite motivates us to define $BL(V)$ where $\dim V$ is infinite as follows:

$$BL(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one and } \dim(V/\operatorname{Im}\alpha) \text{ is infinite} \,\}.$$

Then $BL(V) = M(V) \cap OE(V)$. To show that $BL(V)$ is a subsemigroup of $L(V)$, assume that $\dim V$ is infinite. Let B be a basis of V. Then B is infinite, so there exists $A \subseteq B$ such that $|A| = |B \smallsetminus A| = |B|$. Thus there exists a bijection $\varphi : B \to A$. Define $\alpha \in L(V)$ by $v\alpha = v\varphi$ for all $v \in B$. We thus deduce from Proposition 1.12(vii) that $\alpha \in M(V)$. By Proposition 1.12(iv), we have

$$\dim(V/\operatorname{Im}\alpha) = \dim(V/\langle A\rangle) = |B \smallsetminus A|.$$

This implies that $\alpha \in OE(V)$. Then $\alpha \in M(V) \cap OE(V)$, so $BL(V)$ is a subsemigroup of $L(V)$, as required. Similarly, we consider the "opposite semigroup" of $BL(V)$ which is defined to be

$$OBL(V) = \{\, \alpha \in L(V) \mid \operatorname{Im}\alpha = V \text{ and } \dim \operatorname{Ker}\alpha \text{ is infinite} \,\}$$

where $\dim V$ is infinite. By the definition, we have $OBL(V) = E(V) \cap OM(V)$. To show that $OBL(V)$ is indeed a subsemigroup of $L(V)$, it suffices to show that $OBL(V) \neq \varnothing$. Let B be a basis of V and let A be a subset of B such that $|A| = |B \smallsetminus A| = |B|$. Then there exists a bijection $\varphi : A \to B$. Define $\alpha \in L(V)$ by

$$v\alpha = \begin{cases} v\varphi & \text{if } v \in A, \\ 0 & \text{if } v \in B \smallsetminus A. \end{cases}$$

Then $\operatorname{Im}\alpha = \langle A\varphi\rangle = \langle B\rangle = V$. By Proposition 1.12(viii), $\operatorname{Ker}\alpha = \langle B \smallsetminus A\rangle$, so $\dim \operatorname{Ker}\alpha = |B \smallsetminus A| = |B|$. Hence $\alpha \in OBL(V)$. Then $OBL(V)$ is a subsemigroup of $L(V)$ which is considered as the "opposite semigroup"of $BL(V)$. Note that $0 \notin OBL(V)$. Moreover, $E(BL(V)) = \varnothing = E(OBL(V))$ by Proposition 1.9(i).

Next, the following subsets of $L(V)$ are considered:

$$AM(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker} \alpha \text{ is finite} \,\} \text{ and}$$

$$AE(V) = \{\, \alpha \in L(V) \mid \dim (V/\operatorname{Im} \alpha) \text{ is finite} \,\}.$$

Then $M(V) \subseteq AM(V)$ and $E(V) \subseteq AE(V)$. To show that $AM(V)$ and $AE(V)$ are subsemigroups of $L(V)$, let $\alpha, \beta \in L(V)$. We claim that $\alpha|_{\operatorname{Ker} \alpha\beta}$ is a linear transformation from $\operatorname{Ker} \alpha\beta$ onto $\operatorname{Ker} \beta \cap \operatorname{Im} \alpha$ with $\operatorname{Ker} (\alpha|_{\operatorname{Ker} \alpha\beta}) = \operatorname{Ker} \alpha$. Since

$$(\operatorname{Ker} \alpha\beta)\alpha|_{\operatorname{Ker} \alpha\beta} = (\operatorname{Ker} \alpha\beta)\alpha \subseteq \operatorname{Im} \alpha \quad \text{and}$$

$$\big((\operatorname{Ker} \alpha\beta)\alpha|_{\operatorname{Ker} \alpha\beta}\big)\beta = (\operatorname{Ker} \alpha\beta)\alpha\beta = \{0\},$$

it follows that $\operatorname{Im} (\alpha|_{\operatorname{Ker} \alpha\beta}) \subseteq \operatorname{Ker} \beta \cap \operatorname{Im} \alpha$. Let $v \in \operatorname{Ker} \beta \cap \operatorname{Im} \alpha$. Then $u\alpha = v$ for some $u \in V$ and $v\beta = 0$. This implies that $u\alpha\beta = v\beta = 0$. Thus $u \in \operatorname{Ker} \alpha\beta$ and so $v = u\alpha = u(\alpha|_{\operatorname{Ker} \alpha\beta}) \in \operatorname{Im} (\alpha|_{\operatorname{Ker} \alpha\beta})$. Hence we have $\operatorname{Im} (\alpha|_{\operatorname{Ker} \alpha\beta}) = \operatorname{Ker} \beta \cap \operatorname{Im} \alpha$. Since

$$\begin{aligned}
\operatorname{Ker} (\alpha|_{\operatorname{Ker} \alpha\beta}) &= \{\, v \in \operatorname{Ker} \alpha\beta \mid v\alpha = 0 \,\} \\
&\subseteq \{\, v \in V \mid v\alpha = 0 \,\} \quad (\text{since } \operatorname{Ker} \alpha\beta \subseteq V) \\
&= \operatorname{Ker} \alpha \\
&= \{\, v \in V \mid v\alpha = 0 \,\} \\
&= \{\, v \in V \mid v\alpha\beta = 0 \,\} \cap \{\, v \in V \mid v\alpha = 0 \,\} \quad (\text{since } 0\beta = 0) \\
&= \{\, v \in \operatorname{Ker} \alpha\beta \mid v\alpha = 0 \,\} \\
&= \operatorname{Ker} (\alpha|_{\operatorname{Ker} \alpha\beta}),
\end{aligned}$$

we have $\operatorname{Ker} (\alpha|_{\operatorname{Ker} \alpha\beta}) = \operatorname{Ker} \alpha$. Hence we have the claim. It then follows from Proposition 1.12(i) that

$$\begin{aligned}
\dim \operatorname{Ker} \alpha\beta &= \dim \operatorname{Ker} \alpha + \dim (\operatorname{Ker} \beta \cap \operatorname{Im} \alpha) \\
&\leq \dim \operatorname{Ker} \alpha + \dim \operatorname{Ker} \beta. \qquad (1)
\end{aligned}$$

Define $\beta^* : V/\mathrm{Im}\,\alpha \to \mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta$ by

$$(v + \mathrm{Im}\,\alpha)\beta^* = v\beta + \mathrm{Im}\,\alpha\beta \quad \text{for all } v \in V.$$

Clearly, β^* is well-defined and onto. Since β is linear, β^* is linear. Hence β^* is a linear transformation of $V/\mathrm{Im}\,\alpha$ onto $\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta$. Then we have that $\dim(V/\mathrm{Im}\,\alpha) \geq \dim(\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta)$. Since $\mathrm{Im}\,\alpha\beta \subseteq \mathrm{Im}\,\beta$, we have by Proposition 1.12(iii) and (ii) that

$$\dim(V/\mathrm{Im}\,\beta) = \dim((V/\mathrm{Im}\,\alpha\beta)/(\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta)) \quad \text{and}$$

$$\dim(V/\mathrm{Im}\,\alpha\beta) = \dim(\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta) + \dim((V/\mathrm{Im}\,\alpha\beta)/(\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha\beta)),$$

respectively. These facts imply that

$$\dim(V/\mathrm{Im}\,\alpha\beta) \leq \dim(V/\mathrm{Im}\,\alpha) + \dim(V/\mathrm{Im}\,\beta). \tag{2}$$

We have respectively from (1) and (2) that $AM(V)$ and $AE(V)$ are subsemigroups of $L(V)$, as required. The semigroups $AM(V)$ and $AE(V)$ can be referred to respectively as the semigroup of all "almost one-to-one linear transformations" of V and the semigroup of all "almost onto linear transformations" of V. Observe that if $\dim V$ is finite, then $AM(V) = AE(V) = L(V)$.

Finally, we consider the following subsets of $L(V)$:

$MAE(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one and } \dim(V/\mathrm{Im}\,\alpha) \text{ is finite} \,\}$,

$EAM(V) = \{\, \alpha \in L(V) \mid \mathrm{Im}\,\alpha = V \text{ and } \dim \mathrm{Ker}\,\alpha \text{ is finite} \,\}$ and

$AME(V) = \{\, \alpha \in L(V) \mid \dim \mathrm{Ker}\,\alpha \text{ and } \dim(V/\mathrm{Im}\,\alpha) \text{ are finite} \,\}$.

Then we have that

$$G(V) \subseteq M(V) \cap AE(V) = MAE(V),$$

$$G(V) \subseteq E(V) \cap AM(V) = EAM(V) \quad \text{and}$$

$$G(V) \subseteq AM(V) \cap AE(V) = AME(V),$$

so the above three subsets of $L(V)$ are subsemigroups of $L(V)$. Note that if $\dim V$ is finite, then $MAE(V) = EAM(V) = G(V)$ and $AME(V) = L(V)$.

The aim of Chapter III is to characterize in terms of dimensions of V when the subsemigroups of $L(V)$ mentioned above belong to $\boldsymbol{BQ}$.

Next, let $(X, \leq)$ be a partially ordered set. For $\alpha \in T_X$, α is said to be *order-preserving* if for all $x, y \in X$, $x \leq y$ implies that $x\alpha \leq y\alpha$. For partially ordered sets $(X, \leq)$ and $(Y, \leq')$, we say that $(X, \leq)$ and $(Y, \leq')$ are *order-isomorphic* if there is a bijection $\varphi : X \to Y$ such that for $x_1, x_2 \in X$, $x_1 \leq x_2$ if and only if $x_1\varphi \leq' x_2\varphi$. The *opposite partial order* $\leq_{opp}$ on X of $\leq$ is defined by

$$x \leq_{opp} y \quad \text{if and only if} \quad y \leq x \text{ for all } x, y \in X.$$

Clearly, $\leq_{opp}$ is really a partial order on X. It is clear that for a nonempty interval I of $\mathbb{R}$, $(I, \leq)$ and $(-I, \leq_{opp})$ are order-isomorphic by $x \mapsto -x$ where $\leq$ is the usual order of real numbers and $-I = \{-x \mid x \in I\}$. Hence we have

Proposition 1.13. *Let $\leq$ be the usual partial order on $\mathbb{R}$.*

(i) *For $a \in \mathbb{R}$, $((-\infty, a), \leq)$ is order-isomorphic to $((-a, \infty), \leq_{opp})$.*

(ii) *For $a \in \mathbb{R}$, $((-\infty, a], \leq)$ is order-isomorphic to $([-a, \infty), \leq_{opp})$.*

(iii) *For $a, b \in \mathbb{R}$ and $a < b$, $((a, b], \leq)$ is order-isomorphic to $([-b, -a), \leq_{opp})$.*

Let $T_{OP}(X)$ denote the set of all order-preserving transformations of X. Then $T_{OP}(X)$ is a subsemigroup of T_X. In [14], $T_{OP}(X)$ is said to be the *full order-preserving transformation semigroup* on X. Y. Kemprasit and T. Changphas [10] characterized when $T_{OP}(I)$ is regular where I is a nonempty interval of $\mathbb{R}$, as follows:

Proposition 1.14. ([10]). *For a nonempty interval I of $\mathbb{R}$, $T_{OP}(I)$ is regular if and only if I is closed and bounded.*

Then we can conclude from Proposition 1.5 and Proposition 1.14 that if I is closed and bounded, then $T_{OP}(I)$ is a BQ-semigroup. By making use of Proposition 1.8 and Proposition 1.13, we show in the last chapter that the converse of this statement holds. Hence we obtain the fact that for a nonempty interval I of $\mathbb{R}$, $T_{OP}(I) \in \boldsymbol{BQ}$ if and only if I is closed and bounded.

CHAPTER II

GENERALIZED TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to characterize when the generalized transformation semigroups mentioned in Chapter I belong to $\boldsymbol{BQ}$.

Let us recall the notations which are used throughout this chapter. Let X and Y be sets and

$$P_X = \text{the partial transformation semigroup on } X,$$

$$T_X = \text{the full transformation semigroup on } X,$$

$$I_X = \text{the one-to-one partial transformation semigroup on } X,$$

$$M_X = \text{the semigroup of one-to-one transformations of } X,$$

$$E_X = \text{the semigroup of onto transformations of } X,$$

$$G_X = \text{the symmetric group on } X,$$

$$P(X,Y) = \{\, \alpha : A \to Y \mid A \subseteq X \,\},$$

$$T(X,Y) = \{\, \alpha \in P(X,Y) \mid \operatorname{Dom}\alpha = X \,\},$$

$$I(X,Y) = \{\, \alpha \in P(X,Y) \mid \alpha \text{ is one-to-one} \,\},$$

$$M(X,Y) = \{\, \alpha \in T(X,Y) \mid \alpha \text{ is one-to-one} \,\},$$

$$E(X,Y) = \{\, \alpha \in T(X,Y) \mid \operatorname{Im}\alpha = Y \,\}.$$

As was mentioned in Chapter I, P_X, T_X and I_X are regular. In fact, if $|X| > 1$, T_X is not left [right] simple and P_X and I_X are not left [right] 0-simple. That is because the set

$$\{\, \alpha \in S_X \mid |\operatorname{Im}\alpha| \leq 1 \,\}$$

is clearly a proper ideal of S_X where S_X is P_X, T_X or I_X. Then $(T(X,Y),\theta)$ need not be left [right] simple and $(P(X,Y),\theta)$, $(I(X,Y),\theta)$ need not be left [right] 0-simple. Moreover, these three semigroups need not be regular. To see this, let $X = Y = \mathbb{N}$ and define $\theta : Y \to X$ by $x\theta = 2x$ for all $x \in Y$. Then $\theta \in S(Y,X) = S(\mathbb{N},\mathbb{N})$. Since for every $\alpha \in S(\mathbb{N},\mathbb{N})$, $1_{\mathbb{N}}\theta\alpha\theta 1_{\mathbb{N}} = \theta\alpha\theta$ which implies that $\operatorname{Im}(1_{\mathbb{N}}\theta\alpha\theta 1_{\mathbb{N}}) = \operatorname{Im}(\theta\alpha\theta) \subseteq \operatorname{Im}\theta = 2\mathbb{N} \neq \operatorname{Im} 1_{\mathbb{N}}$. Hence $1_{\mathbb{N}} \in S(\mathbb{N},\mathbb{N})$ which is not regular in $(S(\mathbb{N},\mathbb{N}),\theta)$.

The first theorem requires the facts that P_Y, T_Y, and I_Y are regular and every regular semigroup is a BQ-semigroup.

Theorem 2.1. *If $S(X,Y)$ is any one of $T(X,Y), P(X,Y)$ and $I(X,Y)$ and $\theta \in S(Y,X)$, then $(S(X,Y),\theta) \in \boldsymbol{BQ}$.*

Proof. We know that $(A)_b \subseteq (A)_q$ for any nonempty subset A of $S(X,Y)$ (Proposition 1.3). To prove that $(S(X,Y),\theta) \in \boldsymbol{BQ}$, by Proposition 1.4 and Proposition 1.7, it suffices to show that for any nonempty subset A of $S(X,Y)$,

$$S(X,Y)\theta A \cap A\theta S(X,Y) \subseteq A\theta S(X,Y)\theta A.$$

For this purpose, let A be a nonempty subset of $S(X,Y)$ and $\alpha \in S(X,Y)\theta A \cap A\theta S(X,Y)$. Then we have

$$\alpha = \beta\theta\lambda = \gamma\theta\mu \tag{1}$$

for some $\beta,\mu \in S(X,Y)$ and $\lambda,\gamma \in A$. But $\theta\lambda \in S(Y,Y)$ and T_Y, P_Y and I_Y are all regular, so there exists $\eta \in S(Y,Y)$ such that

$$\theta\lambda = \theta\lambda\eta\theta\lambda. \tag{2}$$

It thus follows from (1) and (2) that

$$\alpha = \beta\theta\lambda\eta\theta\lambda = \gamma\theta\mu\eta\theta\lambda = \gamma\theta(\mu\eta)\theta\lambda. \tag{3}$$

Since $\mu \in S(X,Y)$, $\eta \in S(Y,Y)$ and $\lambda, \gamma \in A$, we have from (3) that $\alpha \in A\theta S(X,Y)\theta A$. This completes the proof. $\square$

The next two theorems are the second main results of this chapter. We first prove three lemmas which will be used to determine when the semigroups $(M(X,Y),\theta)$ where $\theta \in M(Y,X)$ and $(E(X,Y),\theta)$ where $\theta \in E(Y,X)$ are in the class $\boldsymbol{BQ}$.

For convenience, we denote the semigroup $(M(X,Y),\theta)$ where $\theta \in M(Y,X)$ by (M_X,θ) if $X = Y$. The notion (E_X,θ) is defined similarly. Also, the notation (G_X,θ) where $\theta \in G_X$ is used for the semigroup G_X with the operation $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in G_X$. Clearly, (G_X,θ) is a group having θ^{-1} as its identity.

Lemma 2.2. *If $\theta \in G_X$, then $(G_X,\theta) \cong G_X$, $(M_X,\theta) \cong M_X$ and $(E_X,\theta) \cong E_X$.*

Proof. Define $\varphi : T_X \to T_X$ by $\alpha\varphi = \alpha\theta$ for all $\alpha \in T_X$. Then for $\alpha, \beta \in T_X$, $(\alpha\theta\beta)\varphi = \alpha\theta\beta\theta = (\alpha\varphi)(\beta\varphi)$, $\alpha\theta^{-1} \in T_X$, $(\alpha\theta^{-1})\varphi = \alpha$ and $\alpha\theta = \beta\theta$ implies that $\alpha = \alpha\theta\theta^{-1} = \beta\theta\theta^{-1} = \beta$. Hence φ is an isomorphism from (T_X,θ) onto T_X. Since $\theta \in G_X$, $G_X\theta = G_X$, $M_X = M_X\theta^{-1}\theta \subseteq M_X\theta \subseteq M_X$ and $E_X = E_X\theta^{-1}\theta \subseteq E_X\theta \subseteq E_X$. It then follows that $\varphi|_{G_X}$, $\varphi|_{M_X}$ and $\varphi|_{E_X}$ are respectively isomorphisms of (G_X,θ), (M_X,θ) and (E_X,θ) onto G_X, M_X and E_X. $\square$

Lemma 2.3. *The following statements hold.*

(i) $M(X,Y) \neq \varnothing$ and $M(Y,X) \neq \varnothing$ if and only if $|X| = |Y|$.

(ii) $E(X,Y) \neq \varnothing$ and $E(Y,X) \neq \varnothing$ if and only if $|X| = |Y|$.

Proof. If $\alpha \in M(X,Y)$ and $\beta \in M(Y,X)$, then

$$|X| = |\operatorname{Im}\alpha| \leq |Y| \quad \text{and} \quad |Y| = |\operatorname{Im}\beta| \leq |X|,$$

so $|X| = |Y|$. If $\gamma \in E(X,Y)$ and $\lambda \in E(Y,X)$, then

$$|X| \geq |\operatorname{Im}\gamma| = |Y| \quad \text{and} \quad |Y| \geq |\operatorname{Im}\lambda| = |X|,$$

which implies that $|X| = |Y|$.

If $|X| = |Y|$, then there is a bijection μ of X onto Y, then $\mu \in M(X,Y) \cap E(X,Y)$ and $\mu^{-1} \in M(Y,X) \cap E(Y,X)$.

Hence (i) and (ii) are proved. □

Lemma 2.4. *Assume that $|X| = |Y|$. If φ is a bijection of X onto Y, then*

(i) $(M(X,Y),\theta) \cong (M_X, \varphi\theta)$ where $\theta \in M(Y,X)$ and

(ii) $(E(X,Y),\theta) \cong (E_X, \varphi\theta)$ where $\theta \in E(Y,X)$.

Proof. Define $\psi_1 : M(X,Y) \to M_X$ and $\psi_2 : E(X,Y) \to E_X$ by

$$\alpha\psi_1 = \alpha\varphi^{-1} \text{ for all } \alpha \in M(X,Y) \quad \text{and} \quad \beta\psi_2 = \beta\varphi^{-1} \text{ for all } \beta \in E(X,Y).$$

Let $\theta \in M(Y,X)$. Then $\varphi\theta \in M_X$ and for $\alpha, \beta \in M(X,Y)$,

$$(\alpha\theta\beta)\psi_1 = (\alpha\theta\beta)\varphi^{-1} = (\alpha\varphi^{-1})(\varphi\theta)(\beta\varphi^{-1}) = (\alpha\psi_1)(\varphi\theta)(\beta\psi_1).$$

and if $\alpha\psi_1 = \beta\psi_1$, then $\alpha = (\alpha\varphi^{-1})\varphi = (\alpha\psi_1)\varphi = (\beta\psi_1)\varphi = (\beta\varphi^{-1})\varphi = \beta$. If $\alpha \in M_X$, then $\alpha\varphi \in M(X,Y)$ and $(\alpha\varphi)\psi_1 = (\alpha\varphi)\varphi^{-1} = \alpha$. This proves that ψ_1 is an isomorphism of $(M(X,Y),\theta)$ onto $(M_X, \varphi\theta)$. We can show similarly that ψ_2 is an isomorphism of $(E(X,Y),\theta)$ onto $(E_X, \varphi\theta)$ where $\theta \in E(Y,X)$. Hence (i) and (ii) are proved, as desired. □

Theorem 2.5. *For $\theta \in M(Y,X)$, the semigroup $(M(X,Y),\theta)$ belongs to $\boldsymbol{BQ}$ if and only if $|X| = |Y| < \infty$.*

Proof. Assume that $|X| = |Y| < \infty$. Then $\theta : Y \to X$ is a bijection, hence $\theta^{-1} : X \to Y$ is also a bijection. By Lemma 2.4(i), $(M(X,Y),\theta) \cong (M_X, \theta^{-1}\theta) = (M_X, 1_X) = M_X$. Since $|X| < \infty$, $M_X = G_X$, so $M_X \in \boldsymbol{BQ}$ by Proposition 1.5. Hence $(M(X,Y),\theta) \in \boldsymbol{BQ}$.

Conversely, assume that $(M(X,Y),\theta) \in \boldsymbol{BQ}$. By Lemma 2.3(i), $|X| = |Y|$. Let $\varphi : X \to Y$ be a bijection. Then $\varphi\theta \in M_X$. To show that $|X| < \infty$, suppose that X is infinite. Therefore $G_X \subsetneq M_X$.

Case 1: $\varphi\theta \in G_X$. Then $(M_X, \varphi\theta) \cong M_X$ by Lemma 2.2. But $M_X \notin \boldsymbol{BQ}$ by Proposition 1.10(i), so we have $(M_X, \varphi\theta) \notin \boldsymbol{BQ}$.

Case 2: $\varphi\theta \in M_X \smallsetminus G_X$. Then by Proposition 1.9(iii), $(\varphi\theta)^n \in M_X \smallsetminus G_X$ for every $n \in \mathbb{N}$. It thus follows from Proposition 1.9(i) that $(\varphi\theta)^n \neq (\varphi\theta)^m$ for all distinct $n, m \in \mathbb{N}$. In particular,

$$(\varphi\theta)^2 \neq \varphi\theta \quad \text{and} \quad (\varphi\theta)^2 \neq (\varphi\theta)^3. \tag{1}$$

We have from Proposition 1.4 that in $(M_X, \varphi\theta)$,

$$(\varphi\theta)_q = \big(M_X(\varphi\theta)^2 \cap (\varphi\theta)^2 M_X\big) \cup \{\,\varphi\theta\,\}, \tag{2}$$

$$(\varphi\theta)_b = (\varphi\theta)^2 M_X (\varphi\theta)^2 \cup \{\,\varphi\theta, (\varphi\theta)^3\,\}. \tag{3}$$

By (2), $(\varphi\theta)^2 \in (\varphi\theta)_q$ in $(M_X, \varphi\theta)$. Since $(\varphi\theta)^2 \notin G_X$, by Proposition 1.9(ii), $(\varphi\theta)^2$ is not regular in M_X. Thus

$$(\varphi\theta)^2 \notin (\varphi\theta)^2 M_X (\varphi\theta)^2. \tag{4}$$

From (1), (3) and (4), we conclude that $(\varphi\theta)^2 \notin (\varphi\theta)_b$ in $(M_X, \varphi\theta)$. It then follows from Proposition 1.8 that $(M_X, \varphi\theta) \notin \boldsymbol{BQ}$.

Now we have $(M_X, \varphi\theta) \notin \boldsymbol{BQ}$ from Case 1 and Case 2. But since $(M(X,Y),\theta) \cong (M_X, \varphi\theta)$ by Lemma 2.4(i), we deduce that $(M(X,Y),\theta) \notin \boldsymbol{BQ}$, a contradiction. It then follows that $|X| = |Y| < \infty$.

Hence the theorem is proved, as required. $\square$

Theorem 2.6. *For $\theta \in E(Y,X)$, the semigroup $(E(X,Y),\theta)$ belongs to $\boldsymbol{BQ}$ if and only if $|X| = |Y| < \infty$.*

Proof. Assume that $|X| = |Y| < \infty$. Then we have that $\theta : Y \to X$ is a bijection, so $\theta^{-1} : X \to Y$ is a bijection. By Lemma 2.4(ii), $(E(X,Y),\theta) \cong (E_X, \theta^{-1}\theta) = (E_X, 1_X) = E_X$. But $E_X = G_X$ because $|X| < \infty$, so $E_X \in \boldsymbol{BQ}$ by Proposition 1.5. Consequently, $(E(X,Y),\theta) \in \boldsymbol{BQ}$.

For the converse, assume that $(E(X,Y),\theta) \in \boldsymbol{BQ}$. By Lemma 2.3(ii), $|X| = |Y|$. Let $\varphi : X \to Y$ be a bijection. Then $\varphi\theta \in E_X$. To show that $|X| < \infty$, suppose on the contrary that X is infinite. Thus $G_X \subsetneq E_X$.

Case 1: $\varphi\theta \in G_X$. From Lemma 2.2, $(E_X, \varphi\theta) \cong E_X$. By Proposition 1.10(ii), $E_X \notin \boldsymbol{BQ}$. Thus $(E_X, \varphi\theta) \notin \boldsymbol{BQ}$.

Case 2: $\varphi\theta \in E_X \smallsetminus G_X$. Then by Proposition 1.9(iii), $(\varphi\theta)^n \in E_X \smallsetminus G_X$ for every $n \in \mathbb{N}$. From Proposition 1.9(i), we have that $(\varphi\theta)^n \neq (\varphi\theta)^m$ for all distinct $n, m \in \mathbb{N}$. In particular,

$$(\varphi\theta)^2 \neq \varphi\theta \quad \text{and} \quad (\varphi\theta)^2 \neq (\varphi\theta)^3. \tag{1}$$

We can see from Proposition 1.4 that in $(E_X, \varphi\theta)$,

$$(\varphi\theta)_q = \big(E_X(\varphi\theta)^2 \cap (\varphi\theta)^2 E_X\big) \cup \{\,\varphi\theta\,\}, \tag{2}$$

$$(\varphi\theta)_b = (\varphi\theta)^2 E_X (\varphi\theta)^2 \cup \{\,\varphi\theta, (\varphi\theta)^3\,\}. \tag{3}$$

We then have from (2) that $(\varphi\theta)^2 \in (\varphi\theta)_q$ in $(E_X, \varphi\theta)$. From Proposition 1.9(ii), $(\varphi\theta)^2$ is not regular in E_X, that is,

$$(\varphi\theta)^2 \notin (\varphi\theta)^2 E_X (\varphi\theta)^2. \tag{4}$$

It then follows from (1), (3) and (4) that $(\varphi\theta)^2 \notin (\varphi\theta)_b$ in $(E_X, \varphi\theta)$. Hence $(E_X, \varphi\theta) \notin \boldsymbol{BQ}$ by Proposition 1.8.

From the above two cases, we have $(E_X, \varphi\theta) \notin \boldsymbol{BQ}$. But $(E(X,Y), \theta) \cong (E_X, \varphi\theta)$ by Lemma 2.4(ii), we deduce that $(E(X,Y), \theta) \notin \boldsymbol{BQ}$ which is a contradiction. Therefore we have that $|X| = |Y| < \infty$.

Hence the proof is complete. $\square$

CHAPTER III

LINEAR TRANSFORMATION SEMIGROUPS

In this chapter, we give necessary and sufficient conditions for dimensions of a vector space V over a division ring in order that various linear transformation semigroups on V belong to $\boldsymbol{BQ}$.

We first recall the following subsemigroups of $L(V)$ previously mentioned in Chapter I:

$$G(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is an isomorphism} \,\},$$

$$M(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one} \,\},$$

$$E(V) = \{\, \alpha \in L(V) \mid \operatorname{Im}\alpha = V \,\},$$

$$OM(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker}\alpha \text{ is infinite} \,\}$$

where $\dim V$ is infinite,

$$OE(V) = \{\, \alpha \in L(V) \mid \dim (V/\operatorname{Im}\alpha) \text{ is infinite} \,\}$$

where $\dim V$ is infinite,

$$OME(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker}\alpha \text{ and } \dim (V/\operatorname{Im}\alpha) \text{ are infinite} \,\}$$

where $\dim V$ is infinite,

$$BL(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one and } \dim (V/\operatorname{Im}\alpha) \text{ is infinite} \,\}$$

where $\dim V$ is infinite,

$$OBL(V) = \{\, \alpha \in L(V) \mid \operatorname{Im}\alpha = V \text{ and } \dim \operatorname{Ker}\alpha \text{ is infinite} \,\}$$

where $\dim V$ is infinite,

$$AM(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker} \alpha \text{ is finite} \,\},$$

$$AE(V) = \{\, \alpha \in L(V) \mid \dim (V/\operatorname{Im} \alpha) \text{ is finite} \,\},$$

$$MAE(V) = \{\, \alpha \in L(V) \mid \alpha \text{ is one-to-one and } \dim (V/\operatorname{Im} \alpha) \text{ is finite} \,\},$$

$$EAM(V) = \{\, \alpha \in L(V) \mid \operatorname{Im} \alpha = V \text{ and } \dim \operatorname{Ker} \alpha \text{ is finite} \,\} \text{ and}$$

$$AME(V) = \{\, \alpha \in L(V) \mid \dim \operatorname{Ker} \alpha \text{ and } \dim (V/\operatorname{Im} \alpha) \text{ are finite} \,\}.$$

In the remainder, let V be a vector space over a division ring R.

We first introduce the following lemmas which will be used.

Lemma 3.1. *If B is a basis of V, $A \subseteq B$ and $\alpha \in L(V)$ is one-to-one, then*

$$dim\,(Im\,\alpha/\langle A\alpha\rangle) = |B \smallsetminus A|.$$

Proof. Assume that α is one-to-one. Then we have that $\alpha \colon V \to \operatorname{Im} \alpha$ is an isomorphism. Define $\bar{\alpha} : V/\langle A\rangle \to \operatorname{Im} \alpha/\langle A\rangle\alpha$ by

$$(v + \langle A\rangle)\bar{\alpha} = v\alpha + \langle A\rangle\alpha \quad \text{ for all } v \in V.$$

Clearly, $\bar{\alpha}$ is well-defined and onto. Since α is linear, it follows that $\bar{\alpha}$ is linear. Also $\bar{\alpha}$ is one-to-one since α is one-to-one. Hence $\bar{\alpha}$ is an isomorphism from $V/\langle A\rangle$ onto $\operatorname{Im} \alpha/\langle A\rangle\alpha$. Thus $\operatorname{Im} \alpha/\langle A\rangle\alpha \cong V/\langle A\rangle$. But $\dim (V/\langle A\rangle) = |B \smallsetminus A|$ by Proposition 1.12(iv), so $\dim (\operatorname{Im} \alpha/\langle A\alpha\rangle) = |B \smallsetminus A|$. $\square$

Lemma 3.2. *Assume that B is a linearly independent subset of V. If $v_1, v_2, \ldots, v_n \in B$ are distinct and $u_1, u_2, \ldots, u_n \in \langle B \smallsetminus \{v_1, v_2, \ldots, v_n\}\rangle$, then $v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n$ are linearly independent over R.*

Proof. It is clear that for any subset A of B, $\langle A\rangle \cap \langle B \smallsetminus A\rangle = \{0\}$. Let $r_1, r_2, \ldots, r_n \in R$ be such that

$$r_1(v_1 - u_1) + r_2(v_2 - u_2) + \ldots + r_n(v_n - u_n) = 0.$$

Then $r_1v_1 + r_2v_2 + \ldots + r_nv_n = r_1u_1 + r_2u_2 + \ldots + r_nu_n \in \langle\{v_1, v_2, \ldots, v_n\}\rangle \cap \langle B \smallsetminus \{v_1, v_2, \ldots, v_n\}\rangle = \{0\}$. Since $v_1, v_2, \ldots, v_n$ are linearly independent over R, we have that $r_i = 0$ for every $i \in \{1, 2, \ldots, n\}$. This proves that $v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n$ are linearly independent over R. $\square$

We know from Proposition 1.10 that for any set X, $M_X \in \boldsymbol{BQ}$ if and only if $|X| < \infty$ and this is also true for E_X. Following the technique of the given proofs for these facts, we obtain the same results for $M(V)$ and $E(V)$ by replacing $|X|$ by $\dim V$. However, our proofs are more complicated.

Theorem 3.3. *The semigroup $M(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. If $\dim V < \infty$, then $M(V) = G(V)$ which implies by Proposition 1.5 that $M(V) \in \boldsymbol{BQ}$.

For the converse, assume that $\dim V$ is infinite. Let B be a basis of V. Then B is infinite. Let $A = \{\, u_n \mid n \in \mathbb{N} \,\}$ be a subset of B where for any distinct $i, j \in \mathbb{N}$, $u_i \neq u_j$. Let α, β, $\gamma \in L(V)$ be defined by

$$v\alpha = \begin{cases} u_{2n} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} u_{n+1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A \end{cases}$$

and

$$v\gamma = \begin{cases} u_{n+2} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A. \end{cases}$$

By Proposition 1.12(vii), α, β, $\gamma \in M(V)$. From the definitions of α, β and γ, we have that

$$u_n\beta\alpha = u_{2n+2} = u_n\alpha\gamma \quad \text{for all} \quad n \in \mathbb{N}$$

and

$$v\beta\alpha = v = v\alpha\gamma \quad \text{for all } v \in B \smallsetminus A.$$

This implies that $\alpha \neq \beta\alpha = \alpha\gamma$, so $\beta\alpha \in M(V)\alpha \cap \alpha M(V) = (\alpha)_q$ by Proposition 1.4. Suppose that $\beta\alpha \in (\alpha)_b$. Since $\alpha \neq \beta\alpha$, by Proposition 1.4, $\beta\alpha \in \alpha M(V)\alpha$. Let $\lambda \in M(V)$ be such that $\beta\alpha = \alpha\lambda\alpha$. From Proposition 1.9(i), $\beta = \alpha\lambda$. It then follows that

$$B \smallsetminus \{\, u_1 \,\} = B\beta = B\alpha\lambda = (B \smallsetminus \{\, u_{2n-1} \mid n \in \mathbb{N} \,\})\lambda. \tag{1}$$

We have by Lemma 3.1 that

$$\dim\left(\operatorname{Im}\lambda / \langle (B \smallsetminus \{\, u_{2n-1} \mid n \in \mathbb{N} \,\})\lambda \rangle\right) = |\{\, u_{2n-1} \mid n \in \mathbb{N} \,\}|. \tag{2}$$

Thus from (1) and (2) yield that

$$\dim\left(\operatorname{Im}\lambda / \langle B \smallsetminus \{\, u_1 \,\} \rangle\right) = |\{\, u_{2n-1} \mid n \in \mathbb{N} \,\}|. \tag{3}$$

But $\dim\left(V / \langle B \smallsetminus \{\, u_1 \,\} \rangle\right) = |\{u_1\}| = 1$ by Proposition 1.12(iv), so

$$\dim\left(\operatorname{Im}\lambda / \langle B \smallsetminus \{\, u_1 \,\} \rangle\right) \leq \dim\left(V / \langle B \smallsetminus \{\, u_1 \,\} \rangle\right) = 1. \tag{4}$$

We have a contradiction because of (3) and (4). Then $\beta\alpha \notin (\alpha)_b$, so by Proposition 1.8, $M(V) \notin \boldsymbol{BQ}$.

Hence the theorem is completely proved. $\square$

Theorem 3.4. *The semigroup $E(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. If $\dim V < \infty$, then $E(V) = G(V)$, so $E(V) \in \boldsymbol{BQ}$ by Proposition 1.5.

Conversely, assume that $\dim V$ is infinite. Let B be an infinite basis of V and $A = \{ u_n \mid n \in \mathbb{N} \} \subseteq B$ where $u_i \neq u_j$ if $i \neq j$. Define α, β, $\gamma \in L(V)$ by

$$v\alpha = \begin{cases} 0 & \text{if } v = u_n \text{ for some odd } n \in \mathbb{N}, \\ u_{\frac{n}{2}} & \text{if } v = u_n \text{ for some even } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} 0 & \text{if } v = u_1 \text{ or } u_2, \\ u_{n-2} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{ 1, 2 \}, \\ v & \text{if } v \in B \smallsetminus A \end{cases}$$

and

$$v\gamma = \begin{cases} 0 & \text{if } v = u_1, \\ u_{n-1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{ 1 \}, \\ v & \text{if } v \in B \smallsetminus A. \end{cases}$$

Then $\operatorname{Im} \alpha = \operatorname{Im} \beta = \operatorname{Im} \gamma = \langle B \cup \{ 0 \} \rangle = V$, so α, β, $\gamma \in E(V)$. Moreover,

$$\begin{aligned} u_n\beta\alpha = 0 = u_n\alpha\gamma \qquad & \text{if } n = 2 \text{ or } n \text{ is odd}, \\ u_n\beta\alpha = u_{\frac{n-2}{2}} = u_n\alpha\gamma \qquad & \text{if } n > 2 \text{ and } n \text{ is even} \quad \text{and} \\ v\beta\alpha = v = v\alpha\gamma \qquad & \text{for all } v \in B \smallsetminus A. \end{aligned}$$

Consequently, $\alpha \neq \beta\alpha = \alpha\gamma \in E(V)\alpha \cap \alpha E(V)$. By Proposition 1.4, $\alpha\gamma \in (\alpha)_q$. Suppose that $\alpha\gamma \in (\alpha)_b$. By Proposition 1.4, $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in E(V)$. By Proposition 1.9(i), we have $\gamma = \lambda\alpha$. By the definition of γ, we have from Proposition 1.12(viii) that

$$\dim \operatorname{Ker} (\lambda\alpha) = \dim \operatorname{Ker} \gamma = 1. \tag{1}$$

Since $\operatorname{Im}\lambda = V$, for each odd $n \in \mathbb{N}$, $u_n'\lambda = u_n$ for some $u_n' \in V$. Then from Proposition 1.12(v),

$$\{\, u_{2n-1}' \mid n \in \mathbb{N} \,\} \text{ is a linearly independent subset of } V \text{ and}$$
$$\text{for every } n \in \mathbb{N},\ u_{2n-1}'\gamma = u_{2n-1}'\lambda\alpha = u_{2n-1}\alpha = 0. \tag{2}$$

Hence (1) and (2) yield a contradiction. Consequently, $\alpha\gamma \notin (\alpha)_b$. Therefore $E(V) \notin \boldsymbol{BQ}$ by Proposition 1.8.

Therefore the theorem is proved. $\square$

For the study of $OM(V)$, $OE(V)$, $OME(V)$, $BL(V)$ and $OBL(V)$, we always assume that $\dim V$ is infinite. We will show that the semigroups $OM(V)$ and $OE(V)$ are not regular and neither left 0-simple nor right 0-simple but they are always in $\boldsymbol{BQ}$.

Proposition 3.5. *The semigroup $OM(V)$ is not regular.*

Proof. Let B be a basis of V and $A \subseteq B$ such that $|A| = |B \smallsetminus A| = |B|$. Then there exists a bijection $\varphi : B \smallsetminus A \to B$. Define $\alpha \in L(V)$ by

$$v\alpha = \begin{cases} v\varphi & \text{if } v \in B \smallsetminus A, \\ 0 & \text{if } v \in A. \end{cases}$$

Then $\operatorname{Ker}\alpha = \langle A\rangle$ by Proposition 1.12(viii) and $\operatorname{Im}\alpha = \langle \operatorname{Im}\varphi\rangle = \langle B\rangle = V$. Therefore $\alpha \in OM(V)$. Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in L(V)$. Since $\alpha \in E(V)$ by Proposition 1.9(i), $1_V = \beta\alpha$ where 1_V is the identity map on V. This implies that β is one-to-one, so $\beta \notin OM(V)$. This proves that α is not regular in $OM(V)$. Hence $OM(V)$ is not a regular semigroup. $\square$

Proposition 3.6. *The semigroup $OM(V)$ is neither left 0-simple nor right 0-simple.*

Proof. For each $k \in \mathbb{N}$, let

$$A_k = \{\, \alpha \in L(V) \mid \dim \operatorname{Im} \alpha \leq k \,\}.$$

Clearly, $0 \in A_k \neq \{0\}$ for all $k \in \mathbb{N}$. Since $\dim V = \dim \operatorname{Ker} \alpha + \dim \operatorname{Im} \alpha$ for all $\alpha \in L(V)$ (Proposition 1.12(i)) and $\dim V$ is infinite, it follows that $\dim \operatorname{Ker} \alpha$ is infinite for all $\alpha \in A_k$ and for all $k \in \mathbb{N}$. Then $A_k \subseteq OM(V)$. Since for $\alpha, \beta \in L(V)$, $\operatorname{rank}(\alpha\beta) \leq \min\{\operatorname{rank} \alpha, \operatorname{rank} \beta\}$, it follows that A_k is an ideal of $L(V)$. Hence A_k is a nonzero ideal of $OM(V)$.

We can see that $\alpha \in OM(V)$ defined in the proof of Proposition 3.5 is not an element of A_k for all $k \in \mathbb{N}$. Hence A_k is a nonzero proper ideal of $OM(V)$ for every $k \in \mathbb{N}$.

Therefore $OM(V)$ is neither left 0-simple nor right 0-simple. □

As an immediate consequence of the fact that $\operatorname{Ker} \alpha\beta \supseteq \operatorname{Ker} \alpha$ for all $\alpha, \beta \in L(V)$, we have

Lemma 3.7. *The semigroup $OM(V)$ is a right ideal of $L(V)$.*

Theorem 3.8. *The semigroup $OM(V)$ always belongs to $\boldsymbol{BQ}$.*

Proof. To show that $OM(V) \in \boldsymbol{BQ}$ from Proposition 1.3, Proposition 1.4 and Proposition 1.7, it suffices to show that for every nonempty subset X of $OM(V)$, $OM(V)X \cap XOM(V) \subseteq XOM(V)X$.

Let X be a nonempty subset of $OM(V)$ and let $\alpha \in OM(V)X \cap XOM(V)$. Then

$$\alpha = \beta\gamma = \lambda\eta \;\text{ for some }\; \beta, \eta \in OM(V) \;\text{ and }\; \gamma, \lambda \in X. \tag{1}$$

But $L(V)$ is regular, so $\gamma = \gamma\mu\gamma$ for some $\mu \in L(V)$. It then follows from (1) that

$$\alpha = \beta\gamma\mu\gamma = \lambda\eta\mu\gamma = \lambda(\eta\mu)\gamma. \tag{2}$$

By Lemma 3.7, $\eta\mu \in OM(V)$, so from (2), we have $\alpha \in XOM(V)X$. This proves that $OM(V)X \cap XOM(V) \subseteq XOM(V)X$. Hence $OM(V) \in \boldsymbol{BQ}$, as required. $\square$

Proposition 3.9. *The semigroup $OE(V)$ is not regular.*

Proof. Let B be an infinite basis of V and $A \subseteq B$ such that $|A| = |B \smallsetminus A| = |B|$. Then there exists a bijection $\varphi : B \to A$. Define $\alpha \in L(V)$ by $v\alpha = v\varphi$ for all $v \in B$. Then $\operatorname{Im}\alpha = \langle A \rangle$ and by Proposition 1.12(vii), α is one-to-one. By Proposition 1.12(iv), $\dim(V/\operatorname{Im}\alpha) = \dim(V/\langle A \rangle) = |B \smallsetminus A|$. Thus, we have $\alpha \in OE(V)$. If $\alpha = \alpha\beta\alpha$ for some $\beta \in L(V)$, then $\alpha\beta = 1_V$ since α is one-to-one which implies that $\operatorname{Im}\beta = V$, so $\beta \notin OE(V)$. Hence, we deduce that α is not regular in $OE(V)$. We therefore conclude that $OE(V)$ is not a regular semigroup. $\square$

Proposition 3.10. *The semigroup $OE(V)$ is neither left 0-simple nor right 0-simple.*

Proof. For each $k \in \mathbb{N}$, let

$$A_k = \{\, \alpha \in L(V) \mid \dim \operatorname{Im}\alpha \leq k \,\}.$$

As in the proof of Proposition 3.6. We have that A_k is a nonzero ideal of $L(V)$ for all $k \in \mathbb{N}$. Since $\dim V = \dim \operatorname{Im}\alpha + \dim(V/\operatorname{Im}\alpha)$ (Proposition 1.12(ii)) and $\dim V$ is infinite, it follows that $\dim(V/\operatorname{Im}\alpha)$ is infinite for all $\alpha \in A_k$ and for all $k \in \mathbb{N}$. Thus $A_k \subseteq OE(V)$ for every $k \in \mathbb{N}$.

Let $\alpha \in OE(V)$ be defined as in the proof of Proposition 3.9. Since A is an infinite subset of B and $\operatorname{Im} \alpha = \langle A \rangle$, it follows that $\alpha \notin A_k$ for every $k \in \mathbb{N}$. Therefore each A_k is a nonzero proper ideal of $OE(V)$. Hence $OE(V)$ is neither left 0-simple nor right 0-simple. $\square$

Lemma 3.11. *The semigroup $OE(V)$ is a left ideal of $L(V)$.*

Proof. This is clear because of Proposition 1.12(iii) and the fact that $\operatorname{Im} \alpha\beta \subseteq \operatorname{Im} \beta$ for all $\alpha, \beta \in L(V)$. $\square$

Theorem 3.12. *The semigroup $OE(V)$ is always in $\boldsymbol{BQ}$.*

Proof. To prove the theorem, by Proposition 1.3, Proposition 1.4 and Proposition 1.7, it suffices to show that $OE(V)X \cap XOE(V) \subseteq XOE(V)X$ for any nonempty subset X of $OE(V)$.

Let X be a nonempty subset of $OE(V)$ and let $\alpha \in OE(V)X \cap XOE(V)$. Then

$$\alpha = \beta\gamma = \lambda\eta \text{ for some } \beta, \eta \in OE(V) \text{ and } \gamma, \lambda \in X. \tag{1}$$

Since $L(V)$ is regular, $\lambda = \lambda\mu\lambda$ for some $\mu \in L(V)$. Then from (1),

$$\alpha = \lambda\mu\lambda\eta = \lambda\mu\beta\gamma = \lambda(\mu\beta)\gamma. \tag{2}$$

By Lemma 3.11, $\mu\beta \in OE(V)$. It then follows from (2) that $\alpha \in XOE(V)X$.

Therefore the theorem is proved. $\square$

We will show that $OME(V)$ is always regular and hence it is in $\boldsymbol{BQ}$ by Proposition 1.5.

Lemma 3.13. *The semigroup $OME(V)$ is a regular semigroup.*

Proof. To show that $OME(V)$ is regular, let $\alpha \in OME(V)$. Let B_1 and B_2 be respectively bases of $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$. Then B_1 is infinite. Let B be a basis of V containing B_2. By Proposition 1.12(iv), $\dim(V/\operatorname{Im}\alpha) = \dim(V/\langle B_2\rangle) = |B \smallsetminus B_2|$. But $\alpha \in OE(V)$, so $B \smallsetminus B_2$ is infinite. For each $v \in B_2$, there exists an element $u_v \in V$ such that $u_v\alpha = v$. It then follows that $|\{\, u_v \mid v \in B_2 \,\}| = |B_2|$. Moreover, $B_1 \cup \{\, u_v \mid v \in B_2 \,\}$ is a basis of V by Proposition 1.12(vi). Define $\beta \in L(V)$ by

$$v\beta = \begin{cases} u_v & \text{if } v \in B_2, \\ 0 & \text{if } v \in B \smallsetminus B_2. \end{cases}$$

Then $\operatorname{Ker}\beta = \langle B \smallsetminus B_2\rangle$ by Proposition 1.12(viii) and $\operatorname{Im}\beta = \langle\{\, u_v \mid v \in B_2 \,\}\rangle$. We therefore have $\dim\operatorname{Ker}\beta = |B \smallsetminus B_2|$. Since $B_1 \cup \{\, u_v \mid v \in B_2 \,\}$ is a basis of V by Proposition 1.12(vi), we have by Proposition 1.12(iv) that

$$\dim(V/\operatorname{Im}\beta) = |(B_1 \cup \{\, u_v \mid v \in B_2 \,\}) \smallsetminus \{\, u_v \mid v \in B_2 \,\}| = |B_1|.$$

It then follows that $\beta \in OME(V)$. Since $B_1 \cup \{\, u_v \mid v \in B_2 \,\}$ is a basis of V and

$$v\alpha\beta\alpha = (v\alpha)\beta\alpha = 0\beta\alpha = 0 = v\alpha \text{ for all } v \in B_1$$

$$u_v\alpha\beta\alpha = (u_v\alpha)\beta\alpha = v\beta\alpha = (v\beta)\alpha = u_v\alpha \text{ for all } v \in B_2,$$

we deduce that $\alpha\beta\alpha = \alpha$. This proves that $OME(V)$ is regular, as required. □

The following theorem is obtained directly from Lemma 3.13 and Proposition 1.5.

Theorem 3.14. *The semigroup $OME(V)$ always belongs to* $\boldsymbol{BQ}$.

Observe that $OM(V)$ and $OE(V)$ are not regular but $OME(V)(= OM(V) \cap OE(V))$ is. However, $OM(V)$ and $OE(V)$ are neither left 0-simple nor right 0-simple and neither is $OME(V)$ as shown in the following proposition.

Proposition 3.15. *The semigroup $OME(V)$ is neither left 0-simple nor right 0-simple.*

Proof. For each $k \in \mathbb{N}$, define

$$A_k = \{\, \alpha \in L(V) \mid \dim \operatorname{Im} \alpha \leq k \,\} \tag{1}$$

as in the proof of Proposition 3.6. By the proof of Proposition 3.6 and Proposition 3.10, each A_k is a nonzero ideal of $OM(V)$ and $OE(V)$, respectively. Then A_k is a nonzero ideal of $OME(V)(= OM(V) \cap OE(V))$. From (1), we have

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

Let B be a basis of V and let $u_1, u_2, u_3, \dots$ be distinct elements of B. For each positive integer k, define $\alpha_k \in L(V)$ by

$$v\alpha_k = \begin{cases} v & \text{if } v \in \{\, u_1, u_2, \dots u_k \,\}, \\ 0 & \text{if } v \in B \smallsetminus \{\, u_1, u_2, \dots u_k \,\}. \end{cases}$$

Then for every $k \in \mathbb{N}$, $\operatorname{Im} \alpha_k = \langle u_1, u_2, \dots u_k \rangle$ and hence $\dim \operatorname{Im} \alpha_k = k$. Thus for every $k > 1$, $\alpha_k \in A_k \smallsetminus A_{k-1}$. Consequently,

$$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots .$$

Therefore each A_k is a nonzero proper ideal of $OME(V)$.

Hence $OME(V)$ is neither left 0-simple nor right 0-simple. $\square$

As was mentioned in Chapter I, BL_X is right simple if X is a countably infinite set. This is also true that $BL(V)$ is right simple if $\dim V = \aleph_0$. We give this fact as a lemma in order to prove the next theorem, analogous to Proposition 1.11. That is, to prove that $BL(V) \in \boldsymbol{BQ}$ if and only if $\dim V = \aleph_0$. The technique of the proof Proposition 1.11 is helpful for the proof of this theorem.

Lemma 3.16. *If* $\dim V = \aleph_0$*, then* $BL(V)$ *is right simple .*

Proof. By assumption, we have that for every $\alpha \in BL(V)$, $\dim(V/\operatorname{Im}\alpha) = \aleph_0$. We will show that $BL(V)$ is right simple by Proposition 1.1. This is equivalent to show that for all α, $\beta \in BL(V)$, there exists $\gamma \in BL(V)$ such that $\alpha\gamma = \beta$. Let α, $\beta \in BL(V)$ be arbitrary fixed. Since α and β are one-to-one linear transformations of V, we have that $\alpha^{-1}\beta\colon \operatorname{Im}\alpha \to \operatorname{Im}\beta$ is an isomorphism.

Let B_1 be a basis of $\operatorname{Im}\alpha$ and $B_2 = B_1\alpha^{-1}\beta$. Then B_2 is a basis of $\operatorname{Im}\beta$. Let B and B' be bases of V such that $B_1 \subseteq B$ and $B_2 \subseteq B'$. It then follows from Proposition 1.12(iv), that

$$\dim(V/\operatorname{Im}\alpha) = |B \smallsetminus B_1| \quad \text{and} \quad \dim(V/\operatorname{Im}\beta) = |B' \smallsetminus B_2|.$$

Consequently, $|B \smallsetminus B_1| = |B| = |B' \smallsetminus B_2| = \aleph_0$. Let $A \subseteq B' \smallsetminus B_2$ be such that $|A| = |B' \smallsetminus B_2| = |(B' \smallsetminus B_2) \smallsetminus A|$. Thus

$$|(B' \smallsetminus B_2) \smallsetminus A| = \aleph_0. \tag{1}$$

Then there exists a bijection $\varphi\colon B \smallsetminus B_1 \to A$. Define $\gamma \in L(V)$ by

$$v\gamma = \begin{cases} v\alpha^{-1}\beta & \text{if } v \in B_1, \\ v\varphi & \text{if } v \in B \smallsetminus B_1. \end{cases} \tag{2}$$

Since $B_1\alpha^{-1}\beta \cap (B \smallsetminus B_1)\varphi = B_2 \cap A = \varnothing$, it follows that $\gamma|_B : B \to B_2 \cup A \subseteq B'$ is a bijection. Hence γ is one-to-one by Proposition 1.12(vii), so $\gamma \in M(V)$. Also, we have from Proposition 1.12(iv) that

$$\dim(V/\operatorname{Im}\gamma) = \dim(V/\langle B_2 \cup A\rangle) = |B' \smallsetminus (B_2 \cup A)| = |(B' \smallsetminus B_2) \smallsetminus A|. \tag{3}$$

Then $\gamma \in OE(V)$ by (1) and (3). But since $BL(V) = M(V) \cap OE(V)$, $\gamma \in BL(V)$. Because $\operatorname{Im}\alpha = \langle B_1\rangle$, we deduce from (2) that $\gamma|_{\operatorname{Im}\alpha} = \alpha^{-1}\beta$. This implies that $\alpha\gamma = \beta$.

Hence $BL(V)$ is right simple, as desired. □

Theorem 3.17. *The semigroup $BL(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V = \aleph_0$.*

Proof. If $\dim V = \aleph_0$, then $BL(V) \in \boldsymbol{BQ}$ by Lemma 3.16 and Proposition 1.6.

For the converse, assume that $\dim V \neq \aleph_0$. Since $\dim V$ is infinite, $\dim V > \aleph_0$. Let B be a basis of V. Then B is uncountable. Let A and C be subsets of B such that

$$A \subseteq C,\ |B \smallsetminus C| = |C| = |B| \quad \text{and} \quad |C \smallsetminus A| = |A| = |C|. \tag{1}$$

Let D be a countably infinite subset of B. Since B is uncountable,

$$|B \smallsetminus D| = |B|.$$

Then there are α, $\beta \in L(V)$ such that $\alpha|_B \colon B \to B \smallsetminus C$ and $\beta|_B \colon B \to B \smallsetminus D$ are bijections. By Proposition 1.12(vii), we have that α, $\beta \in M(V)$. By Proposition 1.12(iv),

$$\begin{aligned} &\dim(V/\operatorname{Im}\alpha) = \dim(V/\langle B \smallsetminus C\rangle) = |C| \quad \text{and} \\ &\dim(V/\operatorname{Im}\beta) = \dim(V/\langle B \smallsetminus D\rangle) = |D|, \end{aligned} \tag{2}$$

so we have that α, $\beta \in OE(V)$. Thus α, $\beta \in BL(V)$. Also we have

$$(B \smallsetminus C)\alpha^{-1}\beta\alpha = (B \smallsetminus D)\alpha \subseteq B \smallsetminus C. \tag{3}$$

Since $|C| = |A|$, there is a bijection $\varphi \colon C \to A$. By (1), $C\varphi \subseteq C$. Define $\gamma \in L(V)$ by

$$v\gamma = \begin{cases} v\alpha^{-1}\beta\alpha & \text{if } v \in B \smallsetminus C, \\ v\varphi & \text{if } v \in C. \end{cases} \tag{4}$$

Because of (3) and $C\varphi \subseteq A$, we have $(B \smallsetminus C)\alpha^{-1}\beta\alpha \cap C\varphi \subseteq (B \smallsetminus C) \cap C = \varnothing$. Then $\gamma|_B : B \to (B \smallsetminus C) \cup A \subseteq B$ is one-to-one. Hence $\gamma \in M(V)$ by Proposition

1.12(vii) and $\operatorname{Im}\gamma \subseteq \langle (B \smallsetminus C) \cup A \rangle$. Since

$$\begin{aligned} \dim(V/\operatorname{Im}\gamma) &\geq \dim(V/\langle (B \smallsetminus C) \cup A \rangle) && \text{from Proposition 1.12(iii)} \\ &= |B \smallsetminus ((B \smallsetminus C) \cup A)| && \text{from Proposition 1.12(iv)} \\ &= |C \smallsetminus A| \\ &= |B| && \text{from (1),} \end{aligned}$$

we have $\gamma \in OE(V)$. Hence $\gamma \in BL(V)$. We have by (4) that $\gamma|_{\operatorname{Im}\alpha} = \alpha^{-1}\beta\alpha$. Consequently, $\beta\alpha = \alpha\gamma \in BL(V)\alpha \cap \alpha BL(V)$. By Proposition 1.4, $\beta\alpha \in (\alpha)_q$. To show that $\beta\alpha \notin (\alpha)_b$, suppose on the contrary that $\beta\alpha \in (\alpha)_b$. By Proposition 1.4, $\beta\alpha = \alpha$, $\beta\alpha = \alpha^2$ or $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in BL(V)$. By Proposition 1.9(i), $\beta = 1_V$, $\beta = \alpha$ or $\beta = \alpha\lambda$. Since C is uncountable, D is countable, $B\alpha = B \smallsetminus C$ and $B\beta = B \smallsetminus D$, we deduce that $\beta \neq 1_V$ and $\beta \neq \alpha$. Then $\beta = \alpha\lambda$. Hence

$$\operatorname{Im}\beta = \operatorname{Im}(\alpha\lambda) = (\operatorname{Im}\alpha)\lambda = \langle B \smallsetminus C \rangle \lambda = \langle (B \smallsetminus C)\lambda \rangle. \tag{5}$$

Consequently,

$$\begin{aligned} |D| &= \dim(V/\operatorname{Im}\beta) && \text{from (2)} \\ &= \dim(V/\langle B \smallsetminus C \rangle \lambda) && \text{from (5)} \\ &\geq \dim(\operatorname{Im}\lambda/\langle B \smallsetminus C \rangle \lambda) \\ &= |B \smallsetminus (B \smallsetminus C)| && \text{from Lemma 3.1} \\ &= |C|. \end{aligned}$$

This contradicts the facts that D is countable but C is uncountable. Therefore $\beta\alpha \notin (\alpha)_b$. By Proposition 1.8, $BL(V) \notin \boldsymbol{BQ}$.

Hence the theorem is completely proved. $\square$

If $BL(V)$ is right simple, by Proposition 1.6, $BL(V) \in \boldsymbol{BQ}$ which implies by Theorem 3.17 that $\dim V = \aleph_0$. That is, the converse of Lemma 3.16 holds.

Corollary 3.18. *$\dim V = \aleph_0$ if and only if $BL(V)$ is a right simple semigroup.*

Next, to show that $\dim V = \aleph_0$ is also necessary and sufficient for the semigroup $OBL(V)$ belongs to $\boldsymbol{BQ}$, we first show as a lemma that if $\dim V = \aleph_0$, then $OBL(V)$ is left simple. Recall that $OBL(V) = OM(V) \cap E(V)$.

Lemma 3.19. *If $\dim V = \aleph_0$, then $OBL(V)$ is left simple.*

Proof. To show that $OBL(V)$ is left simple by Proposition 1.1 which is equivalent to show that for all $\alpha, \beta \in OBL(V)$, $\gamma\alpha = \beta$ for some $\gamma \in OBL(V)$.

Let $\alpha, \beta \in OBL(V)$ and let B be a basis of V. Then B is countably infinite. Then for every infinite subset A of B, $|A| = |B|$. Since $\operatorname{Im}\alpha = \operatorname{Im}\beta = V$, for every $v \in B$, there exist $u_v, w_v \in V$ such that

$$u_v\alpha = w_v\beta = v. \tag{1}$$

Then for distinct $v_1, v_2 \in B$, $u_{v_1} \neq u_{v_2}$ and $w_{v_1} \neq w_{v_2}$. This implies that

$$|\{\, u_v \mid v \in B \,\}| = |B| = |\{\, w_v \mid v \in B \,\}|.$$

Let B_1 and B_2 be respectively bases of $\operatorname{Ker}\alpha$ and $\operatorname{Ker}\beta$. Then B_1 and B_2 are countably infinite. By Proposition 1.12(vi), we have that $B_1 \cup \{\, u_v \mid v \in B \,\}$ and $B_2 \cup \{\, w_v \mid v \in B \,\}$ are both bases of V. Next, let C be a subset of B_2 such that $|C| = |B_2| = |B_2 \smallsetminus C|$. Then there is a bijection $\varphi : C \to B_1$. Note that $C \cup (B_2 \smallsetminus C) \cup \{\, w_v \mid v \in B \,\}$ is a disjoint union and it is a basis of V. Define $\gamma \in L(V)$ by

$$\begin{aligned} &\text{for every } v \in C, v\gamma = v\varphi, \\ &\text{for every } v \in B_2 \smallsetminus C, v\gamma = 0, \\ &\text{for every } v \in B, w_v\gamma = u_v. \end{aligned} \tag{2}$$

Since $B_1 \cup \{ u_v \mid v \in B \}$ is a basis of V,

$$\operatorname{Im} \gamma = \langle C\varphi \cup \{ u_v \mid v \in B \} \rangle = \langle B_1 \cup \{ u_v \mid v \in B \} \rangle = V.$$

Also, $\operatorname{Ker} \gamma = \langle B_2 \smallsetminus C \rangle$ by Proposition 1.12(viii) and hence $\dim \operatorname{Ker} \gamma = |B_2 \smallsetminus C| = |B_2| = \dim \operatorname{Ker} \beta$. Thus $\gamma \in OBL(V)$. To show that $\gamma\alpha = \beta$, let $v \in B$. Then $v \in C$, $v \in B_2 \smallsetminus C$ or $v = w_z$ for some $z \in B$.

Case 1: $v \in C$. Since $C \subseteq B_2 \subseteq \operatorname{Ker} \beta$, $v\beta = 0$. From (2), $v\gamma = v\varphi \in B_1 \subseteq \operatorname{Ker} \alpha$. This implies that $v\gamma\alpha = 0$. Hence $v\gamma\alpha = v\beta$.

Case 2: $v \in B_2 \smallsetminus C$. Since $B_2 \smallsetminus C \subseteq B_2 \subseteq \operatorname{Ker} \beta$, $v\beta = 0$. By (2), $v\gamma = 0$. Thus $v\gamma\alpha = 0 = v\beta$.

Case 3: $v = w_z$ for some $z \in B$. Then by (1), $u_z\alpha = z = w_z\beta$. From (2), $v\gamma = w_z\gamma = u_z$ and thus $v\gamma\alpha = u_z\alpha = z = w_z\beta = v\beta$.

Hence $\gamma\alpha = \beta$.

This proves that $OBL(V)$ is left simple, as required. □

Theorem 3.20. *The semigroup $OBL(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V = \aleph_0$.*

Proof. First, assume that $\dim V \neq \aleph_0$. Then $\dim V > \aleph_0$ since $\dim V$ is infinite. Let B be a basis of V and let $C \subseteq B$ be such that $|B \smallsetminus C| = |C| = |B|$. Let D_1 and D_2 be countably infinite subsets of C and $B \smallsetminus C$, respectively. Since C and $B \smallsetminus C$ are uncountable, we have

$$|(B \smallsetminus C) \smallsetminus D_2| = |B \smallsetminus C| = |B \smallsetminus D_1| = |B| \text{ and } |C \smallsetminus D_1| = |C| = |B|.$$

Then there are bijections $\varphi_1 : D_2 \to D_1$, $\varphi_2 : (B \smallsetminus C) \smallsetminus D_2 \to B \smallsetminus D_1$, $\varphi_3 : C \smallsetminus D_1 \to C$ and $\varphi_4 : (B \smallsetminus C) \smallsetminus D_2 \to B \smallsetminus C$. By the choices of C, D_1 and D_2, we have

$$B = ((B \smallsetminus C) \smallsetminus D_2) \cup D_2 \cup C = (C \smallsetminus D_1) \cup ((B \smallsetminus C) \smallsetminus D_2) \cup D_1 \cup D_2$$

$$\text{which are disjoint unions .} \tag{1}$$

We also have $\varphi_2^{-1}\varphi_4 : B \smallsetminus D_1 \to B \smallsetminus C$ is a bijection. It then follows that

$$(B \smallsetminus D_1)\varphi_2^{-1}\varphi_4 = ((B \smallsetminus C) \smallsetminus D_2) \cup D_2$$

$$\text{which is a disjoint union .} \tag{2}$$

Next, define $\alpha, \beta, \gamma \in L(V)$ by

$$v\alpha = \begin{cases} v\varphi_2 & \text{if } v \in (B \smallsetminus C) \smallsetminus D_2, \\ v\varphi_1 & \text{if } v \in D_2, \\ 0 & \text{if } v \in C, \end{cases}$$

$$v\beta = \begin{cases} v\varphi_3 & \text{if } v \in C \smallsetminus D_1, \\ v\varphi_4 & \text{if } v \in (B \smallsetminus C) \smallsetminus D_2, \\ 0 & \text{if } v \in D_1 \cup D_2 \end{cases}$$

and

$$v\gamma = \begin{cases} v\varphi_2^{-1}\varphi_4\varphi_1 & \text{if } v \in B \smallsetminus D_1 \text{ and } v\varphi_2^{-1}\varphi_4 \in D_2, \\ v\varphi_2^{-1}\varphi_4\varphi_2 & \text{if } v \in B \smallsetminus D_1 \text{ and } v\varphi_2^{-1}\varphi_4 \in (B \smallsetminus C) \smallsetminus D_2, \\ 0 & \text{if } v \in D_1. \end{cases}$$

We have that α and β are well-defined by (1) and γ is well-defined by (2). From Proposition 1.12(viii) and the definitions of α and β, we get

$$\operatorname{Im} \alpha = \langle ((B \smallsetminus C) \smallsetminus D_2)\varphi_2 \cup D_2\varphi_1 \rangle = \langle (B \smallsetminus D_1) \cup D_1) \rangle = \langle B \rangle = V,$$

$$\dim \operatorname{Ker} \alpha = \dim \langle C \rangle = |C|, \tag{3}$$

$$\operatorname{Im} \beta = \langle (C \smallsetminus D_1)\varphi_3 \cup ((B \smallsetminus C) \smallsetminus D_2)\varphi_4 \rangle = \langle C \cup (B \smallsetminus C) \rangle = \langle B \rangle = V,$$

$$\dim \operatorname{Ker} \beta = \dim \langle D_1 \cup D_2 \rangle = |D_1 \cup D_2|.$$

By (2), Proposition1.12(viii) and the definitions of φ_1, φ_2 and γ, we have

$$\operatorname{Im}\gamma = \langle D_2\varphi_1 \cup ((B \smallsetminus C) \smallsetminus D_2))\varphi_2\rangle = \langle D_1 \cup (B \smallsetminus D_1)\rangle = \langle B\rangle = V,$$

$$\dim \operatorname{Ker}\gamma = \dim\langle D_1\rangle = |D_1|. \tag{4}$$

Consequently, $\alpha, \beta, \gamma \in OBL(V)$. We claim that $\beta\alpha = \alpha\gamma$. Let $v \in B$. Then v belongs to one of the following subsets of B: $D_1, D_2, C \smallsetminus D_1$ and $(B \smallsetminus C) \smallsetminus D_2$.

Case 1: $v \in D_1$. Then $v\beta\alpha = 0\alpha = 0$. Since $D_1 \subseteq C$, $v\alpha\gamma = 0\gamma = 0$.

Case 2: $v \in D_2$. Then $v\beta\alpha = 0\alpha = 0$. Since $v\alpha = v\varphi_1 \in D_1$, $v\alpha\gamma = 0$.

Case 3: $v \in C \smallsetminus D_1$. Then $v\alpha\gamma = 0\gamma = 0$. But $v\beta = v\varphi_3 \in C$, so $v\beta\alpha = 0$.

Case 4: $v \in (B \smallsetminus C) \smallsetminus D_2$. Then $v\beta = v\varphi_4 \in B \smallsetminus C$, so

$$v\beta\alpha = \begin{cases} v\varphi_4\varphi_1 & \text{if } v\varphi_4 \in D_2, \\ v\varphi_4\varphi_2 & \text{if } v\varphi_4 \in (B \smallsetminus C) \smallsetminus D_2. \end{cases}$$

Since $v\alpha = v\varphi_2 \in B \smallsetminus D_1$, we have

$$v\alpha\gamma = \begin{cases} v\alpha\varphi_2^{-1}\varphi_4\varphi_1 & \text{if } v\alpha\varphi_2^{-1}\varphi_4 \in D_2, \\ v\alpha\varphi_2^{-1}\varphi_4\varphi_2 & \text{if } v\alpha\varphi_2^{-1}\varphi_4 \in (B \smallsetminus C) \smallsetminus D_2. \end{cases}$$

But $v\alpha\varphi_2^{-1}\varphi_4 = v\varphi_2\varphi_2^{-1}\varphi_4 = v\varphi_4$, so

$$v\alpha\varphi_2^{-1}\varphi_4\varphi_1 = v\varphi_4\varphi_1 \text{ if } v\varphi_4 \in D_2,$$

$$v\alpha\varphi_2^{-1}\varphi_4\varphi_2 = v\varphi_4\varphi_2 \text{ if } v\varphi_4 \in (B \smallsetminus C) \smallsetminus D_2.$$

We then conclude that

$$v\alpha\gamma = \begin{cases} v\varphi_4\varphi_1 & \text{if } v\varphi_4 \in D_2, \\ v\varphi_4\varphi_2 & \text{if } v\varphi_4 \in (B \smallsetminus C) \smallsetminus D_2. \end{cases}$$

This proves that $v\beta\alpha = v\alpha\gamma$ for every $v \in B$. Hence $\beta\alpha = \alpha\gamma \in OBL(V)\alpha \cap \alpha OBL(V)$. By Proposition 1.4, $\alpha\gamma \in (\alpha)_q$. Suppose that $\alpha\gamma \in (\alpha)_b$. By Proposition 1.4, $\alpha\gamma = \alpha$, $\alpha\gamma = \alpha^2$ or $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in OBL(V)$. By Proposition

1.9(i), $\gamma = 1_V$, $\gamma = \alpha$ or $\gamma = \lambda\alpha$. By the definition of γ, we have that $\gamma \neq 1_V$. Since D_1 is countable and C is uncountable, from (3) and (4), $\gamma \neq \alpha$. Then $\gamma = \lambda\alpha$. Since $\operatorname{Im}\lambda = V$, for each $v \in C$, there exists an element $u_v \in V$ such that $u_v\lambda = v$. Then $|\{ u_v \mid v \in C \}| = |C|$, so $\{ u_v \mid v \in C \}$ is uncountable. Since C is a linearly independent subset of V over R, by Proposition 1.12(v), $\{ u_v \mid v \in C \}$ is linearly independent over R. But since $\operatorname{Ker}\alpha = \langle C \rangle$, so for every $v \in C$, $u_v\lambda\alpha = v\alpha = 0$. It then follows that $\{ u_v \mid v \in C \} \subseteq \operatorname{Ker}\lambda\alpha$. Hence $\dim \operatorname{Ker}\lambda\alpha$ is uncountable. Then by (4), that $\gamma = \lambda\alpha$ is impossible. Therefore $\alpha\gamma \notin (\alpha)_b$. Thus $(\alpha)_b \neq (\alpha)_q$. By Proposition 1.8, we have that $OBL(V)$ is not in $\boldsymbol{BQ}$. This proves that if $OBL(V)$ belongs to $\boldsymbol{BQ}$, then $\dim V = \aleph_0$.

The converse of the theorem follows directly from Lemma 3.19 and Proposition 1.6. □

Corollary 3.21. *$\dim V = \aleph_0$ if and only if $OBL(V)$ is a left simple semigroup.*

Proof. Assume that $OBL(V)$ is left simple. Then $OBL(V) \in \boldsymbol{BQ}$ by Proposition 1.6. Therefore we have by Theorem 3.20 that $\dim V = \aleph_0$.

The converse is Lemma 3.19. □

Next, assume that V is a vector space over a division ring R of any dimension. Recall from Chapter I, page 14, that if $\dim V < \infty$, then $AM(V) = AE(V) = L(V)$. Since $L(V)$ is a regular semigroup, it follows from Proposition 1.5 that

(1) if $\dim V < \infty$, then $AM(V) \in \boldsymbol{BQ}$ and

(2) if $\dim V < \infty$, then $AE(V) \in \boldsymbol{BQ}$.

The next two theorems show that the converses of (1) and (2) are also true.

Theorem 3.22. *The semigroup $AM(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. If $\dim V < \infty$, then $AM(V) \in \boldsymbol{BQ}$, as was mentioned above.

For the converse, assume that $\dim V$ is infinite. Let B be a basis of V and $A = \{ u_n \mid n \in \mathbb{N} \} \subseteq B$ where $u_i \neq u_j$ if $i \neq j$. Define α, β, $\gamma \in L(V)$ by

$$v\alpha = \begin{cases} u_{2n} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} u_{n+1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

and

$$v\gamma = \begin{cases} u_{n+2} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A. \end{cases}$$

Then α, β, $\gamma \in M(V)$ by Proposition 1.12(vii), so α, β, $\gamma \in AM(V)$. Since

$$\text{for every } n \in \mathbb{N},\ u_n\beta\alpha = u_{2n+2} = u_n\alpha\gamma \quad \text{and}$$

$$\text{for every } v \in B \smallsetminus A,\ v\beta\alpha = v = v\alpha\gamma,$$

we deduce that $\alpha \neq \beta\alpha = \alpha\gamma$. Thus $\beta\alpha \in AM(V)\alpha \cap \alpha AM(V) = (\alpha)_q$ by Proposition 1.4. Suppose that $\beta\alpha \in (\alpha)_b$. By Proposition 1.4, $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in AM(V)$. Since α is one-to-one, we conclude that $\beta = \alpha\lambda$. This implies by the definitions of α and β that

$$B \smallsetminus \{ u_1 \} = B\beta = B\alpha\lambda = (B\alpha)\lambda = (B \smallsetminus \{ u_1, u_3, u_5, \ldots \})\lambda.$$

For convenience, for each $n \in \mathbb{N}$, let $w_n = u_{2n-1}$. Thus

$$\langle B \smallsetminus \{ w_n \mid n \in \mathbb{N} \}\rangle\lambda = \langle (B \smallsetminus \{ w_n \mid n \in \mathbb{N} \})\lambda\rangle = \langle B \smallsetminus \{u_1\}\rangle. \tag{1}$$

But $V = \langle B \smallsetminus \{u_1\}\rangle + \langle u_1\rangle$, so by (1), we have

$$V = \langle B \smallsetminus \{ w_n \mid n \in \mathbb{N} \}\rangle\lambda + Ru_1 \tag{2}$$

We then have by (2) that for each $n \in \mathbb{N}$, there exist $v_n \in \langle B \smallsetminus \{ w_n \mid n \in \mathbb{N} \} \rangle$ and $a_n \in R$ such that $w_n\lambda = v_n\lambda + a_n u_1$. Therefore

$$\text{for every } n \in \mathbb{N}, (w_n - v_n)\lambda \in \langle u_1 \rangle. \tag{3}$$

It follows from Lemma 3.2 that the set $\{ w_n - v_n \mid n \in \mathbb{N} \}$ is linearly independent over R and $w_i - v_i \neq w_j - v_j$ if $i \neq j$. Set $W = \langle \{ w_n - v_n \mid n \in \mathbb{N} \} \rangle$. Then $\dim W$ is infinite. From (3), $\lambda|_W : W \to \langle u_1 \rangle$ and hence $\dim \operatorname{Im}(\lambda|_W) \leq 1$. By Proposition 1.12(i),

$$\dim W = \dim \operatorname{Ker}(\lambda|_W) + \dim \operatorname{Im}(\lambda|_W).$$

We thus conclude that $\dim \operatorname{Ker}(\lambda|_W)$ is infinite. But $\operatorname{Ker}\lambda \supseteq \operatorname{Ker}(\lambda|_W)$, so $\dim \operatorname{Ker}\lambda$ is infinite. It is a contradiction since $\lambda \in AM(V)$. This proves that $(\alpha)_q \neq (\alpha)_b$. Therefore $AM(V) \notin \boldsymbol{BQ}$ by Proposition 1.8.

Hence the proof of the theorem is complete. □

Theorem 3.23. *The semigroup $AE(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. If $\dim V < \infty$, then $AE(V) \in \boldsymbol{BQ}$, as mentioned previously.

On the other hand, assume that $\dim V$ is infinite. Let B be a basis of V and $A = \{ u_n \mid n \in \mathbb{N} \} \subseteq B$ where $u_i \neq u_j$ if $i \neq j$. Define α, β, $\gamma \in L(V)$ by

$$v\alpha = \begin{cases} 0 & \text{if } v = u_n \text{ for some odd } n \in \mathbb{N}, \\ u_{\frac{n}{2}} & \text{if } v = u_n \text{ for some even } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} 0 & \text{if } v = u_1 \text{ or } u_2, \\ u_{n-2} & \text{if } v = u_n \text{ for some } n > 2, \\ v & \text{if } v \in B \smallsetminus A \end{cases}$$

and

$$v\gamma = \begin{cases} 0 & \text{if } v = u_1, \\ u_{n-1} & \text{if } v = u_n \text{ for some } n > 1, \\ v & \text{if } v \in B \smallsetminus A. \end{cases}$$

Then $\operatorname{Im}\alpha = \operatorname{Im}\beta = \operatorname{Im}\gamma = \langle B \cup \{0\}\rangle = B$. Thus α, β, $\gamma \in E(V) \subseteq AE(V)$ and

$$\begin{aligned} u_n\beta\alpha &= 0 = u_n\alpha\gamma && \text{if } n = 2 \text{ or } n \text{ is odd,} \\ u_n\beta\alpha &= u_{\frac{n-2}{2}} = u_n\alpha\gamma && \text{if } n > 2 \text{ and } n \text{ is even} \quad \text{and} \\ v\beta\alpha &= v = v\alpha\gamma && \text{for all } v \in B \smallsetminus A. \end{aligned}$$

Thus $\alpha \neq \beta\alpha = \alpha\gamma$. Consequently, $\alpha\gamma \in AE(V)\alpha \cap \alpha AE(V) = (\alpha)_q$ by Proposition 1.4. Suppose that $\alpha\gamma \in (\alpha)_b$. By Proposition 1.4, $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in AE(V)$. Since $\alpha \in E(V)$, by Proposition 1.9(i), $\gamma = \lambda\alpha$. Then $u_1\lambda\alpha = u_1\gamma = 0$, so $u_1\lambda \in \operatorname{Ker}\alpha$. By the definition of α and Proposition 1.12(viii),

$$\operatorname{Ker}\alpha = \langle\{\, u_n \mid n \in \mathbb{N} \text{ and } n \text{ is odd} \,\}\rangle.$$

For convenience, let $w_n = u_{2n-1}$ for every $n \in \mathbb{N}$. Thus

$$\operatorname{Ker}\alpha = \langle\{\, w_n \mid n \in \mathbb{N} \,\}\rangle. \tag{1}$$

Since $u_1\lambda \in \operatorname{Ker}\alpha$, there are $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in R$ such that

$$u_1\lambda = \sum_{n=1}^{k} a_n w_n. \tag{2}$$

We claim that $\{\, w_{k+n} + \operatorname{Im}\lambda \mid n \in \mathbb{N} \,\}$ is a linearly independent infinite subset of $V/\operatorname{Im}\lambda$. To prove this, let $l \in \mathbb{N}$ and $b_1, b_2, \ldots, b_l \in R$ be such that

$$\sum_{n=1}^{l} b_n(w_{k+n} + \operatorname{Im}\lambda) = \operatorname{Im}\lambda.$$

It then follows that $\sum_{n=1}^{l} b_n w_{k+n} \in \operatorname{Im}\lambda$, so there exists $z \in V$ such that

$$z\lambda = \sum_{n=1}^{l} b_n w_{k+n} \tag{3}$$

Since $V = \langle B\rangle = \langle A\rangle + \langle B \smallsetminus A\rangle$, there are $m \in \mathbb{N}$, $c_1, c_2, \ldots, c_m \in R$ and $u' \in \langle B \smallsetminus A\rangle$ such that

$$z = \sum_{n=1}^{m} c_n u_n + u'. \tag{4}$$

We may choose $m > 1$. From (3) and (4), we have

$$\sum_{n=1}^{l} b_n w_{k+n} = \sum_{n=1}^{m} c_n(u_n\lambda) + u'\lambda. \tag{5}$$

By (1) and (5), we get

$$0 = (\sum_{n=1}^{l} b_n w_{k+n})\alpha = \sum_{n=1}^{m} c_n(u_n\lambda\alpha) + u'\lambda\alpha.$$

Since $\lambda\alpha = \gamma$ and $u_1\gamma = 0$, we have

$$\sum_{n=2}^{m} c_n(u_n\gamma) = -u'\gamma. \tag{6}$$

From the definition of γ, we have

$$\sum_{n=2}^{m} c_n(u_n\gamma) = \sum_{n=2}^{m} c_n u_{n-1} \in \langle A\rangle \quad \text{and} \quad u'\gamma = u' \in \langle B \smallsetminus A\rangle. \tag{7}$$

But $\langle A\rangle \cap \langle B \smallsetminus A\rangle = \{0\}$, so (6) and (7) yield $u' = 0$ and $c_2 = c_3 = \ldots = c_m = 0$. It then follows from (5) that

$$\sum_{n=1}^{l} b_n w_{k+n} = c_1(u_1\lambda).$$

From this equality and (2), we obtain the following equality.

$$\sum_{n=1}^{l} b_n w_{k+n} = c_1 \sum_{n=1}^{k} a_n w_n.$$

This implies that $c_1a_1w_1 + c_1a_2w_2 + \ldots + c_1a_kw_k - b_1w_{k+1} - b_2w_{k+2} - \ldots - b_lw_{k+l} = 0$. But since $w_1, w_2, \ldots, w_k, \ldots, w_{k+l}$ are linearly independent over R, so $b_n = 0$ for all $n \in \{1, 2, \ldots, l\}$. Hence we have the claim. This contradicts that $\dim(V/\operatorname{Im}\lambda)$ is finite. This proves that $(\alpha)_q \neq (\alpha)_b$. By Proposition 1.8, $AE(V) \notin \boldsymbol{BQ}$.

Hence the theorem is completely proved. $\square$

Finally, we show that the finiteness of $\dim V$ is also necessary and sufficient for each of the linear transformation semigroups $MAE(V)$ and $EAM(V)$ to belong to $\boldsymbol{BQ}$.

Theorem 3.24. *The semigroup $MAE(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. Assume that $\dim V$ is infinite. Let B be a basis of V and $A = \{ u_n \mid n \in \mathbb{N} \} \subseteq B$ where $u_i \neq u_j$ if $i \neq j$. Define $\alpha,\ \beta,\ \gamma \in L(V)$ by

$$v\alpha = \begin{cases} u_{n+2} & \text{if } v = u_n \text{ for some } n \in \mathbb{N}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} u_{n+1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{ 1 \}, \\ v & \text{if } v \in (B \smallsetminus A) \cup \{ u_1 \} \end{cases}$$

and

$$v\gamma = \begin{cases} u_{n+1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{ 1,2,3 \}, \\ v & \text{if } v \in (B \smallsetminus A) \cup \{ u_1, u_2, u_3 \}. \end{cases}$$

Then α, β and γ are one-to-one by Proposition 1.12(vii), $\operatorname{Im}\alpha = \langle B \smallsetminus \{ u_1, u_2 \} \rangle$, $\operatorname{Im}\beta = \langle B \smallsetminus \{ u_2 \} \rangle$ and $\operatorname{Im}\gamma = \langle B \smallsetminus \{ u_4 \} \rangle$. Thus from Proposition 1.12(iv), $\dim(V/\operatorname{Im}\alpha) = 2$, $\dim(V/\operatorname{Im}\beta) = 1$ and $\dim(V/\operatorname{Im}\gamma) = 1$. Hence $\alpha,\ \beta,\ \gamma \in MAE(V)$. By the definitions of α, β and γ, we have that

$$u_1\beta\alpha = u_1\alpha = u_3 = u_3\gamma = u_1\alpha\gamma,$$

$$u_n\beta\alpha = u_{n+1}\alpha = u_{n+3} = u_{n+2}\gamma = u_n\alpha\gamma \quad \text{for any } n > 1,$$

$$v\beta\alpha = v = v\alpha\gamma \quad \text{for any } v \in B \smallsetminus A.$$

It then follows that $\beta\alpha = \alpha\gamma$, so $\beta\alpha \in MAE(V)\alpha \cap \alpha MAE(V)$. By Proposition 1.4, $\beta\alpha \in (\alpha)_q$. Suppose that $\beta\alpha \in (\alpha)_b$. Since $u_2\beta\alpha = u_5 \neq u_4 = u_2\alpha$, $\beta\alpha \neq \alpha$.

Then by Proposition 1.4, $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in MAE(V)$. By Proposition 1.9(i), we have $\beta = \alpha\lambda$. Then

$$B \setminus \{ u_2 \} = B\beta = B\alpha\lambda = (B \setminus \{ u_1, u_2 \})\lambda. \tag{1}$$

Since λ is one-to-one, $\lambda : V \to V\lambda$ is an isomorphism. Consequently, $V/W \cong V\lambda/W\lambda$ for every subspace W of V (see the proof of Lemma 3.1). Hence

$$\begin{aligned} 2 &= \dim\left(V/\langle B \setminus \{ u_1, u_2 \}\rangle\right) && \text{from Proposition 1.12(iv)} \\ &= \dim\left(V\lambda/\langle B \setminus \{ u_1, u_2 \}\rangle\lambda\right) \\ &\le \dim\left(V/\langle B \setminus \{ u_1, u_2 \}\rangle\lambda\right) \\ &= \dim\left(V/\langle (B \setminus \{ u_1, u_2 \})\lambda\rangle\right) \\ &= \dim\left(V/\langle B \setminus \{ u_2 \}\rangle\right) = 1 && \text{from (1) and Proposition 1.12(iv)} \end{aligned}$$

which is a contradiction. Thus $\beta\alpha \notin (\alpha)_b$, so $(\alpha)_q \neq (\alpha)_b$. By Proposition 1.8, $MAE(V)$ does not belong to $\boldsymbol{BQ}$. This proves that if $MAE(V)$ is in $\boldsymbol{BQ}$, then $\dim V < \infty$.

As was mentioned in Chapter I, page 15, $MAE(V) = G(V)$ if $\dim V < \infty$, so $MAE(V) \in \boldsymbol{BQ}$ if $\dim V < \infty$.

Hence the proof is complete. $\square$

Theorem 3.25. *The semigroup $EAM(V)$ is in $\boldsymbol{BQ}$ if and only if $\dim V < \infty$.*

Proof. Assume that $\dim V$ is infinite. Let B be a basis of V and $A = \{ u_n \mid n \in$

$\mathbb{N}\,\} \subseteq B$ where $u_i \neq u_j$ for all distinct $i \neq j$. Define α, β, $\gamma \in L(V)$ by

$$v\alpha = \begin{cases} 0 & \text{if } v \in \{\, u_1, u_2 \,\}, \\ u_{n-2} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{\, 1, 2 \,\}, \\ v & \text{if } v \in B \smallsetminus A, \end{cases}$$

$$v\beta = \begin{cases} 0 & \text{if } v = u_1, \\ u_{n-1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{\, 1 \,\}, \\ v & \text{if } v \in B \smallsetminus A \end{cases}$$

and

$$v\gamma = \begin{cases} 0 & \text{if } v = u_1, \\ u_{n-1} & \text{if } v = u_n \text{ for some } n \in \mathbb{N} \smallsetminus \{\, 1 \,\}, \\ v & \text{if } v \in B \smallsetminus A. \end{cases}$$

Then $\operatorname{Im}\alpha = \operatorname{Im}\beta = \operatorname{Im}\gamma = \langle B \cup \{\, 0 \,\}\rangle = V$ and by Proposition 1.12(viii), $\operatorname{Ker}\alpha = \langle u_1, u_2\rangle$, $\operatorname{Ker}\beta = \langle u_1\rangle$ and $\operatorname{Ker}\gamma = \langle u_1\rangle$, so we have that α, β, $\gamma \in EAM(V)$. From the definitions of α, β and γ, we have the following equalities.

$$\begin{aligned} & u_1\beta\alpha = 0\alpha = 0 = u_1\alpha = u_1\alpha\gamma, \\ & u_2\beta\alpha = u_1\alpha = 0 = 0\gamma = u_2\alpha\gamma, \\ & u_3\beta\alpha = u_2\alpha = 0 = u_1\gamma = u_3\alpha\gamma, \\ & u_n\beta\alpha = u_{n-1}\alpha = u_{n-3} = u_{n-2}\gamma = u_n\alpha\gamma \qquad \text{if } n > 3 \text{ and} \\ & v\beta\alpha = v = v\alpha\gamma \quad \text{for all } v \in B \smallsetminus A. \end{aligned}$$

It then follows that $\beta\alpha = \alpha\gamma \in EAM(V)\alpha \cap \alpha EAM(V)$. By Proposition 1.4, $\alpha\gamma \in (\alpha)_q$. Suppose that $\alpha\gamma \in (\alpha)_b$. Since $u_3\alpha\gamma = 0 \neq u_1 = u_3\alpha$. By Proposition 1.4, $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in EAM(V)$. By Proposition 1.9(i), $\gamma = \lambda\alpha$. From the definition of γ,

$$\dim \operatorname{Ker}(\lambda\alpha) = \dim \operatorname{Ker}\gamma = 1. \tag{1}$$

Since $\operatorname{Im}\lambda = V$, there are $u_1', u_2' \in V$ such that $u_1'\lambda = u_1$ and $u_2'\lambda = u_2$. Since u_1 and u_2 are linearly independent, we have from Proposition 1.12(v) that

$$u_1' \text{ and } u_2' \text{ are linearly independent.} \tag{2}$$

We also have

$$\{\, u_1', u_2' \,\}\gamma = \{\, u_1', u_2' \,\}\lambda\alpha = \{\, u_1, u_2 \,\}\alpha = \{0\}. \tag{3}$$

Therefore (1), (2) and (3) yield a contradiction. Consequently, $\alpha\gamma \notin (\alpha)_b$. By Proposition 1.8, $EAM(V) \notin \boldsymbol{BQ}$. This proves that if $EAM(V) \in \boldsymbol{BQ}$, then $\dim V < \infty$.

As was mentioned in Chapter I, page 15, $EAM(V) = G(V)$ if $\dim V$ is finite, hence the converse holds. □

Finally, we shall show that $AME(V)$ is also regular which implies that it is a BQ-semigroup.

Proposition 3.26. *The semigroup $AME(V)$ is a regular semigroup.*

Proof. Let $\alpha \in AME(V)$. Then $\dim \operatorname{Ker}\alpha$ and $\dim (V/\operatorname{Im}\alpha)$ are finite. Let B_1 and B_2 be bases of $\operatorname{Ker}\alpha$ and and $\operatorname{Im}\alpha$, respectively. Let B be a basis of V containing B_2. For each $v \in B_2$, let $u_v \in V$ be such that $u_v\alpha = v$. It then follows from Proposition 1.12(vi) that $B_1 \cup \{\, u_v \mid v \in B_2 \,\}$ is a basis of V. By Proposition 1.12(iv),

$$\dim (V/\operatorname{Im}\alpha) = \dim (V/\langle B_2 \rangle) = |B \smallsetminus B_2|,$$

so $|B \smallsetminus B_2| < \infty$. Define $\beta \in L(V)$ by

$$v\beta = \begin{cases} u_v & \text{if } v \in B_2, \\ 0 & \text{if } v \in B \smallsetminus B_2. \end{cases}$$

Then $\operatorname{Ker}\beta = \langle B \setminus B_2 \rangle$ by Proposition 1.12(viii) and $\operatorname{Im}\beta = \langle \{ u_v \mid v \in B_1 \} \rangle$. Thus $\dim \operatorname{Ker}\beta = |B \setminus B_2| < \infty$ and we also have by Proposition 1.12(iv) that

$$\begin{aligned}\dim (V/\operatorname{Im}\beta) &= \dim (\langle B_1 \cup \{ u_v \mid v \in B_2 \} \rangle / \langle \{ u_v \mid v \in B_2 \} \rangle) \\ &= |B_1| = \dim \operatorname{Ker}\alpha < \infty.\end{aligned}$$

Hence $\beta \in AME(V)$. Since $B_1 \cup \{ u_v \mid v \in B_2 \}$ is a basis of V,

$$v\alpha\beta\alpha = 0 = v\alpha \quad \text{for all } v \in B_1 \quad \text{and}$$

$$u_v\alpha\beta\alpha = v\beta\alpha = u_v\alpha \quad \text{for all } v \in B_2,$$

we have $\alpha\beta\alpha = \alpha$. This proves that $AME(V)$ is regular, as required. $\square$

Therefore by Proposition 3.26 and Proposition 1.5, we have

Theorem 3.27. *The semigroup $AME(V)$ always belongs to $\boldsymbol{BQ}$.*

Observe that $0 \in AME(V)$ if and only if $\dim V < \infty$. Because $AME(V)$ is always a $\boldsymbol{BQ}$-semigroup, it is natural to ask whether $AME(V)$ is left 0-simple and/or right 0-simple if $\dim V < \infty$ and whether it is left simple and/or right simple if $\dim V$ is infinite (see Proposition 1.6). The following proposition is the answer.

Proposition 3.28. *The following statements hold.*

(i) *If $\dim V < \infty$, then $AME(V)$ is left [right] 0-simple if and only if $\dim V = 1$.*

(ii) *If $\dim V$ is infinite, then $AME(V)$ is neither left simple nor right simple.*

Proof. Since $AME(V) = L(V)$ if $\dim V < \infty$, we have that

$$AME(V) \begin{cases} = \{ 0 \} & \text{if } \dim V = 0, \\ \cong R & \text{if } \dim V = 1. \end{cases}$$

Since R is a division ring, it follows that if $\dim V = 1$, then $AME(V)$ is both left 0-simple and right 0-simple. Next, we assume that $\dim V > 1$ and let

$$C = \{\, \alpha \in AME(V) \mid \dim \operatorname{Ker} \alpha > 1 \,\},$$

$$D = \{\, \alpha \in AME(V) \mid \dim (V/\operatorname{Im} \alpha) > 1 \,\}.$$

Let B be a basis of V and $u, w \in B$ such that $u \neq w$. Define $\beta, \gamma \in L(V)$ by

$$v\beta = \begin{cases} v & \text{if } v \in B \smallsetminus \{\, u, w \,\}, \\ 0 & \text{if } v = u \text{ or } v = w \end{cases}$$

and

$$v\gamma = \begin{cases} v & \text{if } v \in B \smallsetminus \{u\}, \\ 0 & \text{if } v = u. \end{cases}$$

Then by Proposition 1.12(viii), $\operatorname{Ker} \beta = \langle u, w \rangle$ and $\operatorname{Ker} \gamma = \langle u \rangle$. Moreover, $\operatorname{Im} \beta = \langle B \smallsetminus \{\, u, w \,\} \rangle$ and $\operatorname{Im} \gamma = \langle B \smallsetminus \{u\} \rangle$. Therefore we have that $\dim \operatorname{Ker} \beta = 2$, $\dim (V/\operatorname{Im} \beta) = 2$, $\dim \operatorname{Ker} \gamma = 1$ and $\dim (V/\operatorname{Im} \gamma) = 1$. It thus follows that $\beta \in C$, $\beta \in D$, $\gamma \in AME(V) \smallsetminus C$ and $\gamma \in AME(V) \smallsetminus D$. This shows that C and D are nonempty proper subsets of $AME(V)$. Since $\operatorname{Ker} \alpha\beta \supseteq \operatorname{Ker} \alpha$ and $\operatorname{Im} \beta\alpha \subseteq \operatorname{Im} \alpha$ for all $\alpha, \beta \in AME(V)$, we deduce that C is a proper right ideal and D is a proper left ideal of $AME(V)$. This proves that (i) and (ii) hold. $\square$

Remark 3.29. The known result in Proposition 1.10 about M_X and E_X motivates us to study $M(V)$ and $E(V)$. Also, our study on $BL(V)$ in Theorem 3.17 is motivated by the known result in Proposition 1.11. After that, many other linear transformation semigroups are considered. As can be seen in this chapter, many linear transformation semigroups are characterized when to be BQ-semigroups. Our technique of proofs use some knowledge of cardinalities of sets and linear algebra. Especially, suitable constructions of linear transformations to achieve our goals are really important. It is quite clearly seen that if we define the transformation semigroups on a sets which were not defined in [9] in the similar way as in this chapter, the expected results will be obtained by replacing $\dim V$ with $|X|$. Moreover, the proofs will be about the same or easier.

CHAPTER IV

ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

In this chapter, we are concerned with any order-preserving transformation semigroups $T_{OP}(I)$ on a nonempty interval I of real numbers under usual ordering. The aim is to characterize when $T_{OP}(I)$ is in $\boldsymbol{BQ}$ in terms of I. Proposition 1.8, Proposition 1.13 and suitable constructions of mappings are important tools.

It is obvious that there are 9 types of nonempty intervals of $\mathbb{R}$ as follows where $a, b \in \mathbb{R}$.

(1) $\mathbb{R}$,

(2) (a, ∞), (3) $[a, \infty)$,

(4) $(-\infty, a)$, (5) $(-\infty, a]$,

(6) (a, b) where $a < b$, (7) $(a, b]$ where $a < b$,

(8) $[a, b)$ where $a < b$, (9) $[a, b]$ where $a \leq b$.

To provide the main result, a series of following lemmas are required.

Lemma 4.1. *$T_{OP}(\mathbb{R})$ is not in $\boldsymbol{BQ}$.*

Proof. Define $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ by

$$x\alpha = 2^x, \quad x\beta = 3x \quad \text{and} \quad x\gamma = x^3 \text{ for all } x \in \mathbb{R}.$$

Then all of α, β and γ are one-to-one and increasing on $\mathbb{R}$, so $\alpha,\ \beta,\ \gamma \in T_{OP}(\mathbb{R})$.

Moreover, $\operatorname{Im}\alpha = (0,\infty)$ and $\operatorname{Im}\beta = \operatorname{Im}\gamma = \mathbb{R}$. Since for every $x \in \mathbb{R}$,

$$x\beta\alpha = (x\beta)\alpha = (3x)\alpha = 2^{3x},$$

$$x\alpha\gamma = (x\alpha)\gamma = (2^x)\gamma = (2^x)^3 = 2^{3x},$$

it follows that $\alpha \neq \beta\alpha = \alpha\gamma \in T_{OP}(\mathbb{R})\alpha \cap \alpha T_{OP}(\mathbb{R})$. By Proposition 1.4, $\beta\alpha \in (\alpha)_q$. To show that $\beta\alpha \notin (\alpha)_b$, suppose that $\beta\alpha \in (\alpha)_b$. From Proposition 1.4, $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in T_{OP}(\mathbb{R})$. Since α is one-to-one, $\beta = \alpha\lambda$ (Proposition 1.9(i)). It then follows that

$$\lambda|_{\operatorname{Im}\alpha} = \lambda|_{(0,\infty)} = \alpha^{-1}\beta, \tag{1}$$

$$\mathbb{R} = \mathbb{R}\beta = \mathbb{R}\alpha\lambda = (\mathbb{R}\alpha)\lambda = (0,\infty)\lambda. \tag{2}$$

From (1), we have that $\lambda|_{(0,\infty)}$ is one-to-one. Because $0\lambda \in \mathbb{R}$, by (2), there exists $d \in (0,\infty)$ such that $0\lambda = d\lambda$. But since λ is order-preserving, it follows that $(0,d]\lambda = \{0\lambda\}$. This is a contradiction because $\lambda|_{(0,\infty)}$ is one-to-one. Hence $\beta\alpha \notin (\alpha)_b$. By Proposition 1.8, we have $T_{OP}(\mathbb{R}) \notin \boldsymbol{BQ}$. □

Lemma 4.2. *If $a \in \mathbb{R}$ and $I = (a,\infty)$ or $[a,\infty)$, then $T_{OP}(I)$ is not in $\boldsymbol{BQ}$.*

Proof. Define $\alpha, \beta, \gamma : [a,\infty) \to \mathbb{R}$ by

$$\begin{aligned} x\alpha &= \frac{x-a}{x-a+1} + a, \\ x\beta &= 2x - a, \\ x\gamma &= \frac{2x-2a}{x-a+1} + a \\ &\quad \text{for all } x \in [a,\infty). \end{aligned} \tag{1}$$

Then we have

$$\alpha, \beta \text{ and } \gamma \text{ are continuous on } I, \tag{2}$$

$$x\alpha' = \frac{(x-a+1)-(x-a)}{(x-a+1)^2} = \frac{1}{(x-a+1)^2} > 0,$$
$$x\beta' = 2 > 0, \tag{3}$$
$$x\gamma' = \frac{(x-a+1)(2)-(2x-2a)}{(x-a+1)^2} = \frac{2}{(x-a+1)^2} > 0$$

for all $x \in [a, \infty)$. It then follows from (3) that α, β, γ are strictly increasing on $[a, \infty)$. From (1), we have that

$$a\alpha = a\beta = a\gamma = a. \tag{4}$$

Consequently, α, β, $\gamma \in T_{OP}(I)$ and all of them are one-to-one. Let

$$\alpha_1 = \alpha|_I, \quad \beta_1 = \beta|_I \quad \text{and} \quad \gamma_1 = \gamma|_I.$$

Then $\alpha_1, \beta_1, \gamma_1 \in T_{OP}(I)$ from (3) and (4). Observe that if $a \in I$, then $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $\gamma_1 = \gamma$. We claim that $\beta_1\alpha_1 = \alpha_1\gamma_1$. To show this, let $x \in I$. Then

$$\begin{aligned}
x\beta_1\alpha_1 = (x\beta_1)\alpha_1 &= (2x-a)\alpha_1 \\
&= \frac{(2x-a)-a}{(2x-a)-a+1} + a \\
&= \frac{2x-2a}{2x-2a+1} + a, \\
x\alpha_1\gamma_1 = (x\alpha_1)\gamma_1 &= \Big(\frac{x-a}{x-a+1} + a\Big)\gamma \\
&= \frac{2\Big(\frac{x-a}{x-a+1} + a\Big) - 2a}{\Big(\frac{x-a}{x-a+1} + a\Big) - a + 1} + a \\
&= \frac{2\Big(\frac{x-a}{x-a+1}\Big)}{\frac{x-a}{x-a+1} + 1} + a \\
&= \frac{2x-2a}{2x-2a+1} + a.
\end{aligned}$$

Thus $\beta_1\alpha_1 = \alpha_1\gamma_1 \in T_{OP}(I)\alpha_1 \cap \alpha_1 T_{OP}(I)$. By Proposition 1.4, $\beta_1\alpha_1 \in (\alpha_1)_q$. Since $a+1 \in I$, $(a+1)\alpha_1 = \dfrac{a+1-a}{a+1-a+1} + a = \dfrac{1}{2} + a$ and

$$(a+1)\beta_1\alpha_1 = (2a+2-a)\alpha_1 = (a+2)\alpha_1 = \frac{a+2-a}{a+2-a+1} + a = \frac{2}{3} + a,$$

we have that $\beta_1\alpha_1 \neq \alpha_1$. Suppose that $\beta_1\alpha_1 \in (\alpha_1)_b$. From Proposition 1.4, $\beta_1\alpha_1 = \alpha_1\lambda\alpha_1$ for some $\lambda \in T_{OP}(I)$. But since α_1 is one-to-one, $\beta_1 = \alpha_1\lambda$. From (1), (3) and (4), we have $\operatorname{Im}\beta = [a, \infty)$. Since

$$\lim_{x\to\infty}(x\alpha) = \lim_{x\to\infty}\Big(\frac{1-\frac{a}{x}}{1-\frac{a}{x}+\frac{1}{x}} + a\Big) = a+1,$$

we have from (2)-(4) that $\operatorname{Im}\alpha = [a, a+1)$. Hence we have

$$\operatorname{Im}\beta_1 = I \quad \text{and} \quad \operatorname{Im}\alpha_1 = I \cap [a, a{+}1). \tag{5}$$

Since $\beta_1 = \alpha_1\lambda$ and both α_1 and β_1 are one-to-one, from (5) we conclude that

$$\lambda|_{I\cap[a,a+1)} = \alpha_1^{-1}\beta_1 \text{ which is one-to-one} \tag{6}$$

Moreover, from (5),

$$I = I\beta_1 = I\alpha_1\lambda = (I\alpha_1)\lambda = (I \cap [a, a+1))\lambda. \tag{7}$$

Since $a{+}1 \in I$, $(a{+}1)\lambda \in I$, so by (7) $(a{+}1)\lambda = d\lambda$ for some $d \in I \cap [a, a{+}1)$. Then $d < a{+}1$ and $d\lambda = (a{+}1)\lambda$. But λ is order-preserving, thus $[d, a{+}1)\lambda = \{(a{+}1)\lambda\}$. Now, we have

$$d < a{+}1, \;\; [d, a{+}1) \subseteq I \cap [a, a{+}1) \quad \text{and} \quad [d, a{+}1)\lambda = \{(a{+}1)\lambda\}. \tag{8}$$

Then (6) and (8) yield a contradiction. Hence $\beta_1\alpha_1 \notin (\alpha_1)_b$. We therefore have from Proposition 1.8 that $T_{OP}(I) \notin \boldsymbol{BQ}$. □

The following lemma is directly obtained from Lemma 4.2 and Proposition 1.13(i) and (ii).

Lemma 4.3. *If $a \in \mathbb{R}$ and $I = (-\infty, a)$ or $(-\infty, a]$, then $T_{OP}(I) \notin \boldsymbol{BQ}$.*

Lemma 4.4. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let I be (a, b) or $(a, b]$. Then $T_{OP}(I) \notin \boldsymbol{BQ}$.*

Proof. Define $\alpha, \beta, \gamma : [a, b] \to \mathbb{R}$ by

$$x\alpha = \frac{x}{2} + \frac{b}{2} \text{ for all } x \in [a, b] \tag{1}$$

$$x\beta = \begin{cases} \dfrac{2x}{3} + \dfrac{a}{3} & \text{if } a \le x < \dfrac{a+b}{2}, \\ \dfrac{4x}{3} - \dfrac{b}{3} & \text{if } \dfrac{a+b}{2} \le x \le b, \end{cases} \tag{2}$$

$$x\gamma = \begin{cases} \dfrac{2x}{3} + \dfrac{a+b}{6} & \text{if } a \le x < \dfrac{a+3b}{4}, \\ \dfrac{4x}{3} - \dfrac{b}{3} & \text{if } \dfrac{a+3b}{4} \le x \le b. \end{cases} \tag{3}$$

We then have respectively from (1), (2) and (3) that

$$a\alpha = \frac{a+b}{2} \in (a, b), \quad b\alpha = b \quad \text{and} \quad x\alpha' = \frac{1}{2} \text{ for all } x \in [a, b], \tag{4}$$

$$a\beta = a, \quad b\beta = b,$$

$$x\beta' = \begin{cases} \dfrac{2}{3} & \text{if } a \le x < \dfrac{a+b}{2}, \\ \dfrac{4}{3} & \text{if } \dfrac{a+b}{2} < x \le b, \end{cases} \tag{5}$$

$$\lim_{x \to (\frac{a+b}{2})^-} (x\beta) = \frac{2}{3}\left(\frac{a+b}{2}\right) + \frac{a}{3} = \frac{2a+b}{3} \quad \text{and}$$

$$\left(\frac{a+b}{2}\right)\beta = \frac{4}{3}\left(\frac{a+b}{2}\right) - \frac{b}{3} = \frac{2a+b}{3},$$

$$a\gamma = \frac{5a+b}{6} \in (a, b), \quad b\gamma = b,$$

$$x\gamma' = \begin{cases} \dfrac{2}{3} & \text{if } a \le x < \dfrac{a+3b}{4}, \\ \dfrac{4}{3} & \text{if } \dfrac{a+3b}{4} < x \le b, \end{cases} \tag{6}$$

$$\lim_{x \to (\frac{a+3b}{4})^-} (x\gamma) = \frac{2}{3}\left(\frac{a+3b}{4}\right) + \frac{a+b}{6} = \frac{a+2b}{3} \quad \text{and}$$

$$\left(\frac{a+3b}{4}\right)\gamma = \frac{4}{3}\left(\frac{a+3b}{4}\right) - \frac{b}{3} = \frac{a+2b}{3}.$$

From (4), (5) and (6), we have that α, β and γ are continuous, order-preserving and one-to-one and α, β, $\gamma : [a,b] \to [a,b]$. This implies that α_1, β_1 ,$\gamma_1 \in T_{OP}(I)$ where $\alpha_1 = \alpha|_I$, $\beta_1 = \beta|_I$ and $\gamma_1 = \gamma|_I$. To show that $\beta_1\alpha_1 = \alpha_1\gamma_1$. Let $x \in I$.

Case 1: $x < \dfrac{a+b}{2}$. Then by (1)-(3),

$$x\beta_1\alpha_1 = (x\beta_1)\alpha_1 = (\frac{2x+a}{3})\alpha_1 = \frac{1}{2}(\frac{2x+a}{3}) + \frac{b}{2} = \frac{2x+a+3b}{6},$$
$$x\alpha_1\gamma_1 = (x\alpha_1)\gamma_1 = (\frac{x+b}{2})\gamma_1 = \frac{2}{3}(\frac{x+b}{2}) + \frac{a+b}{6} = \frac{2x+a+3b}{6}$$

since $x < \dfrac{a+b}{2} \Rightarrow \dfrac{x+b}{2} < \dfrac{\frac{a+b}{2}+b}{2} = \dfrac{a+3b}{4}$.

Case 2: $x \geq \dfrac{a+b}{2}$. Then $\dfrac{x+b}{2} \geq \dfrac{\frac{a+b}{2}+b}{2} = \dfrac{a+3b}{4}$, so we have from (1)-(3) that

$$x\beta_1\alpha_1 = (x\beta_1)\alpha_1 = (\frac{4x-b}{3})\alpha_1 = \frac{1}{2}(\frac{4x-b}{3}) + \frac{b}{2} = \frac{2x+b}{3},$$
$$x\alpha_1\gamma_1 = (x\alpha_1)\gamma_1 = (\frac{x+b}{2})\gamma_1 = \frac{4}{3}(\frac{x+b}{2}) - \frac{b}{3} = \frac{2x+b}{3}$$

Hence $\beta_1\alpha_1 = \alpha_1\gamma_1 \in T_{OP}(I)\alpha_1 \cap \alpha_1 T_{OP}(I) = (\alpha_1)_q$. Suppose that $\beta_1\alpha_1 \in (\alpha_1)_b$. Since α_1 is one-to-one and $\beta_1 \neq 1_X$, by Proposition 1.9(i), $\beta_1\alpha_1 \neq \alpha_1$. By Proposition 1.4, $\beta_1\alpha_1 = \alpha_1\lambda\alpha_1$ for some $\lambda \in T_{OP}(I)$. Then $\beta_1 = \alpha_1\lambda$ since α_1 is one-to-one. Since $a \notin I$, from (4) and (5) and the continuity of α and β, we have

$$\operatorname{Im}\alpha_1 = I \cap (\frac{a+b}{2}, b] \quad \text{and} \quad \operatorname{Im}\beta_1 = I \tag{7}$$

respectively. Since $\beta_1 = \alpha_1\lambda$, we have from (7) that

$$I = I\beta_1 = I\alpha_1\lambda = (I\alpha_1)\lambda = \big(I \cap (\frac{a+b}{2}, b]\big)\lambda. \tag{8}$$

We also have

$$\lambda|_{\operatorname{Im}\alpha_1} = \alpha_1^{-1}\beta_1 \text{ which is one-to-one} \tag{9}$$

since α_1 and β_1 are one-to-one. For any case of I, $I \cap (a, \frac{a+b}{2}) \neq \varnothing$. Let $c \in I \cap (a, \frac{a+b}{2})$. By (8), $c\lambda = d\lambda$ for some $d \in I \cap (\frac{a+b}{2}, b]$. Then

$$c < \frac{a+b}{2} < d \begin{cases} < b \text{ if } I = (a, b), \\ \leq b \text{ if } I = (a, b]. \end{cases} \tag{10}$$

Since λ is order-preserving, we have $[c, d]\lambda = \{c\lambda\}$ which implies that $(\frac{a+b}{2}, d]\lambda = \{d\lambda\}$. By (7) and (10), $(\frac{a+b}{2}, d] \subseteq \operatorname{Im} \alpha_1$. Now, we have

$$\frac{a+b}{2} < d, \ (\frac{a+b}{2}, d] \subseteq \operatorname{Im} \alpha_1 \quad \text{and} \quad (\frac{a+b}{2}, d]\lambda = \{d\lambda\}. \tag{11}$$

From (9) and (11), we have a contradiction. Therefore, $\beta_1\alpha_1 \notin (\alpha_1)_b$. By Proposition 1.8, we have $T_{OP}(I) \notin \boldsymbol{BQ}$, as desired. $\square$

We also have the following lemma from Lemma 4.4 and Proposition 1.13(iii).

Lemma 4.5. *If $a, b \in \mathbb{R}$ are such that $a < b$, then $T_{OP}([a, b)) \notin \boldsymbol{BQ}$.*

Now we are ready to give the main result of this chapter.

Theorem 4.6. *For a nonempty interval I of $\mathbb{R}$, $T_{OP}(I) \in \boldsymbol{BQ}$ if and only if I is closed and bounded.*

Proof. If I is closed and bounded, by Proposition 1.14 and Proposition 1.5, $T_{OP}(I) \in \boldsymbol{BQ}$.

On the other hand, assume that I is neither closed nor bounded. Then I is one of the types (1)–(8) mentioned at the beginning of this chapter. Hence by Lemma 4.1–Lemma 4.5, we conclude that $T_{OP}(I) \notin \boldsymbol{BQ}$. $\square$

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